The Macroeconomics of Irreversibility

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Online Appendix

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A Proofs

A.1 Auxiliary Theorems

Auxiliary Theorem 1 (**Optional Sampling Theorem, OST**)**.** *Let $\hat{k}$ be a martingale on the filtered space $(\Omega, \mathbb{P}, \mathcal{F})$ and let τ be a stopping time. If $(\{\hat{k}_t\}_t, \tau)$ is a well-defined stopping process, then*

$$\mathbb{E}[\hat{k}_\tau] \;=\; \mathbb{E}[\hat{k}_0]. \tag{A.1}$$

See Theorem 4.4 in Stokey (2009). This result establishes that, under certain conditions, the expected value of a martingale at a stopping time is equal to its initial expected value. We use this result to derive the mappings between the cross-sectional moments of adjusters and non-adjusters.

Auxiliary Theorem 2 (**Occupancy Measure Theorem, OMT**)**.** *Let $\hat{k}_t$ be a Brownian motion, τ a stopping time, and $\hat{k}_\tau = \hat{k}^*$ a constant reset state. Let G be the ergodic distribution of $\hat{k}$. Consider a real-valued function $f(\hat{k})$ such that $\int f(\hat{k})\,\mathrm{d}G(\hat{k}) = \lim_{T\to\infty} T^{-1}\int_0^T f(\hat{k}_t)\,\mathrm{d}t$ for all initial $\hat{k}_0$. Then the following relationship holds:*

$$\underbrace{\mathbb{E}\left[\int_0^\tau f(\hat{k}_t)\,\mathrm{d}t\middle|\hat{k}_0 = \hat{k}^*\right]}_{\text{occupancy measure}} \;=\; \underbrace{\int f(\hat{k})\,\mathrm{d}G(\hat{k})}_{\text{steady-state mass}} \quad \underbrace{\mathbb{E}\left[\tau\middle|\hat{k}_0 = \hat{k}^*\right]}_{\text{proportionality constant}}. \tag{A.2}$$

See Stokey (2009) and the Green measure in Chapter 9 of Oksendal (2007)). This result establishes the equivalence between the occupancy measure—the average time an agent's state spends at a given value—and the stationary mass of agents at that particular state, with a proportionality constant equal to the expected time between adjustments. E.g., if $f(\hat{k}) = \hat{k}^m$, then $\mathbb{E}\left[\int_0^\tau \hat{k}_t^m\,\mathrm{d}t|\hat{k}_0 = \hat{k}^*\right] = \mathbb{E}[\hat{k}^m]\mathbb{E}[\tau|\hat{k}_0 = \hat{k}^*]$. We use this theorem to convert occupancy measures, scaled by frequency, into steady-state cross-sectional moments. We use $\overline{\mathbb{E}}[\cdot] = \mathbb{E}[\cdot|\hat{k}_0 = \hat{k}^*]$.

Auxiliary Theorem 3 (**Equivalence between sequential and recursive formulations**)**.** *Let $\hat{k}_s$ be a Brownian motion ($\mathrm{d}\hat{k}_s = -\nu\,\mathrm{d}s + \sigma\,\mathrm{d}W_s$) inside a continuation region $\mathcal{R}$. Let $g(\hat{k})$ be the flow payoff and $\phi(\hat{k}_s)$ be the terminal payoff(s). Define a (non-optimal) value $w(\hat{k})$ using a sequential formulation as follows:*

$$w(\hat{k}) \;\equiv\; \mathbb{E}\left[\int_0^{\tau_\mathcal{R}} e^{-\rho\tau_\mathcal{R}} g(\hat{k}_s)\,\mathrm{d}s\middle|\hat{k}_0 = \hat{k}\right] \;+\; \mathbb{E}\left[e^{-\rho\tau_\mathcal{R}}\phi(\hat{k}_{\tau_\mathcal{R}})\middle|\hat{k}_0 = \hat{k}\right], \quad \forall \hat{k} \in \mathcal{R}, \tag{A.3}$$

where $\tau_\mathcal{R}$ is <u>any stopping time</u>. Under certain regularity conditions over $\mathcal{R}$, $g(\hat{k})$, and $\phi(\hat{k})$, we have that $\forall \hat{k} \in \mathcal{R}$:

$$\text{(HJB)} \qquad \rho w(\hat{k}) \;=\; g(\hat{k}) \;-\; \nu w'(\hat{k}) \;+\; \frac{\sigma^2}{2} w''(\hat{k}), \tag{A.4}$$

$$\text{(Value Matching)} \qquad \lim_{t\uparrow\tau_\mathcal{R}} w(\hat{k}_t) \;=\; \phi(\hat{k}_{\tau_\mathcal{R}}), \quad a.s. \tag{A.5}$$

If $\tau_\mathcal{R}$ is an <u>optimal stopping time</u> then the smooth-pasting condition also holds:

$$\text{(Smooth Pasting)} \qquad \lim_{t\uparrow\tau_\mathcal{R}} w'(\hat{k}_t) \;=\; \phi(\hat{k}_{\tau_\mathcal{R}}), \quad a.s. \tag{A.6}$$

If there exist a function $w_1 \in \mathbb{C}^2(\mathcal{R})$ and w_1 satisfies (A.4) and (A.5) (and (A.6) if optimal), then $w_1 = w$.

See Chapters 9 and 10 in Oksendal (2007). These results allow us to go back and forth between w's sequential formulation—given by the cumulative flow payoff during inaction plus the value at termination—and the recursive formulations—with an HJB in the interior of the inaction region, value-matching conditions when stopping, and smooth pasting conditions if the stopping policy is optimal.

A.2 Proof of Proposition 1

Proposition 1. ***(Optimal policy)*** *Marginal $q(\hat{k})$ and the optimal policy $\{\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+\}$ is characterized by the following sufficient optimality conditions:*

(i) Inside the inaction region $\mathcal{R}$, $q(\hat{k})$ solves the Hamilton-Jacobini-Bellman (HJB) equation:

$$\mathcal{U}q(\hat{k}) = \frac{\alpha e^{(\alpha-1)\hat{k}}}{p} - \nu q'(\hat{k}) + \frac{\sigma^2}{2}q''(\hat{k}), \qquad \forall\, \hat{k} \in (\hat{k}^-, \hat{k}^+). \tag{11}$$

(ii) In the outer inaction regions, $q(\hat{k})$ satisfies the value-matching conditions:

$$\frac{\theta}{p} = \int_{\hat{k}^-}^{\hat{k}^{*-}} e^{\hat{k}} \left(q(\hat{k}) - 1\right) \mathrm{d}\hat{k}, \qquad \forall\, \hat{k} \in [\hat{k}^-, \hat{k}^{*-}], \tag{12}$$

$$\frac{\theta}{p} = \int_{\hat{k}^{*+}}^{\hat{k}^{+}} e^{\hat{k}} \left((1-\omega) - q(\hat{k})\right) \mathrm{d}\hat{k} \qquad \forall\, \hat{k} \in [\hat{k}^{*+}, \hat{k}^{+}]. \tag{13}$$

(iii) At the borders of the inaction region and reset points, $q(\hat{k})$ satisfies the optimality conditions:

$$q(\hat{k}) = 1, \qquad \hat{k} \in \left\{\hat{k}^-, \hat{k}^{*-}\right\}, \tag{14}$$

$$q(\hat{k}) = 1-\omega, \qquad \hat{k} \in \left\{\hat{k}^{*+}, \hat{k}^{+}\right\}. \tag{15}$$

From these conditions, q's stopping-time formulation is given by

$$q(\hat{k}) = \mathbb{E}\left[\int_0^{\tau} \frac{\alpha e^{-\mathcal{U}s + (\alpha-1)\hat{k}_s}}{p}\, \mathrm{d}s + e^{-\mathcal{U}\tau} q(\hat{k}_\tau)\right]. \tag{16}$$

Proof's strategy We divide this proof into three steps.

1. In Step 1, we characterize the two-state value function $V(k,u)$ and optimal policies $(k^-(u), k^{*-}(u), k^{*+}(u), k^+(u))$ through the Hamilton-Jacobi-Bellman (HJB) equation, value matching, optimality, and smooth pasting conditions.

2. In Step 2, we guess that $V(k,u) = uv(\hat{k})$, where $v(\hat{k})$ is a function of the log capital-to-productivity ratio $\hat{k} \equiv \log(k/u)$. Using the guess, we exploit homotheticity in the firm's programming problem to express optimality conditions as joint conditions between $v(\hat{k})$ and the firm's policy $(\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+)$. We use the sufficient conditions that characterize the two-state value function and optimal policies from Step 1 to reduce the state space into one dimension. The corresponding policies are $(\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+)$. We verify the guess by showing that the sufficient conditions for $V(\cdot)$ are equivalent to those satisfied by $v(\cdot)$.

3. In Step 3, we note that $q(\hat{k}) = v'(\hat{k})/(pe^{\hat{k}})$ and establish its sufficient optimality conditions reexpressing the HJB and optimality conditions for v in Step 2. An advantage of characterizing the policy with $q(\hat{k})$ is that it shares the infinitesimal generator with $\hat{k}$; this is not the case with $v(\hat{k})$ as its drift equals $-(\nu + \sigma^2)$.

A.2.1 Step 1: Characterize the two-state value function $V(k,u)$

Substitute output $y_s = u_s^{1-\alpha}k_s^\alpha$ from (1) and the adjustment costs $\theta_s = \theta u_s$ from (3) into the firm problem in (5):

$$V(k_0, u_0) = \max_{\{T_h,\, i_{T_h}\}_{h=1}^\infty} \mathbb{E}\left[\int_0^\infty e^{-rs} u_s^{1-\alpha} k_s^\alpha\, \mathrm{d}s - \sum_{h=1}^{\infty} e^{-rT_h}\left(\theta u_{T_h} + p\,(\Delta k_{T_h})\, i_{T_h}\right)\right]. \tag{A.7}$$

Using the principle of optimality, we get a recursive stopping-time problem with initial conditions $(k_0, u_0) = (k, u)$:

$$V(k,u) \;=\; \max_{\tau,\ \Delta k_\tau} \;\mathbb{E}\left[\int_0^\tau e^{-rs} u_s^{1-\alpha} k_s^\alpha \,\mathrm{d}s \;+\; e^{-r\tau}\left(-\theta u_\tau - \left(p(\Delta k_\tau)\right)\Delta k_\tau + V\left(k_{\tau^-} + \Delta k_\tau, u_\tau\right)\right)\right], \tag{A.8}$$

where we change the notation to $i_{T_h} = \Delta k_\tau = k_\tau - k_{\tau^-}$.

Let $\mathcal{R}$ be the firms' inaction region that is equal to $\mathcal{R} \equiv \{(k,u) : k^-(u) < k < k^+(u)\}$, where $k^-(u)$ is the lower inaction threshold that triggers a positive investment, and $k^+(u)$ is the upper inaction thresholds that trigger a negative investment. For each level of productivity u, let $\mathcal{R}^- \equiv \{(k,u) : k = k^-(u)\}$ denote the lower border of the inaction set and $k^{*-}(u)$ the reset capital after positive adjustment, where $\Delta k(u) = k^{*-}(u) - k^-(u) > 0$. Analogously, we denote the upper border of inaction set as $\mathcal{R}^+ \equiv \{(k,u) : k = k^+(u)\}$ with a reset capital after negative adjustment as $k^{*+}(u)$, where $\Delta k(u) = k^{*+}(u) - k^+(u) < 0$.

