Advanced Macroeconomics II, Part I (2016)
Problem Set 1: Investment, Tobin's q, Convex Adjustment Costs

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1 Time to build

We want to compare two formulations of the investment problem without frictions that differ in the time to build the investment. In class, we assumed that capital purchases became productive immediately. Now suppose that there is a one period lag for investments to become productive, so capital purchased in period t only becomes productive in period $t+1$. Assume all functions are differentiable and create Euler equations for each timing assumption. Explain the differences (if any).

2 Persistence, volatility and the q model

Consider a firm that solves the following investment problem with quadratic adjustment costs:

$$\max_{i_t} \quad \mathbb{E}_0 \left[\sum_{t=0}^{\infty} R^{-t} \left\{ e^{\theta_t} k_t^{\alpha} - p_t i_t - \frac{\gamma}{2} i_t^2 \right\} \right]$$

$$i_t = k_t - (1-\delta)\, k_{t-1}$$

where k_t and i_t are capital and investment per capita, and θ_t follows an AR(1) productivity process:

$$\begin{aligned} \theta_t &= \rho\theta_{t-1} + \varepsilon_t, \qquad 0 \le \rho < 1 \\ \varepsilon_t &\sim_{iid} \mathcal{N}(0, \sigma_\varepsilon^2), \qquad \mathbb{E}[\varepsilon_t \varepsilon_{t+k}] = 0 \ \ \forall k \end{aligned}$$

a) Derive a static expression for investment as a function of marginal q (all variables are in period t).

b) Derive the Euler equation for investment. *Hint: Find an Euler equation for $p_t q_t$ and substitute the static $i - q$ relationship you found in* a).

c) Give an expression for investment that depends explicitly on productivity persistence ρ and variance of shocks σ_ε^2.

d) True or false.

- *The response of optimal investment to a shock ε_t increases with persistence ρ.*
- *Optimal investment is an increasing and convex function of volatility σ_ε^2.*

Justify your answers and show explicitly any additional assumptions you make.

3 More on the Q model

Consider a risk-neutral firm that chooses net investment and capital to maximize her value, subject to quadratic adjustment costs

$$\max_{\{I_t,\ K_{t+1}\}} \quad \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t \left\{ A_t F(K_t) - I_t - \frac{\gamma}{2} I_t^2 \right\} \right]$$

The constraint of capital accumulation is given by:

$$K_{t+1} = K_t + I_t$$

$A_t F(K_t)$ is the gross profit function, where A_t is the profitability of the firm, and the function F satisfies the following conditions: $F' > 0, F'' < 0, F'(0) = \infty$, and $F'(\infty) = 0$. $\mathbb{E}_0$ denotes expected value conditional on date 0 information.

a) Write the Lagrangian for this problem, derive the necessary conditions for optimal investment and an Euler equation for q.

b) First assume profitability is constant $A_t = A$.

- Derive a non-linear system of two difference equations for K_{t+1} and q_{t+1}.
- Find the steady state of the model ($\Delta K_{t+1} = I_t = 0, \Delta q_{t+1} = 0$).
- Represent the system in a phase diagram with K in the x-axis and q in the y-axis. Illustrate the saddle path.

c) Suppose now that A_t fluctuates deterministically between A_L and A_H, where $0 < A_L < A_H$, every T periods (with $T \geq 3$).

- Illustrate with the use of the phase diagram the paths for net investment, the capital stock and Tobin's q.
- Does the firm experience periods of low profitability but high investment (and high profitability but low investment)?

d) Now suppose A_t fluctuates stochastically and persistently between A_L and A_H. How would your answer to the previous question change?

4 Entrepreneurial risk and investment

4.1 Risk neutral manager

Consider one firm which is managed by a risk neutral manager who maximizes the net present value of dividends d_t:

$$\max_{\{k_{t+1},a_{t+1}\}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \frac{1}{R^t} d_t \right] \tag{1}$$

The budget constraint is the following:

$$a_{t+1} = Ra_t + \pi_t - d_t, \qquad a_0 \ \ given \tag{2}$$

where a_t are the financial assets of the firm and π_t are the profits defined as follows:

$$\pi_t = \theta_t k_t^{\alpha} - k_{t+1} \tag{3}$$

The productivity shock θ_t follows a stationary process whose realization is observed at the beginning of period t. Notice that equation (3) implies that capital fully depreciates in one period, and that capital k_{t+1} is chosen in period t before the shock θ_{t+1} is observed. The usual transversality conditions apply.

i) Use the budget constraint to substitute d_t from the value function (1), then derive the first order condition, at a generic time t, with respect to k_{t+1}. Find the optimal level of capital chosen by the risk neutral manager.

ii) Provide a brief economic intuition of the results.

4.2 Risk averse manager

Now consider the same firm, but managed instead by a risk averse owner-manager. The owner wants to maximize the stream of utility from consumption:

$$\max_{\{k_{t+1},a_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \tag{4}$$

where $u\left(c_t\right)$ is strictly concave, and $\beta < 1$ is the subjective discount. The firm's profits are the only source of revenues for the owner, so his consumption is equal to the firm's dividends $c_t = d_t$ (that is, the owner can lend and borrow only through the firm). The budget constraint is the following:

$$a_{t+1} = Ra_t + \pi_t - c_t \tag{5}$$

where profits are still defined by (3).

i) Use the budget constraint (5) to substitute c_t from the value function (4), and then derive the first order condition for k_{t+1}. Find the optimal level of capital chosen by the risk averse owner.

ii) How do your answers differ from point a)? Explain the economic intuition.