Optimality conditions for $V(k,u)$ The value $V(k,u)$ and the optimal policy $(k^-(u), k^{*-}(u), k^{*+}(u), k^+(u))$ satisfy the system of sufficient conditions in (A.9) to (A.17):

1. Inside the inaction region $\mathcal{R}$, $V(k,u)$ solves the HJB equation:

$$rV(k,u) \;=\; u\left(\frac{k}{u}\right)^\alpha - \xi k \frac{\partial V(k,u)}{\partial k} + \left(\mu + \frac{\sigma^2}{2}\right) u \frac{\partial V(k,u)}{\partial u} + \frac{\sigma^2 u^2}{2}\frac{\partial^2 V(k,u)}{\partial u^2} \qquad \forall (k,u) \in \mathcal{R}. \tag{A.9}$$

2. Value matching conditions equalize the value of action and inaction at the borders of the inaction region:

$$V(k^{*-}(u),u) \;-\; p \quad \Delta k(u) - \theta u \;=\; V(k^-(u),u) \quad \forall (k,u) \in \mathcal{R}^-, \tag{A.10}$$

$$V(k^{*+}(u),u) \;-\; p(1-\omega)\Delta k(u) - \theta u \;=\; V(k^+(u),u) \quad \forall (k,u) \in \mathcal{R}^+. \tag{A.11}$$

3. The two optimality conditions for the reset capitals $\{k^{*-}, k^{*+}\}$ are:

$$\frac{\partial V(k^{*-}(u),u)}{\partial k} \;=\; p, \tag{A.12}$$

$$\frac{\partial V(k^{*+}(u),u)}{\partial k} \;=\; p(1-\omega). \tag{A.13}$$

4. The four smooth pasting conditions are:

$$\frac{\partial V(k,u)}{\partial k} \;=\; p \qquad \forall (k,u) \in \mathcal{R}^-, \tag{A.14}$$

$$\frac{\partial V(k,u)}{\partial k} \;=\; p(1-\omega) \quad \forall (k,u) \in \mathcal{R}^+, \tag{A.15}$$

$$\frac{\partial V(k^{*-}(u),u)}{\partial u} \;=\; \theta + \frac{\partial V(k,u)}{\partial u} \quad \forall (k,u) \in \mathcal{R}^-, \tag{A.16}$$

$$\frac{\partial V(k^{*+}(u),u)}{\partial u} \;=\; \theta + \frac{\partial V(k,u)}{\partial u} \quad \forall (k,u) \in \mathcal{R}^+, \tag{A.17}$$

For additional details on the sufficiency of these conditions, see Baley and Blanco (2019, 2021).

A.2.2 Step 2: Characterize the one-state value $v(\hat{k}) = V(k,u)/u$

We guess that $V(k,u)$ is separable:

$$V(k,u) \;=\; u \times v\left(\log\left(\frac{k}{u}\right)\right) \;=\; uv(\hat{k}), \tag{A.18}$$

with associated policies

$$(k^-(u), k^{*-}(u), k^{*+}(u), k^+(u)) \;=\; u \times (e^{\hat{k}^-}, e^{\hat{k}^{*-}}, e^{\hat{k}^{*+}}, e^{\hat{k}^+}). \tag{A.19}$$

Given the guess (A.18), the derivatives of $V(k, u)$ and the derivatives of $v(\hat{k})$ satisfy the following relationships:

$$\frac{\partial V(k,u)}{\partial k} \;=\; \frac{u}{k} v'\left(\log\left(\frac{k}{u}\right)\right) = \frac{u}{k} v'(\hat{k}), \tag{A.20}$$

$$\frac{\partial V(k,u)}{\partial u} \;=\; v\left(\log\left(\frac{k}{u}\right)\right) - v'\left(\log\left(\frac{k}{u}\right)\right) = v(\hat{k}) - v'(\hat{k}), \tag{A.21}$$

$$\frac{\partial^2 V(k,u)}{\partial u^2} \;=\; -\frac{v'\left(\log\left(\frac{k}{u}\right)\right)}{u} + \frac{v''\left(\log\left(\frac{k}{u}\right)\right)}{u} = -\frac{v'(\hat{k})}{u} + \frac{v''(\hat{k})}{u}. \tag{A.22}$$

2a. HJB Substituting the guess into (A.9):

$$rV(k,u) = u\left(\frac{k}{u}\right)^{\alpha} - \xi k \frac{\partial V(k,u)}{\partial k} + \left(\mu + \frac{\sigma^2}{2}\right) u \frac{\partial V(k,u)}{\partial u} + \frac{\sigma^2 u^2}{2} \frac{\partial^2 V(k,u)}{\partial u^2}, \tag{A.23}$$

$$ruv(\hat{k}) = \; ue^{\alpha \hat{k}} - \xi k \frac{u}{k} v'(\hat{k}) + \left(\mu + \frac{\sigma^2}{2}\right) u(v(\hat{k}) - v'(\hat{k})) + \frac{\sigma^2 u^2}{2}\left(\frac{v''(\hat{k})}{u} - \frac{v'(\hat{k})}{u}\right). \tag{A.24}$$

Joining terms we get:

$$\left(r - \mu - \frac{\sigma^2}{2}\right) uv(\hat{k}) \;=\; ue^{\alpha \hat{k}} - (\mu + \xi + \sigma^2) uv'(\hat{k}) + \frac{\sigma^2}{2} uv''(\hat{k}). \tag{A.25}$$

Defining new parameters $\nu \equiv \mu + \xi$ and $\rho \equiv r - \mu - \sigma^2/2$, and dividing both sides by u, we obtain the HJB:

$$\rho v(\hat{k}) \;=\; e^{\alpha \hat{k}} - (\nu + \sigma^2) v'(\hat{k}) + \frac{\sigma^2}{2} v''(\hat{k}). \tag{A.26}$$

2b. Value matching Substituting the guess into (A.10) and (A.11):

$$u_\tau v(\hat{k}^{*-}) \;-\; p\Delta k \;-\; \theta u_\tau \;=\; u_\tau v(\hat{k}^-)\;, \tag{A.27}$$

$$u_\tau v(\hat{k}^{*+}) \;-\; p(1-\omega)\Delta k \;-\; \theta u_\tau \;=\; u_\tau v(\hat{k}^-)\;. \tag{A.28}$$

Next, we express investment in terms of changes in capital productivity ratios $\hat{k}$. The expression (9), which reads $\Delta \hat{k} \;=\; \log(1 + \Delta k / k_{\tau^-})$, implies $\Delta k \;=\; e^{\Delta \hat{k}} k_{\tau^-} - k_{\tau^-}$; multiplying and dividing by u_τ and substituting the definition of $\hat{k}$ yields: $\Delta k = u_\tau \left(e^{\Delta \hat{k} + \hat{k}_\tau} - e^{\hat{k}_\tau}\right)$. Then we use $\hat{k}_\tau = \hat{k}^+$ or $\hat{k}_\tau = \hat{k}^-$ accordingly. Using this notation, we rewrite positive investment as $\Delta k = u_\tau(e^{\Delta \hat{k} + \hat{k}^-} - e^{\hat{k}^-})$ and negative investment as $\Delta k = u_\tau(e^{\Delta \hat{k} + \hat{k}^+} - e^{\hat{k}^+})$. Substituting into (A.27) and (A.28) and dividing both sides by u_τ

$$v(\hat{k}^{*-}) - p(e^{\hat{k}^{*-}} - e^{\hat{k}^-}) - \theta \;=\; v(\hat{k}^-), \tag{A.29}$$

$$v(\hat{k}^{*+}) - p(1-\omega)(e^{\hat{k}^{*+}} - e^{\hat{k}^+}) - \theta \;=\; v(\hat{k}^+). \tag{A.30}$$

2c. Optimality Substituting the guess into (A.12) and (A.13):

$$\frac{\partial V(k^{*-}(u), u)}{\partial k} \;=\; p \iff \frac{u}{k^{*-}(u)} v'(\hat{k}^{*-}) \;=\; p \iff v'(\hat{k}^{*-}) \;=\; pe^{\hat{k}^{*-}}, \tag{A.31}$$

$$\frac{\partial V(k^{*+}(u), u)}{\partial k} \;=\; p(1-\omega) \iff \frac{u}{k^{*+}(u)} v'(\hat{k}^{*-}) \;=\; p(1-\omega) \iff v'(\hat{k}^{*-}) \;=\; p(1-\omega)e^{\hat{k}^{*+}}. \tag{A.32}$$

2d. Smooth pasting for capital Substituting the guess into (A.14) and (A.15)

$$\text{(A.33)} \qquad \frac{\partial V(k^-(u),u)}{\partial k} = p \iff \frac{u}{k^-(u)} v'(\hat{k}^-) = p \iff v'(\hat{k}^-) = pe^{\hat{k}^-},$$

$$\text{(A.34)} \qquad \frac{\partial V(k^+(u),u)}{\partial k} = p(1-\omega) \iff \frac{u}{k^+(u)} v'(\hat{k}^+) = p(1-\omega) \iff v'(\hat{k}^+) = p(1-\omega)e^{\hat{k}^+}.$$

2e. Smooth pasting for idiosyncratic productivity To verify the smooth-pasting for idiosyncratic productivity, we substitute the guess into (A.16) and (A.17) and then substitute $v'(\hat{k}) = pe^{\hat{k}}$ and $v'(\hat{k}) = p(1-\omega)e^{\hat{k}}$ in the outer inaction regions to rewrite $v'(\cdot)$ in terms of prices, which yields

$$\text{(A.35)} \qquad v(\hat{k}^{*-}) = \theta + v(\hat{k}^-) + p(e^{\hat{k}^{*-}} - e^{\hat{k}^-})$$

$$\text{(A.36)} \qquad v(\hat{k}^{*+}) = \theta + v(\hat{k}^+) + p(1-\omega)(e^{\hat{k}^{*+}} - e^{\hat{k}^+})$$

Summary The value $v(\hat{k})$ and the optimal policy $\{\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+\}$ satisfy the conditions:

(i) In the inaction region $\mathcal{R}$, $v(\hat{k})$ solves the HJB equation:

$$\text{(A.37)} \qquad \rho v(\hat{k}) = e^{\alpha\hat{k}} - (\nu + \sigma^2)v'(\hat{k}) + \frac{\sigma^2}{2}v''(\hat{k}), \ \forall \hat{k} \in (\hat{k}^-, \hat{k}^+).$$

(ii) At the borders of inaction, $v(\hat{k})$ satisfies the value-matching conditions:

$$\text{(A.38)} \qquad v(\hat{k}^-) = v(\hat{k}^{*-}) - \theta - p(e^{\hat{k}^{*-}} - e^{\hat{k}^-}),$$

$$\text{(A.39)} \qquad v(\hat{k}^+) = v(\hat{k}^{*+}) - \theta + p(1-\omega)(e^{\hat{k}^+} - e^{\hat{k}^{*+}}).$$

(iii) At the borders of inaction and reset states, $v(\hat{k})$ satisfies the smooth-pasting and the optimality conditions:

$$\text{(A.40)} \qquad v'(\hat{k}) = pe^{\hat{k}}, \qquad \hat{k} \in \left\{\hat{k}^-, \hat{k}^{*-}\right\},$$

$$\text{(A.41)} \qquad v'(\hat{k}) = p(1-\omega)e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^{*+}, \hat{k}^+\right\}.$$

A.2.3 Step 3: Characterizing $q = v'(\hat{k})/pe^{\hat{k}}$

From the definition $q(\hat{k}) \equiv \frac{\partial V(k,u)}{\partial k}/p$, and the decomposition $V(k,u) = uv(\hat{k})$ from Step 2, we have that $q(\hat{k}) = \frac{v'(\hat{k})}{pe^{\hat{k}}}$. Thus, the following relationships hold:

$$\text{(A.42)} \qquad q'(\hat{k}) = \frac{v''(\hat{k})}{pe^{\hat{k}}} - \frac{v'(\hat{k})}{pe^{\hat{k}}} = \frac{v''(\hat{k})}{pe^{\hat{k}}} - q(\hat{k}) \iff \frac{v''(\hat{k})}{pe^{\hat{k}}} = q'(\hat{k}) + q(\hat{k})$$

$$\text{(A.43)} \qquad q''(\hat{k}) = \frac{v'''(\hat{k})}{pe^{\hat{k}}} - 2\frac{v''(\hat{k})}{pe^{\hat{k}}} + \frac{v'(\hat{k})}{pe^{\hat{k}}} = \frac{v'''(\hat{k})}{pe^{\hat{k}}} - 2q'(\hat{k}) - q(\hat{k}) \iff \frac{v'''(\hat{k})}{pe^{\hat{k}}} = q''(\hat{k}) + 2q'(\hat{k}) + q(\hat{k}).$$

3a. HJB We take the first derivative of the HBJ equation for v in (A.37) and then divide by $pe^{\hat{k}}$:

$$\text{(A.44)} \qquad \rho\frac{v'(\hat{k})}{pe^{\hat{k}}} = \frac{\alpha e^{(\alpha-1)\hat{k}}}{p} - (\nu + \sigma^2)\frac{v''(\hat{k})}{pe^{\hat{k}}} + \frac{\sigma^2}{2}\frac{v'''(\hat{k})}{pe^{\hat{k}}}, \ \forall \hat{k} \in (\hat{k}^-, \hat{k}^+).$$

Substituting q's definition and the second and third derivatives of v in (A.42) and (A.43):

$$\rho q(\hat{k}) \;=\; \frac{\alpha e^{(\alpha-1)\hat{k}}}{p} \;-\; (\nu+\sigma^2)\left(q'(\hat{k})+q(\hat{k})\right) \;+\; \frac{\sigma^2}{2}\left(q''(\hat{k})+2q'(\hat{k})+q(\hat{k})\right). \tag{A.45}$$

Joining common terms:

$$\left(\rho+\nu+\frac{\sigma^2}{2}\right)q(\hat{k}) \;=\; \frac{\alpha e^{(\alpha-1)\hat{k}}}{p} \;-\; \nu q'(\hat{k}) \;+\; \frac{\sigma^2}{2}q''(\hat{k}), \tag{A.46}$$

Since $\rho \equiv r-\mu-\sigma^2/2$, then $\rho+\nu+\frac{\sigma^2}{2} = r+\xi := \mathcal{U}$. Substitute to obtain the final expression for $q's$ HJB:

$$\mathcal{U}q(\hat{k}) \;=\; \frac{\alpha e^{(\alpha-1)\hat{k}}}{p} \;-\; \nu q'(\hat{k}) \;+\; \frac{\sigma^2}{2}q''(\hat{k}), \qquad \forall\;\hat{k}\in(\hat{k}^-,\hat{k}^+). \tag{A.47}$$

3b. Value matching The value matching at $\hat{k}^-$ in (A.38) can be written as a definite integral:

$$\theta \;=\; v(\hat{k}^{*-}) \;-\; pe^{\hat{k}^{*-}} \;-\; \left(v(\hat{k}^-)-pe^{\hat{k}^-}\right) \;=\; \int_{\hat{k}^-}^{\hat{k}^{*-}}\left(v'(\hat{k}) \;-\; pe^{\hat{k}}\right)\mathrm{d}\hat{k}. \tag{A.48}$$

Dividing both sides by p, factoring $e^{\hat{k}}$ on the right, and substituting q, we obtain the value matching for q at $\hat{k}^-$:

$$\frac{\theta}{p} \;=\; \int_{\hat{k}^-}^{\hat{k}^{*-}} e^{\hat{k}}\left(\frac{v'(\hat{k})}{pe^{\hat{k}}} \;-\; 1\right)\mathrm{d}\hat{k} = \int_{\hat{k}^-}^{\hat{k}^{*-}} e^{\hat{k}}\left(q(\hat{k}) \;-\; 1\right)\mathrm{d}\hat{k}. \tag{A.49}$$

Similarly, we use (A.39) to obtain the value matching for q at $\hat{k}^+$:

$$\frac{\theta}{p} \;=\; \int_{\hat{k}^{*+}}^{\hat{k}^+} e^{\hat{k}}\left((1-\omega)-q(\hat{k})\right)\mathrm{d}\hat{k}. \tag{A.50}$$

3c. Optimality Substituting q's definition in the optimality conditions for v in (A.40) and (A.41)

$$v'(\hat{k}) \;=\; pe^{\hat{k}} \iff q(\hat{k})=1 \qquad \hat{k}\in\left\{\hat{k}^-,\hat{k}^{*-}\right\} \tag{A.51}$$

$$v'(\hat{k}) \;=\; p(1-\omega)e^{\hat{k}} \iff q(\hat{k})=(1-\omega) \quad \hat{k}\in\left\{\hat{k}^{*+},\hat{k}^+\right\}. \tag{A.52}$$

3d. Stopping-time formulation Given the sufficient conditions, we write the optimal $q(\hat{k})$ using a stopping-time formulation (note that there is no maximization involved):

$$q(\hat{k}) \;\equiv\; \mathbb{E}\left[\int_0^{\tau}\frac{\alpha e^{-\mathcal{U}s+(\alpha-1)\hat{k}_s}}{p}\,\mathrm{d}s \;+\; e^{-\mathcal{U}\tau}Q^*(\hat{k}_\tau)\right], \tag{A.53}$$

where the reset function takes two values: $Q^*(\hat{k}^{*-})=1$ and $Q^*(\hat{k}^{*+})=1-\omega$.

A.3 Cross-sectional distributions

Consider $\theta_s = \theta u_s$, where $\theta > 0$ is a constant fixed adjustment cost. The density and frequencies solve the KFE

$$\nu g'(\hat{k}) \; + \; \frac{\sigma^2}{2} g''(\hat{k}) \; = \; 0, \quad \text{for all} \quad \hat{k} \in (\hat{k}^-, \hat{k}^+) \setminus \{\hat{k}^{*-}, \hat{k}^{*+}\}; \tag{A.54}$$

three border conditions

$$g(\hat{k}) \; = \; 0, \quad \text{for} \;\; \hat{k} \in \{\hat{k}^{*-}, \hat{k}^{*+}\}, \tag{A.55}$$

$$\int_{\hat{k}^-}^{\hat{k}^+} g(\hat{k}) \, \mathrm{d}\hat{k} \; = \; 1; \tag{A.56}$$

two resetting conditions

$$\underbrace{\frac{\sigma^2}{2} \lim_{\hat{k} \downarrow \hat{k}^-} g'(\hat{k})}_{\mathcal{N}^-} \; = \; \frac{\sigma^2}{2} \left[\lim_{\hat{k} \uparrow \hat{k}^{*-}} g'(\hat{k}) - \lim_{\hat{k} \downarrow \hat{k}^{*-}} g'(\hat{k}) \right], \tag{A.57}$$

$$\underbrace{-\frac{\sigma^2}{2} \lim_{\hat{k} \uparrow \hat{k}^+} g'(\hat{k})}_{\mathcal{N}^+} \; = \; \frac{\sigma^2}{2} \left[\lim_{\hat{k} \uparrow \hat{k}^{*+}} g'(\hat{k}) - \lim_{\hat{k} \downarrow \hat{k}^{*+}} g'(\hat{k}) \right], \tag{A.58}$$

and two continuity conditions at the reset points:

$$g(\hat{k}) \in \mathbb{C}, \mathbb{C}^1(\{\hat{k}^{*-}, \hat{k}^{*+}\}), \mathbb{C}^2(\{\hat{k}^{*-}, \hat{k}^{*+}\}). \tag{A.59}$$

Condition (A.55) sets the mass of firms at the inaction thresholds equal to zero. Condition (A.56) ensures that g is a density. Conditions (A.57) and (A.58) relate the masses of upward and downward adjustments to the discontinuities in the derivative of g at the reset points. In a small period of time ds, the mass $\mathcal{N}^-$ that "exits" the inaction region by hitting the lower threshold—equal to $\frac{\sigma^2}{2} \lim_{\hat{k} \downarrow \hat{k}^-} g'(\hat{k})$—must coincide with the mass of firms that "enters" at the reset point $\hat{k}^{*-}$—equal to the jump in g'. This argument is analogous for $\mathcal{N}^+$; in fact, it is straightforward to verify that conditions (A.54) to (A.57) jointly imply condition (A.58), and thus it is redundant.

A.4 Distributions of stopping times τ

Conditional on current capital-productivity ratio Given the inaction thresholds $\hat{k}^- < \hat{k}^+$, we derive the densities of stopping times (first passage time) when firms hit the ℓower threshold $\ell(\tau|\hat{k})$, the υpper threshold $\upsilon(\tau|\hat{k})$, or $h$itting either threshold $h(\tau|\hat{k})$, conditional on a current capital-productivity ratio $\hat{k}$. The first passage time is set to zero after a reset. We use the formulas of the exit times densities when barriers are flat (15) and (16) in Kolkiewicz (2002), adjusted for the drift $-\nu$ and the volatility σ.

- The measure of times for hitting the *lower threshold* for current $\hat{k}$ is

$$\ell(\tau|\hat{k}) = \left(\frac{\pi\sigma^2}{(\hat{k}^+ - \hat{k}^-)^2}\sum_{n=1}^{\infty} n(-1)^{n-1}\sin\left[\pi n\frac{(\hat{k}^+ - \hat{k})}{(\hat{k}^+ - \hat{k}^-)}\right]\exp\left[-\frac{n^2\pi^2\sigma^2\tau}{2(\hat{k}^+ - \hat{k}^-)^2}\right]\right) \times \exp\left[\frac{-\nu}{2\sigma^2}(2(\hat{k}^+ - \hat{k}) + \nu\tau)\right]. \tag{A.60}$$

- The measure of times for hitting the *upper threshold* for current $\hat{k}$ is

$$\upsilon(\tau|\hat{k}) = \left(\frac{\pi\sigma^2}{(\hat{k}^+ - \hat{k}^-)^2}\sum_{n=1}^{\infty} n(-1)^{n-1}\sin\left[\pi n\frac{(\hat{k} - \hat{k}^-)}{(\hat{k}^+ - \hat{k}^-)}\right]\exp\left[-\frac{n^2\pi^2\sigma^2\tau}{2(\hat{k}^+ - \hat{k}^-)^2}\right]\right) \times \exp\left[\frac{-\nu}{2\sigma^2}(2(\hat{k}^- - \hat{k}) + \nu\tau)\right]. \tag{A.61}$$

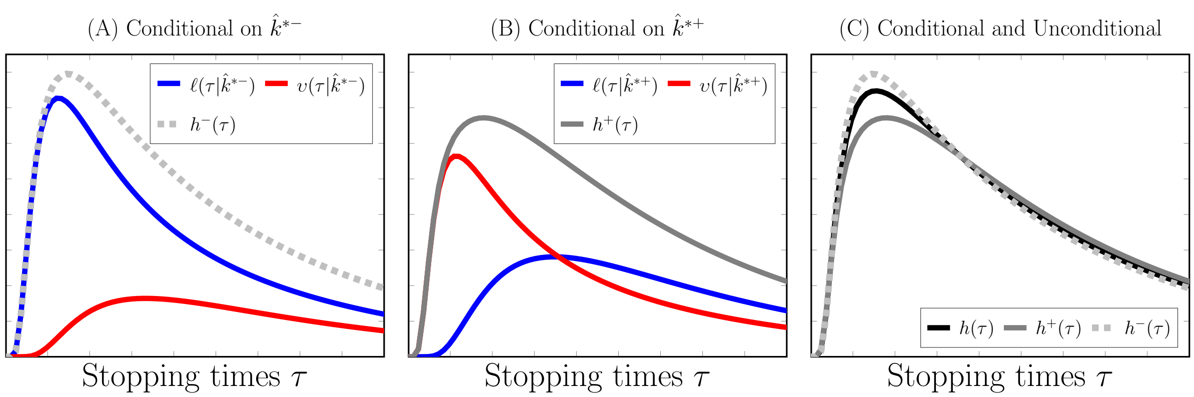
- The density of times for hitting *either threshold* for current $\hat{k}$ is the sum of the two previous measures:

$$h(\tau|\hat{k}) = \ell(\tau|\hat{k}) + \upsilon(\tau|\hat{k}). \tag{A.62}$$

Evaluating (A.60), (A.61), and (A.62) at the reset points, we obtain the duration densities conditional on a previous upsizing $\ell(\tau|\hat{k}^{*-})$, $\upsilon(\tau|\hat{k}^{*-})$ or downsizing $\ell(\tau|\hat{k}^{*+})$, $\upsilon(\tau|\hat{k}^{*+})$. The unconditional duration densities are weighted averages of the conditional densities, averaged using the updating shares:

$$z(\tau) = \frac{\mathcal{N}^-}{\mathcal{N}} z(\tau|\hat{k}^{*-}) + \frac{\mathcal{N}^+}{\mathcal{N}} z(\tau|\hat{k}^{*+}), \quad \text{for} \quad z \in \{\upsilon, \ell, h\}. \tag{A.63}$$

Figure A.1: Distributions of stopping times



Notes: These figures present the conditional densities of stopping times (τ) for hitting the lower and upper thresholds: Panel A depicts the densities following a purchase, and Panel B shows those following a sale. Panel C illustrates the stopping time distribution's conditional and unconditional densities.

A.5 Illustrative example on adjusted shares

Consider an economy where half of the firms adjust their capital every year ($\mathcal{N} = 0.5$), with 80% purchasing ($\mathcal{N}^-/\mathcal{N} = 0.8$) and 20% selling capital ($\mathcal{N}^+/\mathcal{N} = 0.2$). The conditional duration of inaction following a purchase is $\mathbb{E}^-[\tau] = 1.5$ years and following a sale is $\mathbb{E}^+[\tau] = 4$ years. From (24), the economy-wide average duration is computed using shares is $\mathbb{E}[\tau] = (\mathcal{N}^-/\mathcal{N})\mathbb{E}^-[\tau] + (\mathcal{N}^+/\mathcal{N})\mathbb{E}^+[\tau] = 0.8(1.5) + 0.2(4) = 2$ years.[34] The average adjustment $\overline{\mathbb{E}}[\Delta\hat{k}]$ is also computed using these shares.

Now, let us consider the distribution of $\hat{k}$. Assume the average capital-productivity ratio after a purchase is $\mathbb{E}^-[\hat{k}] = -0.2$ (capital is 80% of productivity) and after a sale is $\mathbb{E}^+[\hat{k}] = 0.2$ (capital is 120% of productivity). To compute the economy-wide mean $\mathbb{E}[\hat{k}]$, the naive aggregation using shares is biased as it does not consider the duration of inaction. While only 20% of adjustments are downward, they happen after longer inaction spells with twice the average duration, implying that the capital-productivity ratios generating those adjustments are occupied for more extended periods. According to (25), the renewal weights $r^- = (\mathcal{N}^-/\mathcal{N})(\mathbb{E}^-[\tau]/\mathbb{E}[\tau]) = 0.8(0.75) = 0.6$ and $r^+ = (\mathcal{N}^+/\mathcal{N})(\mathbb{E}^+[\tau]/\mathbb{E}[\tau]) = 0.2(2) = 0.4$ appropriately account for the higher occupancy. Therefore, the average ratio computed with (26) is $\mathbb{E}[\hat{k}] = 0.6(-0.2) + 0.4(0.2) = -0.04$ (capital is 96% of productivity). Using the wrong aggregation delivers a lower mean and biases the estimation of investment frictions.

[34] Note that $\overline{\mathbb{E}}[\tau] = 1/\mathcal{N}$ but $\overline{\mathbb{E}}^{\pm}[\tau] \neq 1/\mathcal{N}^{\pm}$.

B Generalized Hazard Model

In the main text, we specialize investment frictions to a symmetric adjustment cost θ paid indistinctly for positive and negative investments and a price wedge that gives rise to partial irreversibility.

We examine this model mainly for pedagogical reasons, as it simplifies the exposition of the theory. In this section, we expand the scope of the analysis and present an asymmetric generalized hazard model, which follows the contributions by Caballero and Engel (1999, 2007) and examined in contemporary work by Alvarez, Lippi and Oskolkov (2022), which may accommodate other empirically-relevant frictions. All the following proofs in the next sections are shown for the generalized hazard model and thus apply to the parsimonious environments as a special case setting $\Lambda(\hat{k}) = 0$.

B.1 Environment

The generalized hazard function depends mainly on the assumption of the fixed adjustment cost. Therefore, in this section, we assume technology and shocks as in Section 2. Moreover, the firm can control its capital stock through buying and selling investment goods at prices p^{buy} and p^{sell}, with $p^{\text{buy}} > p^{\text{sell}}$.

Adjustment costs The first step generalizes the adjustment cost structure. For every investment $i = \Delta k$, the firm must pay an adjustment cost θ_s proportional to current productivity u_s and measured in consumption units (Caballero and Engel, 1999):

$$\theta_s \;=\; \Theta(i_s, \mathrm{d}N_s^-, \mathrm{d}N_s^+, \vartheta_s^-, \vartheta_s^+)u_s, \tag{B.1}$$

where the function $\Theta(\cdot) > 0$ is described by

$$\Theta(i, \mathrm{d}N^+, \mathrm{d}N^-, \vartheta^-, \vartheta^+) = \begin{cases} 0 & \text{if} \quad i = 0 \\ \bar{\theta}^+(1 - \mathrm{d}N) \;+\; \mathrm{d}N\vartheta^+ & \text{if} \quad i < 0 \\ \bar{\theta}^-(1 - \mathrm{d}N) \;+\; \mathrm{d}N\vartheta^- & \text{if} \quad i > 0. \end{cases} \tag{B.2}$$

Let us describe each element in equation (B.2).

(i) N_s^+ and N_s^- follows Poisson counter with unit increments and arrival rates λ^+ and λ^-;

(ii) $\bar{\theta}^+$ and $\bar{\theta}^-$ are non negative number; and

(iii) ϑ_s^+ and ϑ_s^- are *i.i.d.* random variables with support $\mathbb{S}upp(\vartheta^+) = [0, \bar{\vartheta}^+]$ and $\mathbb{S}upp(\vartheta^-) = [0, \bar{\vartheta}^-]$. We assume that $\vartheta^- \leq \bar{\theta}^-$ and $\vartheta^+ \leq \bar{\theta}^+$. Define $J^+(x) \equiv \Pr(\vartheta^+ < x)$ and $J^-(x) \equiv \Pr(\vartheta^- < x)$ the cumulative distribution for each random variable.

B.1.1 Relationship to the literature

The stochastic process of fixed cost in (B.2) can derive the majority of lumpy adjustment models used in previous work.

1. Setting $\lambda^+ = \lambda^- = 0$ and $\bar{\theta}^+ = \bar{\theta}^-$ yields the standard fixed cost model of adjustment, originally proposed by Scarf (1959) in an inventory model and Sheshinski and Weiss (1977) in a pricing context.
2. Setting $\lambda^+ = \lambda^- > 0$ and $\mathbb{S}upp(\vartheta^+) = \mathbb{S}upp(\vartheta^-) = \{0\}$, and $\bar{\theta}^+ = \bar{\theta}^- > 0$ yields the CalvoPlus model proposed by Nakamura and Steinsson (2010), which nests the standard fixed cost model and the time-dependent Calvo model.

3. Under this case, if $\bar{\theta}^+ \neq \bar{\theta}^-$, then we have the Bernoulli fixed cost model or asymmetric Bernoulli fixed cost model if $\lambda^+ \neq \lambda^-$, see Baley and Blanco (2021).
4. Finally, setting $\lambda^+ = \lambda^- > 0$ and $\bar{\vartheta}^+ = \bar{\vartheta}^- = \bar{\theta}^- = \bar{\theta}^+$ yields the generalized hazard model originally proposed by Caballero and Engel (1993).

B.1.2 Value function and optimal policy

Value Let $V(k,u)$ denote the value of a firm with capital stock k and productivity u. Given initial conditions (k_0, u_0), the firm chooses a sequence of adjustment dates $\{T_h\}_{h=1}^{\infty}$ and investments $\{i_{T_h}\}_{h=1}^{\infty}$, where h counts the number of adjustments, to maximize its expected discounted stream of profits. The sequential problem is

$$\text{(B.3)} \qquad V(k_0,u_0) = \max_{\{T_h, i_{T_h}\}_{h=1}^{\infty}} \mathbb{E}\left[\int_0^{\infty} e^{-\rho s}\pi_s \, \mathrm{d}s - \sum_{h=1}^{\infty} e^{-\rho T_h}\left(\theta_{T_h} + p\left(i_{T_h}\right) i_{T_h}\right)\right],$$

subject to the production technology (1), the idiosyncratic productivity shocks (2), the investment price function (4), the law of motion for the capital stock (6), and the stochastic process of adjustment cost in (B.2).

Capital-productivity ratios $\hat{k}$ As in the main text, it is easy to show that $v(k,u) = uv(\hat{k})$ where

$$\text{(B.4)} \qquad v(\hat{k}) = \max_{\tau, \Delta\hat{k}} \mathbb{E}\left[\int_0^{\tau} Ae^{-rs+\alpha\hat{k}_s} \, \mathrm{d}s + e^{-r\tau}\left(-\theta_\tau(\Delta\hat{k}) - p(\Delta\hat{k})(e^{\hat{k}_\tau + \Delta\hat{k}} - e^{\hat{k}_\tau}) + v(\hat{k}_\tau + \Delta\hat{k})\right) \middle| \hat{k}_0 = \hat{k}\right].$$

Here, $\theta_\tau(\Delta\hat{k})$ is a random variable instead of a number, a function of the adjustment direction—similar to the investment price.

Optimal investment policy The optimal investment policy is characterize by four numbers $\mathcal{K} \equiv \{\hat{k}^- \leq \hat{k}^{*-} \leq \hat{k}^{*+} \leq k^+\}$, and a hazard rate of adjustment $\Lambda(\hat{k})$. The numbers $\hat{k}^-$ and k^+ correspond to the lower and upper borders of the inaction region $\mathcal{R} = \left\{\hat{k} : \hat{k}^- < \hat{k} < \hat{k}^+\right\}$, and $\hat{k}^{*-} < \hat{k}^{*+}$ to the two reset points following a positive and a negative investment, respectively. $\Lambda(\hat{k}) : \overline{\mathcal{R}} \to \mathbb{R}^+$ is a non-negative function corresponding to the arrival rate of a new Poisson counter N^Λ. Given $\mathcal{R}$ and N^Λ, the optimal adjustment dates are

$$\text{(B.5)} \qquad T_h = \inf\left\{s \geq T_{h-1} : \hat{k}_s \notin \mathcal{R} \text{ or } \mathrm{d}N_s^\Lambda(\hat{k}) = 1\right\} \quad \text{with} \quad T_0 = 0.$$

Following Øksendal and Sulem (2005) and Oksendal (2007), Lemma B.1 establishes the optimality conditions that characterize (B.4).

Lemma B.1. *The value function $v(\hat{k})$ and the policy $\mathcal{K} \equiv \{\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+\}$ satisfy:*

(i) For all $\hat{k} \in \mathcal{R}$, $v(\hat{k})$ solves the HJB equation:

$$\text{(B.6)} \qquad rv(\hat{k}) = Ae^{\alpha\hat{k}} - \nu v'(\hat{k}) + \frac{\sigma^2}{2}v''(\hat{k}) + \lambda^- \int_0^{\bar{\vartheta}^-} \max\left\{v^{buy}(\hat{k}) - \vartheta, 0\right\} \mathrm{d}J^-(\vartheta) + \lambda^+ \int_0^{\bar{\vartheta}^+} \max\left\{v^{sell}(\hat{k}) - \vartheta, 0\right\} \mathrm{d}J^+(\vartheta)$$

where the values v^{buy} and v^{sell} are defined as follows:

$$\text{(B.7)} \qquad v^{buy}(\hat{k}) \equiv v(\hat{k}^-) - v(\hat{k}) - p^{buy}(e^{\hat{k}^-} - e^{\hat{k}}),$$

$$\text{(B.8)} \qquad v^{sell}(\hat{k}) \equiv v(\hat{k}^+) - v(\hat{k}) - p^{sell}(e^{\hat{k}^+} - e^{\hat{k}}).$$

(ii) At the borders of the inaction region, $v(\hat{k})$ satisfies the value-matching conditions:

$$v^{buy}(\hat{k}^-) = \bar{\theta}^-; \qquad v^{sell}(\hat{k}^+) = \bar{\theta}^+; \tag{B.9}$$

(iii) At the borders of the inaction region and the two reset states, $v(\hat{k})$ satisfies the smooth-pasting and the optimality conditions:

$$\frac{\mathrm{d}v^{buy}(\hat{k})}{\mathrm{d}\hat{k}} = p^{buy}e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^-, \hat{k}^{*-}\right\}, \tag{B.10}$$

$$\frac{\mathrm{d}v^{sell}(\hat{k})}{\mathrm{d}\hat{k}} = p^{sell}e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^{*+}, \hat{k}^+\right\}. \tag{B.11}$$

Hazard rate of adjustment $\Lambda(\hat{k})$ We are now ready to define $\Lambda(\hat{k})$, which gives the probability of adjustment $\Lambda(\hat{k})\,\mathrm{d}t$ in a time period $\mathrm{d}t$ a firm with $\hat{k} \in \mathcal{R}$. The hazard rate of adjustment is given by

$$\Lambda(\hat{k}) = \lambda^- J^- \left(v^{\text{buy}}(\hat{k})\right) \mathbb{1}_{\left\{\hat{k}\in(\hat{k}^-,\hat{k}^{*-})\right\}} + \lambda^+ J^+ \left(v^{\text{sell}}(\hat{k})\right) \mathbb{1}_{\left\{\hat{k}\in(\hat{k}^{*+},\hat{k}^+)\right\}}. \tag{B.12}$$

The hazard function $\Lambda(\hat{k})$ satisfies the following properties:

1. $\Lambda(\hat{k}) = 0$ in the inner inaction region, i.e., for all $\hat{k} \in (\hat{k}^{*-}, \hat{k}^{*+})$,
2. $\Lambda(\hat{k})$ is weakly decreasing in $(\hat{k}^-, \hat{k}^{*-})$ and weakly increasing in $(\hat{k}^{*+}, \hat{k}^+)$;
3. If $J^-(0) > 0$ then $\Lambda(\hat{k})$ is bounded below in the domain $(\hat{k}^-, \hat{k}^{*-})$ by $\Lambda(\hat{k}) = \lambda^- J^-(0)$.
4. If $J^+(0) > 0$ then $\Lambda(\hat{k})$ is bounded below in the domain $(\hat{k}^{*+}, \hat{k}^+)$ by $\Lambda(\hat{k}) = \lambda^+ J^+(0)$

B.1.3 Cross-sectional distribution

Without irreversibility

$$\Lambda(\hat{k})g(\hat{k}) = \nu g'(\hat{k}) + \frac{\sigma^2}{2} g''(\hat{k}), \quad \forall \hat{k} \in (\hat{k}^-, \hat{k}^+)/\{\hat{k}^*\} \tag{B.13}$$

$$g(\hat{k}^{\pm}) = 0 \tag{B.14}$$

$$\int_{\hat{k}^-}^{\hat{k}^+} g(\hat{k})\,\mathrm{d}\hat{k} = 1, \tag{B.15}$$

$$g(\hat{k}) \in \mathbb{C}, \mathbb{C}^1(\{\hat{k}^*\}), \mathbb{C}^2(\{\hat{k}^*\}) \tag{B.16}$$

With irreversibility

$$\Lambda(\hat{k})g(\hat{k}) = \nu g'(\hat{k}) + \frac{\sigma^2}{2} g''(\hat{k}), \qquad \forall \hat{k} \in (\hat{k}^-, \hat{k}^+)/\{\hat{k}^{*-}, \hat{k}^{*+}\} \tag{B.17}$$

$$g(\hat{k}^{\pm}) = 0 \tag{B.18}$$

$$\int_{\hat{k}^-}^{\hat{k}^+} g(\hat{k})\,\mathrm{d}\hat{k} = 1, \tag{B.19}$$

$$g(\hat{k}) \in \mathbb{C}, \mathbb{C}^1(\{\hat{k}^{*-}, \hat{k}^{*+}\}), \mathbb{C}^2(\{\hat{k}^{*-}, \hat{k}^{*+}\}) \tag{B.20}$$

C Proofs under Generalized Hazard

Proofs' overview. In Proposition 2, we express the CIR as the integral of a value function $m(\hat{k})$ and $g'(\hat{k})$. In Proposition 3, we characterize the terminal value of the value function. In Proposition 4, we characterize the CIR as a function of steady-state moments. In all propositions, we examine cases without and with irreversibility, in that order.

C.1 Proof of Proposition 2

Proposition 2. ***(CIR)*** *Up to the first order, the CIR equals*

$$\frac{CIR(\delta)}{\delta} = -\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} + o(\delta), \tag{40}$$

where $m(\hat{k})$ is a continuously differentiable function equal to the average cumulative deviations of the capital-productivity ratio $\hat{k}$ from the economy's mean $\mathbb{E}[\hat{k}]$, satisfying the HJB

$$0 = \hat{k} - \mathbb{E}[\hat{k}] - \nu m'(\hat{k}) + \frac{\sigma^2}{2}m''(\hat{k}) \qquad \forall \hat{k} \in (\hat{k}^-, \hat{k}^+), \tag{41}$$

with two boundary conditions

$$m(\hat{k}^-) = m(\hat{k}^{*-}), \quad \text{and} \quad m(\hat{k}^+) = m(\hat{k}^{*+}), \tag{42}$$

and a stationarity condition

$$\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} = 0. \tag{43}$$

C.1.1 Step 1: First-order approximation and exchange order of integration

Let $g(\hat{k})$ be the capital-productivity steady-state distribution and $g_t(\hat{k})$ the distribution t-periods after an aggregate productivity shock of size $\delta > 0$, with $g_0(\hat{k}) = g(\hat{k} - \delta)$. Let $f(\hat{k})$ be a continuous function of $\hat{k}$ (in the main text, we take $f(\hat{k}) = \hat{k}$, the proof here is more general). Define the cumulative impulse response of the function f as:

$$\text{CIR}(f,\delta) \equiv \int_0^\infty \int_{\hat{k}^-}^{\hat{k}^+} f(\hat{k}) \, \Big(g_s(\hat{k}) - g(\hat{k})\Big)\,\mathrm{d}\hat{k}\,\mathrm{d}s. \tag{C.21}$$

We show that in a general environment, with or without irreversibility, up to first order, the CIR is equal to

$$\text{CIR}(f,\delta) = -\delta \int_{\hat{k}^-}^{\hat{k}^+} \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} + o(\delta^2). \tag{C.22}$$

where $m_{\mathcal{T}}(\hat{k}_0)$ is defined as

$$m_{\mathcal{T}}(\hat{k}_0) \equiv \int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \Big(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\Big)\, g_s(\hat{k}|\hat{k}_0)\,\mathrm{d}\hat{k}\,\mathrm{d}s. \tag{C.23}$$

Starting from the CIR's definition in (C.21), we do the following steps: Equality (1) operates over the integral; (2) uses the Chapman-Kolmogorov equation to substitute the conditional expectation with respect to $\hat{k}$, with density $g_s(\hat{k})$, with a conditional expectation with respect to the initial condition $\hat{k}_0$, with density $g_s(\hat{k}|\hat{k}_0)g_0(\hat{k}_0)$, where

$g_s(\hat{k}|\hat{k}_0)\,\mathrm{d}\hat{k}$ is the probability of the state $\hat{k}$ at date s with initial condition $\hat{k}_0$; (3) writes the initial density following the shock in terms of the steady-state density $g_0(\hat{k}_0) = g(\hat{k}_0 - \delta)$; (4) applies Fubini's theorem to exchange orders of integration; (5) writes the integral using a limit; (6) defines and substitutes the function $m_{\mathcal{T}}(\hat{k})$ as in (C.23) and changes the variable of integration from $\hat{k}_0$ to $\hat{k}$; and (7) applies a first-order approximation over δ.

$$\begin{aligned}
\mathrm{CIR}(f,\delta) &= \int_0^\infty \int_{\hat{k}^-}^{\hat{k}^+} f(\hat{k}) \left(g_s(\hat{k}) - g(\hat{k})\right) \mathrm{d}\hat{k}\,\mathrm{d}s \\
&=^{(1)} \int_0^\infty \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k})\,\mathrm{d}\hat{k}\,\mathrm{d}s \\
&=^{(2)} \int_0^\infty \int_{\hat{k}^-}^{\hat{k}^+} \left[\int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k}|\hat{k}_0) g_0(\hat{k}_0)\,\mathrm{d}\hat{k}_0\right] \mathrm{d}\hat{k}\,\mathrm{d}s \\
&=^{(3)} \int_0^\infty \int_{\hat{k}^-}^{\hat{k}^+} \left[\int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k}|\hat{k}_0) g(\hat{k}_0 - \delta)\,\mathrm{d}\hat{k}_0\right] \mathrm{d}\hat{k}\,\mathrm{d}s \\
&=^{(4)} \int_{\hat{k}^-}^{\hat{k}^+} \left[\int_0^\infty \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k}|\hat{k}_0)\,\mathrm{d}\hat{k}\,\mathrm{d}s\right] g(\hat{k}_0 - \delta)\,\mathrm{d}\hat{k}_0 \\
&=^{(5)} \int_{\hat{k}^-}^{\hat{k}^+} \left[\lim_{\mathcal{T}\to\infty} \underbrace{\int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k}|\hat{k}_0)\,\mathrm{d}\hat{k}\,\mathrm{d}s}_{\equiv\, m_{\mathcal{T}}(\hat{k}_0)}\right] g(\hat{k}_0 - \delta)\,\mathrm{d}\hat{k}_0 \\
&=^{(6)} \int_{\hat{k}^-}^{\hat{k}^+} \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) g(\hat{k} - \delta)\,\mathrm{d}\hat{k} \\
&=^{(7)} -\delta \int_{\hat{k}^-}^{\hat{k}^+} \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) g'(\hat{k})\,\mathrm{d}\hat{k} + o(\delta^2).
\end{aligned}$$

C.1.2 Step 2: Show that the cross-sectional mean of $m_{\mathcal{T}}$ is zero.

Show that $\int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k} = 0$. Substitute the integral's definition of $m_{\mathcal{T}}(\hat{k})$ from (C.23) into $\int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k}$. In the following equalities, (1) uses Fubini's theorem, (2) uses Bayes' theorem, (3) uses the fact that $g(\hat{k})$ is the steady-state distribution, and (4) solves the first and second integrals.

$$\begin{aligned}
\text{(C.24)} \qquad \int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k} &= \int_{\hat{k}^-}^{\hat{k}^+} \left[\int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k}|\hat{k}_0)\,\mathrm{d}\hat{k}\,\mathrm{d}s\right] g(\hat{k}_0)\,\mathrm{d}\hat{k}_0 \\
&=^{(1)} \int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k}|\hat{k}_0) g(\hat{k}_0)\,\mathrm{d}\hat{k}\,\mathrm{d}\hat{k}_0\,\mathrm{d}s \\
&=^{(2)} \int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) \left[\int_{\hat{k}^-}^{\hat{k}^+} g_s(\hat{k}|\hat{k}_0) g(\hat{k}_0)\,\mathrm{d}\hat{k}_0\right] \mathrm{d}\hat{k}\,\mathrm{d}s \\
&=^{(3)} \int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g(\hat{k})\,\mathrm{d}\hat{k}\,\mathrm{d}s =^{(4)} 0.
\end{aligned}$$

C.1.3 Step 3: Derive HJB and border conditions for $m_{\mathcal{T}}$.

We start from the stopping time definition of $m_{\mathcal{T}}(\hat{k}_0) \equiv \int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k}|\hat{k}_0)\,\mathrm{d}\hat{k}\,\mathrm{d}s$ in equation (C.23), and use the conditions in Auxiliary Theorem (A.3) to characterize its value.

Without irreversibility

$$\text{(C.25)} \qquad 0 = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \frac{\mathrm{d}m_{\mathcal{T}}(\hat{k})}{\mathrm{d}\mathcal{T}} - \nu\frac{\mathrm{d}m_{\mathcal{T}}(\hat{k})}{\mathrm{d}\hat{k}} + \frac{\sigma^2}{2}\frac{\mathrm{d}^2 m_{\mathcal{T}}(\hat{k})}{\mathrm{d}\hat{k}^2} + \Lambda(\hat{k})(m_{\mathcal{T}}(\hat{k}^*) - m_{\mathcal{T}}(\hat{k}))$$

$$\text{(C.26)} \qquad 0 = m_{\mathcal{T}}(\hat{k}^*) - m_{\mathcal{T}}(\hat{k}^{\pm})$$

$$\text{(C.27)} \qquad 0 = \int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k}$$

With irreversibility Using the property that $\Lambda(\hat{k}) = 0$ for all $\hat{k} \in (\hat{k}^{*-}, \hat{k}^{*+})$, we can write in a simple form the HJB and border conditions satisfied by $m_{\mathcal{T}}(\hat{k})$:

$$\text{(C.28)} \qquad 0 = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \frac{\mathrm{d}m_{\mathcal{T}}(\hat{k})}{\mathrm{d}\mathcal{T}} - \nu\frac{\mathrm{d}m_{\mathcal{T}}(\hat{k})}{d\hat{k}} + \frac{\sigma^2}{2}\frac{\mathrm{d}^2 m_{\mathcal{T}}(\hat{k})}{\mathrm{d}\hat{k}^2} + \Lambda(\hat{k})(\mathcal{M}_{\mathcal{T}}(\hat{k}) - m_{\mathcal{T}}(\hat{k}))$$

$$\text{(C.29)} \qquad 0 = \mathcal{M}_{\mathcal{T}}(\hat{k}^{*\pm}) - m_{\mathcal{T}}(\hat{k}^{\pm}),$$

$$\text{(C.30)} \qquad 0 = \int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k}.$$

where $\mathcal{M}_{\mathcal{T}}(\hat{k}) \in \mathbb{C}^2$ is defined as

$$\text{(C.31)} \qquad \mathcal{M}_{\mathcal{T}}(\hat{k}) \equiv \begin{cases} m_{\mathcal{T}}(\hat{k}^{*-}) & \text{if } \hat{k} \in [\hat{k}^-, \hat{k}^{*-}] \\ m_{\mathcal{T}}(\hat{k}^{*+}) & \text{if } \hat{k} \in [\hat{k}^{*+}, \hat{k}^+] \end{cases}$$

C.1.4 Step 4: Show pointwise converge of $m_{\mathcal{T}}$ to m

Let $m(\hat{k})$ be defined as:

$$\text{(C.32)} \qquad m(\hat{k}) \equiv \mathbb{E}\left[\int_0^\infty (\hat{k}_s - \mathbb{E}[\hat{k}])\,\mathrm{d}s \Big| \hat{k}\right] + \mathbb{C}.$$

We show that for each $\hat{k}$, $\lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) = m(\hat{k})$.

Without irreversibility See Baley and Blanco (2021).

With irreversibility Let $\{T_i\}_{i=0}^{N(\mathcal{T})}$ be the adjustment dates between 0 and $\mathcal{T}$, where i denotes the counter of adjustments for all $i = 1, 2, \ldots, N(\mathcal{T})-1$, $N(\mathcal{T})$ is the maximum number of adjustments until $\mathcal{T}$ and $T_0 = 0$. Then, for any $\mathcal{T}$, we rewrite $m_{\mathcal{T}}(\hat{k})$ as a sum between adjustment dates:

$$\text{(C.33)} \qquad m_{\mathcal{T}}(\hat{k}) = \mathbb{E}\left[\sum_{i=1}^{N(\mathcal{T})-1} \int_{T_{i-1}}^{T_i} \Big(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})]\Big)\,\mathrm{d}s + \int_{T_{N(\mathcal{T})-1}}^{\mathcal{T}} \Big(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})]\Big)\,\mathrm{d}s \,\Bigg|\, \hat{k}_0 = \hat{k}\right]$$

We take the limit $\mathcal{T} \to \infty$ to show convergence. We conduct the following steps in the next equalities: (1) splits the sum; (2) uses the indicator function to write the finite sum in the first term; (3) uses the fact that $N(\mathcal{T})$ always exceeds i, thus $\mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\mathbb{I}(N(\mathcal{T})\geq i)\middle|\,\hat{k}_0=\hat{k}\right]=1,\forall i$; (4) recognizes that the first term is independent of $\mathcal{T}$.

$$
\begin{aligned}
\text{(C.34)}\quad & \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) \\
&=^{(1)} \mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\sum_{i=1}^{N(\mathcal{T})-1}\int_{T_{i-1}}^{T_i}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s+\lim_{\mathcal{T}\to\infty}\int_{T_{N(\mathcal{T})-1}}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\middle|\hat{k}_0=\hat{k}\right] \\
&=^{(2)} \mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\sum_{i=1}^{\infty}\mathbb{I}(N(\mathcal{T})\geq i)\int_{T_{i-1}}^{T_i}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s+\lim_{\mathcal{T}\to\infty}\int_{T_{N(\mathcal{T})-1}}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\middle|\hat{k}_0=\hat{k}\right] \\
&=^{(3)} \mathbb{E}\left[\sum_{i=1}^{\infty}\int_{T_{i-1}}^{T_i}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s+\lim_{\mathcal{T}\to\infty}\int_{T_{N(\mathcal{T})-1}}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\middle|\hat{k}_0=\hat{k}\right] \\
&=^{(4)} \text{terms independent of } \mathcal{T}+\underbrace{\mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\int_{T_{\mathcal{N}(\mathcal{T})-1}}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\middle|\hat{k}_0=\hat{k}\right]}_{\text{tail term } \mathcal{E}}.
\end{aligned}
$$

Tail term Next, we show the "tail" term is independent of the initial condition $\hat{k}$ and the $\mathcal{T}$. To do this, we consider tails conditional on the previous reset, defined as:

$$
\text{(C.35)}\qquad \mathcal{E}(\hat{k}^{*\pm},\mathcal{T}) \equiv \mathbb{E}\left[\int_{T_{\mathcal{N}(\mathcal{T})-1}}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\middle|\hat{k}_{T_{\mathcal{N}(\mathcal{T})-1}}=\hat{k}^{*\pm}\right].
$$

Define $\mathbb{P}^+_{\mathcal{T}}(\hat{k}_0)\equiv\mathbb{E}\left[\hat{k}_{T_{\mathcal{N}(\mathcal{T})-1}}\geq\hat{k}^{*+}\middle|\hat{k}_0\right]$ and $\mathbb{P}^-_{\mathcal{T}}(\hat{k}_0)\equiv\mathbb{E}\left[\hat{k}_{T_{\mathcal{N}(\mathcal{T})-1}}\leq\hat{k}^{*-}\middle|\hat{k}_0\right]$ be the probabilities of downsizing or upsizing, given a current $\hat{k}$. In the following equalities, we do the following steps. In Step 1, we use the law of iterated expectations (only two contingencies, upsizing or downsizing) and use conditional expectation to substitute the tails and probabilities, *conditional on the initial condition* $\hat{k}_0$. Step 2 eliminates the dependence of probabilities on the initial condition using the convergence of discrete Markov chains (see chapter 11 of Stokey (1989)) and the proof at the end. In other words, $\lim_{\mathcal{T}\to\infty}\mathbb{P}^+_{\mathcal{T}}(\hat{k}_0)$ is independent of $\mathcal{T}$ and $\hat{k}_0$. Finally, the convergence of $\lim_{\mathcal{T}\to\infty}\mathcal{E}(\hat{k}^{*\pm},\mathcal{T})=\mathcal{E}^{\infty}(\hat{k}^{*\pm})$ is shown in Baley and Blanco (2021) and Alexandrov (2021).

$$
\begin{aligned}
&\mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\int_{T_{\mathcal{N}(\mathcal{T})-1}}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\middle|\hat{k}_0\right] \\
&=^{(1)} \mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\mathcal{E}(\hat{k}^{*+},\mathcal{T})\lim_{\mathcal{T}\to\infty}\mathbb{P}^+_{\mathcal{T}}(\hat{k}_0)+\lim_{\mathcal{T}\to\infty}\mathcal{E}(\hat{k}^{*-},\mathcal{T})\lim_{\mathcal{T}\to\infty}\mathbb{P}^-_{\mathcal{T}}(\hat{k}_0)\middle|\hat{k}_0\right] \\
&=^{(2)} \lim_{\mathcal{T}\to\infty}\mathcal{E}(\hat{k}^{*+},\mathcal{T})\mathbb{P}^{+,\infty}+\lim_{\mathcal{T}\to\infty}\mathcal{E}(\hat{k}^{*-},\mathcal{T})\mathbb{P}^{-,\infty} \\
&=^{(3)} \mathcal{E}^{\infty}(\hat{k}^{*+})\mathbb{P}^{+,\infty}+\mathcal{E}^{\infty}(\hat{k}^{*-})\mathbb{P}^{-,\infty}.
\end{aligned}
$$

Extra: Convergence of discrete Markov chains Let $\mathbb{P}_N(\hat{k})\equiv\left[\mathbb{P}^-_N(\hat{k});\mathbb{P}^+_N(\hat{k})\right]\in\mathbb{R}^{2\times1}$, then

$$
\text{(C.36)}\qquad \mathbb{P}_N(\hat{k})=\mathbb{P}^T\mathbb{P}_{N-1}(\hat{k}),
$$

where $\mathbb{P}=[\mathbb{P}^-(\hat{k}^{*-}),1-\mathbb{P}^-(\hat{k}^{*-});1-\mathbb{P}^+(\hat{k}^{*+});\mathbb{P}^+(\hat{k}^{*+})]\in\mathbb{R}^{2\times2}$ is a 2×2 transition probability where the rows are the transition probability and P^T is its transpose. If $\mathbb{P}^-_1(\hat{k}^{*-}),\mathbb{P}^+_1(\hat{k}^{*+})\in(0,1)$, then

$$
\text{(C.37)}\qquad \lim_{N\to\infty}\mathbb{P}_N(\hat{k})=\lim_{N\to\infty}\mathbb{P}^{N-1}\mathbb{P}_1(\hat{k})=[\mathbb{P}^{-\infty};\mathbb{P}^{+\infty}].
$$

where the last equality comes from Theorem 11.1 in Stokey (1989).

C.1.5 Step 5: Show convergence of CIR

We will show that

(C.38) $$\text{CIR}(f,\delta) = -\delta \int_{\hat{k}^-}^{\hat{k}^+} \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) g'(\hat{k})\,\mathrm{d}\hat{k} + o(\delta^2) = -\delta \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k}) g'(\hat{k})\,\mathrm{d}\hat{k} + o(\delta^2)$$

Without irreversibility We also need to show that the HJB and border conditions converge. For this, we take the limit $\mathcal{T}\to\infty$ of conditions (C.25), (C.26) and (C.27) and use point-wise convergence of $m_{\mathcal{T}}(\hat{k})$ from Step 4:

(C.39) $$0 = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu m'(\hat{k}) + \frac{\sigma^2}{2} m''(\hat{k}) + \Lambda(\hat{k})(m(\hat{k}^*) - m(\hat{k})),$$

(C.40) $$0 = m(\hat{k}^*) - m(\hat{k}^{\pm}),$$

With irreversibility Finally, we take the limit $\mathcal{T}\to\infty$ of conditions (C.28), (C.29) and (C.30) and use point-wise convergence of $m_{\mathcal{T}}(\hat{k})$ from Step 4 to obtain:

(C.41) $$0 = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu m'(\hat{k}) + \frac{\sigma^2}{2} m''(\hat{k}) + \Lambda(\hat{k})(\mathcal{M}(\hat{k}) - \mathbb{C}ov[\hat{k},a] - m(\hat{k})),$$

(C.42) $$0 = \mathcal{M}(\hat{k}^{\pm}) - \mathbb{C}ov[\hat{k},a] - m(\hat{k}^{\pm}),$$

(C.43)

where $\mathcal{M}(\hat{k}) \in \mathbb{C}^2$ is defined as

(C.44) $$\mathcal{M}(\hat{k}) \equiv \begin{cases} m(\hat{k}^{*-}) + \mathbb{C}ov[\hat{k},a] & \text{if } \hat{k} \in [\hat{k}^-, \hat{k}^{*-}] \\ m(\hat{k}^{*+}) + \mathbb{C}ov[\hat{k},a] & \text{if } \hat{k} \in [\hat{k}^{*+}, \hat{k}^+] \end{cases}$$

Stability condition To show the stability condition that require the cross-sectional average of $m(\hat{k}) = 0$ in the cases with and without irreversibility, we write $m(\hat{k}) = \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k})$ inside the integral, pull the limit outside the integral, and use the previous result in (C.24) to get:

(C.45) $$\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k} = \int_{\hat{k}^-}^{\hat{k}^+} \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k} = \lim_{\mathcal{T}\to\infty} \int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k} = 0$$

C.1.6 Step 6: Without general hazard

To obtain the characterization in the baseline model, just set $\Lambda(\hat{k}) = 0$.

C.2 Proof of Proposition 3

Proposition 3. ***(Expected sum of deviations)*** *The expected sum of deviations after upsizing* $\mathcal{M}(\hat{k}^{*-}) \equiv m(\hat{k}^{*-}) + \mathbb{C}ov[\hat{k},a]$ *and after downsizing* $\mathcal{M}(\hat{k}^{*+}) \equiv m(\hat{k}^{*+}) + \mathbb{C}ov[\hat{k},a]$ *are equal to*

(47) $$\mathcal{M}(\hat{k}^{*-}) = (\mathbb{E}^-[\hat{k}] - \mathbb{E}[\hat{k}])\ \overline{\mathbb{E}}^-[\tau]\ \frac{\mathbb{E}[\mathbb{P}^+(\hat{k})]}{\mathbb{P}^{-+}} < 0$$

(48) $$\mathcal{M}(\hat{k}^{*+}) = (\mathbb{E}^+[\hat{k}] - \mathbb{E}[\hat{k}])\ \overline{\mathbb{E}}^+[\tau]\ \frac{\mathbb{E}[\mathbb{P}^-(\hat{k})]}{\mathbb{P}^{+-}} > 0,$$

where the average downsizing and upsizing probabilities are equal to

$$(49) \qquad \mathbb{E}[\mathbb{P}^-(\hat{k})] = \frac{\overline{\mathbb{E}}\left[\tau' \mathbb{1}_{\{\hat{k}_{\tau'}=\hat{k}^-\}}\right]}{\overline{\mathbb{E}}[\tau]}, \qquad \mathbb{E}[\mathbb{P}^+(\hat{k})] = \frac{\overline{\mathbb{E}}\left[\tau' \mathbb{1}_{\{\hat{k}_{\tau'}=\hat{k}^+\}}\right]}{\overline{\mathbb{E}}[\tau]}.$$

C.2.1 Without irreversibility

For any continous function f, we characterize the terminal value $m(\hat{k}^*) = -\mathbb{C}ov[a, f(\hat{k})]$. From Proposition 2, $m(\hat{k})$ satisfies the following recursive representation

$$(C.46) \qquad m(\hat{k}) \;=\; \mathbb{E}\left[\int_0^\tau (f(\hat{k}_s) - \mathbb{E}[f(\hat{k})])\,\mathrm{d}s + m(\hat{k}^*)\middle|\, \hat{k}_0 = \hat{k}\right].$$

For a given parameter $\varphi \geq 0$, define the auxiliary function $z(\hat{k}|\varphi)$ as follows

$$(C.47) \qquad z(\hat{k}|\varphi) \;\equiv\; \mathbb{E}\left[\int_0^\tau e^{\varphi s}(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})])\,\mathrm{d}s + e^{\varphi\tau} m(\hat{k}^*)\middle|\, \hat{k}_0 = \hat{k}\right],$$

Using Auxiliary Theorem A.3, the auxiliary function $z(\hat{k}|\varphi)$ satisfies the following HBJ and border conditions:

$$(C.48) \qquad -\varphi z(\hat{k}|\varphi) \;=\; f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu\frac{\partial z(\hat{k}|\varphi)}{\partial \hat{k}} + \frac{\sigma^2}{2}\frac{\partial^2 z(\hat{k}|\varphi)}{\partial \hat{k}^2} \;+\; \Lambda(\hat{k})\left(m(\hat{k}^*) - z(\hat{k}|\varphi)\right),$$

$$(C.49) \qquad z(\hat{k}^\pm, \varphi) \;=\; m(\hat{k}^*).$$

Taking the derivatives of (C.48) and (C.49) with respect to φ:

$$(C.50) \qquad (\Lambda(\hat{k}) - \varphi)\frac{\partial z(\hat{k}|\varphi)}{\partial \varphi} - z(\hat{k}|\varphi) \;=\; -\nu\frac{\partial^2 z(\hat{k},\varphi)}{\partial \hat{k}\partial\varphi} + \frac{\sigma^2}{2}\frac{\partial^3 z(\hat{k}|\varphi)}{\partial \hat{k}^2\partial\varphi} \quad \text{and} \quad \frac{\partial z(\hat{k}^\pm|\varphi)}{\partial\varphi} \;=\; 0.$$

Using the Schwarz's theorem to exchange partial derivatives, evaluating at $\varphi = 0$, and using $z(\hat{k}|0) = m(\hat{k})$, the two expressions become:

$$(C.51) \qquad \Lambda(\hat{k})\frac{\partial z(\hat{k}|0)}{\partial\varphi} \;=\; m(\hat{k}) - \nu\frac{\partial}{\partial\hat{k}}\left(\frac{\partial z(\hat{k}|0)}{\partial\varphi}\right) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial\hat{k}^2}\left(\frac{\partial z(\hat{k}|0)}{\partial\varphi}\right) \quad \text{and} \quad \frac{\partial z(\hat{k}^\pm|0)}{\partial\varphi} \;=\; 0.$$

From Auxiliary Theorem A.3, equations in (C.51) are the HBJ and border conditions of $\frac{\partial z(\hat{k}|0)}{\partial\varphi}$, and therefore:

$$(C.52) \qquad \frac{\partial z(\hat{k}|0)}{\partial\varphi} \;=\; \mathbb{E}\left[\int_0^\tau m(\hat{k}_s)\,\mathrm{d}s\middle| k_0 = \hat{k}\right]$$

Evaluating at $\hat{k}^*$, using the Auxiliary Theorem OMT in (A.2), providing the equivalence of occupancy measure and steady-state moments, we write the previous equation as:

$$(C.53) \qquad \frac{\partial z(\hat{k}^*|0)}{\partial\varphi} \;=\; \mathbb{E}\left[\int_0^\tau m(\hat{k}_s)\,\mathrm{d}s\middle| k_0 = \hat{k}^*\right] \;=\; \mathbb{E}[\tau]\mathbb{E}[m(\hat{k})] \;=\; 0$$

where we used $\mathbb{E}[m(\hat{k})] = \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} = 0$ by (43) .

At the same time, taking the derivative of (C.47) with respect to φ and evaluating at $\varphi = 0$ yields

$$(C.54) \qquad \frac{\partial z(\hat{k}^*|0)}{\partial\varphi} \;=\; \mathbb{E}\left[\int_0^\tau s\left(f(\hat{k}_s) - \mathbb{E}\left[f(\hat{k})\right]\right)\mathrm{d}s + \tau m(\hat{k}^*)\middle|\, \hat{k}_0 = \hat{k}^*\right].$$

Together (C.53) and (C.54) imply:

$$0 = \mathbb{E}\left[\int_0^\tau s\left(f(\hat{k}_s) - \mathbb{E}\left[f(\hat{k})\right]\right)\mathrm{d}s\middle|\hat{k}_0 = \hat{k}^*\right] + \mathbb{E}\left[\tau\middle|\hat{k}_0 = \hat{k}^*\right]m(\hat{k}^*). \tag{C.55}$$

Solving for $m(\hat{k}^*)$:

$$m(\hat{k}^*) = -\frac{\mathbb{E}\left[\int_0^\tau s\left(f(\hat{k}_s) - \mathbb{E}\left[f(\hat{k})\right]\right)\mathrm{d}s\middle|\hat{k}_0 = \hat{k}^*\right]}{\mathbb{E}\left[\tau\middle|\hat{k}_0 = \hat{k}^*\right]}. \tag{C.56}$$

Note that s captures the time elapsed since the last adjustment, that is, capital age a. Using the OMT in (A.2), we rewrite the occupancy measure as a steady-state moment, which turns out to be equal to minus the covariance of age with the function of capital-productivity ratios $f(\hat{k})$:

$$m(\hat{k}^*) = -\mathbb{E}[a(f(\hat{k}) - \mathbb{E}[f(\hat{k})])] \;=\; -\mathbb{C}ov[a, f(\hat{k})]. \tag{C.57}$$

C.2.2 With irreversibility

Observe that $\mathcal{M}(f, \hat{k})$ satisfies the following recursive representation

$$m(\hat{k}) = \mathbb{E}\left[\int_0^\tau \left(f(\hat{k}_s) - \mathbb{E}\left[f(\hat{k})\right]\right)\mathrm{d}s + m(\hat{k}^*(\hat{k}_\tau))\middle|\hat{k}_0 = \hat{k}\right]. \tag{C.58}$$

Define an auxiliary function $z(\hat{k}|\varphi)$ as follows:

$$z(\hat{k}|\varphi) \equiv \mathbb{E}\left[\int_0^\tau e^{\varphi s}\left(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})]\right)\mathrm{d}s + e^{\varphi\tau}m(\hat{k}^*(\hat{k}_\tau))\middle|\hat{k}_0 = \hat{k}\right]. \tag{C.59}$$

and note the relationship: $z(\hat{k}|0) = m(\hat{k})$, $z(\cdot|\varphi) \in \mathbb{C}^2((\hat{k}^-, \hat{k}^+)) \cap \mathbb{C}$ for all φ, and

$$-\varphi z(\hat{k}|\varphi) + \Lambda(\hat{k})\left(z(\hat{k}|\varphi) - m(\hat{k})\right) = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu z'(\hat{k}|\varphi) + \frac{\sigma^2}{2}z''(\hat{k}|\varphi), \tag{C.60}$$

$$z(\hat{k}^\pm|\varphi) = m(\hat{k}^{*\pm}). \tag{C.61}$$

Since $z(\hat{k}|0) = m(\hat{k})$, then we have $\int_{\hat{k}^-}^{\hat{k}^+} z(\hat{k}|0)g(\hat{k})\,\mathrm{d}\hat{k} = \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} = 0$. Taking the derivative with respect to φ in (C.60), we have that

$$(\Lambda(\hat{k}) - \varphi)\frac{\partial z(\hat{k}|\varphi)}{\partial\varphi} - z(\hat{k}|\varphi) = -\nu\frac{\partial^2 z(\hat{k}|\varphi)}{\partial\hat{k}\partial\varphi} + \frac{\sigma^2}{2}\frac{\partial^3 z(\hat{k}|\varphi)}{\partial\hat{k}^2\partial\varphi} \quad \text{and} \quad \frac{\partial z(\hat{k}^\pm, \varphi)}{\partial\varphi} = 0. \tag{C.62}$$

Using the Schwarz's theorem to exchange partial derivatives and evaluating at $\varphi = 0$:

$$\Lambda(\hat{k})\left.\frac{\partial z(\hat{k}, \varphi)}{\partial\varphi}\right|_{\varphi=0} - m(\hat{k}) = -\nu\frac{\partial\frac{\partial\left.\frac{\partial z(\hat{k},\varphi)}{\partial\varphi}\right|_{\varphi=0}}{\partial\varphi}}{\partial\hat{k}} + \frac{\sigma^2}{2}\frac{\partial^2\left.\frac{\partial z(\hat{k},\varphi)}{\partial\varphi}\right|_{\varphi=0}}{\partial\hat{k}^2} \quad \text{and} \quad \left.\frac{\partial z(\hat{k}, \varphi)}{\partial\varphi}\right|_{\varphi=0} = 0. \tag{C.63}$$

From the previous equation, using OMT in (A.2) and the renewal distribution, we have that

$$\overline{\mathbb{E}}\left[\frac{\partial z(\hat{k}^*(\Delta\hat{k})|0)}{\partial\varphi}\right] \;=\; \overline{\mathbb{E}}\left[\mathbb{E}\left[\int_0^\tau m(\hat{k}_s)\,\mathrm{d}s|\hat{k}_0 = \hat{k}^*\right]\right]\overline{\mathbb{E}}[\tau] \;=\; \overline{\mathbb{E}}[\tau]\mathbb{E}[m(\hat{k})] \;=\; 0. \tag{C.64}$$

Therefore, $\overline{\mathbb{E}}\left[\frac{\partial z(\hat{k}^*(\Delta\hat{k})|0)}{\partial\varphi}\right] = 0$. Using this result, the renewal distribution, the OST, and the the definition of $\overline{\mathbb{E}}$ with shares, we get:

$$\begin{aligned}
0 &= \overline{\mathbb{E}}\left[\frac{\partial z(\hat{k}^*(\Delta\hat{k})|0)}{\partial\varphi}\right] \\
&= \overline{\mathbb{E}}\left[\mathbb{E}\left[\int_0^\tau s\left(f(\hat{k}_s) - \mathbb{E}\left[f(\hat{k})\right]\right)\mathrm{d}s + \tau m(\hat{k}^*(\hat{k}_\tau))\middle|\hat{k}_0 = \hat{k}^*\right]\right] \\
&= \overline{\mathbb{E}}[\tau]\mathbb{E}\left[a\left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right)\right] \;+\; \overline{\mathbb{E}}\left[\mathbb{E}[\tau m(\hat{k}^*(\hat{k}_\tau))|\hat{k}_0 = \hat{k}^*]\right] \\
&= \overline{\mathbb{E}}[\tau]\mathbb{C}ov\left[a, f(\hat{k})\right] \;+\; \frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-\left[\tau m(\hat{k}^*(\hat{k}_\tau))\right] + \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+\left[\tau m(\hat{k}^*(\hat{k}_\tau))\right] \\
&= \overline{\mathbb{E}}[\tau]\mathbb{C}ov\left[a, f(\hat{k})\right] \;+\; \frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-\left[\tau\left(m(\hat{k}^{*+})\mathbb{1}_{\{\hat{k}_\tau \geq \hat{k}^{*+}\}} + m(\hat{k}^{*-})\left(1 - \mathbb{1}_{\{\hat{k}_\tau \geq \hat{k}^{*+}\}}\right)\right)\right] \\
&\qquad + \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+\left[\tau\left(m(\hat{k}^{*+})\mathbb{1}_{\{\hat{k}_\tau \geq \hat{k}^{*+}\}} + m(\hat{k}^{*-})\left(1 - \mathbb{1}_{\{\hat{k}_\tau \geq \hat{k}^{*+}\}}\right)\right)\right] \\
&= \overline{\mathbb{E}}[\tau]\mathbb{C}ov\left[a, f(\hat{k})\right] + m(\hat{k}^{*-}) + (m(\hat{k}^{*+}) - m(\hat{k}^{*-}))\overline{\mathbb{E}}[\tau\mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+})] \qquad \text{(C.65)}
\end{aligned}$$

To characterize the difference in cumulative deviations, $m(\hat{k}^{*+}) - m(\hat{k}^{*-})$, observe that

$$m(\hat{k}^{*-}) = \left(\mathbb{E}^-[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right)\overline{\mathbb{E}}^-[\tau] + (1 - \mathbb{P}^{--})m(\hat{k}^{*+}) + \mathbb{P}^{--}m(\hat{k}^{*-}) \tag{C.66}$$

where $\mathbb{E}^-[f(\hat{k})]$ is the expected $\hat{k}$ conditional of a positive investment. Thus,

$$-\left(m(\hat{k}^{*+}) - m(\hat{k}^{*-})\right) = \frac{\left(\mathbb{E}^-[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right)\overline{\mathbb{E}}^-[\tau]}{1 - \mathbb{P}^{--}} \tag{C.67}$$

From (C.65) and (C.67), we have that

$$m(\hat{k}^{*-}) \;+\; \mathbb{C}ov\left[f(\hat{k}), a\right] \;=\; \frac{\overline{\mathbb{E}}[\tau\mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+})]}{\overline{\mathbb{E}}[\tau]}\frac{\left(\mathbb{E}^-[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right)\overline{\mathbb{E}}^-[\tau]}{1 - \mathbb{P}^{--}}. \tag{C.68}$$

With similar steps as before, it is easy to show that

$$m(\hat{k}^{*+}) \;+\; \mathbb{C}ov\left[f(\hat{k}), a\right] \;=\; \frac{\mathbb{E}[\tau\mathbb{I}(\hat{k}_\tau \leq \hat{k}^{*-})]}{\mathbb{E}[\tau]}\frac{\left(\mathbb{E}^+[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right)\mathbb{E}^+[\tau]}{1 - \mathbb{P}^{++}}. \tag{C.69}$$

C.2.3 Expected probabilities

Next, we characterize the average adjustment probabilities in terms of stopping times: $\mathbb{E}[\mathbb{P}^-(\hat{k})] = \frac{\mathbb{E}[\tau\mathbb{I}(\hat{k}_\tau \leq \hat{k}^{*-})]}{\mathbb{E}[\tau]}$ and $\mathbb{E}[\mathbb{P}^+(\hat{k})] = \frac{\mathbb{E}[\tau\mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+})]}{\mathbb{E}[\tau]}$. Define the function

$$\tilde{P}^+(\hat{k}, \varphi) \equiv \mathbb{E}\left[e^{\varphi\tau}\mathbb{I}[\hat{k}_\tau \geq \hat{k}^{*+}]|\hat{k}_0 = \hat{k}\right], \tag{C.70}$$

which satisfies the following HBJ conditions and border conditions

$$-\varphi \tilde{P}^+(\hat{k},\varphi) + \Lambda(\hat{k}) \left(\tilde{P}^+(\hat{k},\varphi) - \mathbb{I}[\hat{k} \geq \hat{k}^{*+}] \right) = -\nu \frac{\partial \tilde{P}^+(\hat{k},\varphi)}{\partial \hat{k}} + \frac{\sigma^2}{2} \frac{\partial^2 \tilde{P}^+(\hat{k},\varphi)}{\partial \hat{k}^2}, \tag{C.71}$$

$$\tilde{P}^+(\hat{k}^+,\varphi) = 1, \tag{C.72}$$

$$\tilde{P}^+(\hat{k}^-,\varphi) = 0. \tag{C.73}$$

Note that $\tilde{P}^+(\hat{k},0) = \mathbb{P}^+(\hat{k})$. Taking the derivative with φ and evaluating at $\varphi = 0$

$$\Lambda(\hat{k}) \frac{\partial \tilde{P}^+(\hat{k},0)}{\partial \varphi} = \mathbb{P}^+(\hat{k}) - \nu \frac{\partial \frac{\partial \tilde{P}^+(\hat{k},0)}{\partial \varphi}}{\partial \hat{k}} + \frac{\sigma^2}{2} \frac{\partial^2 \frac{\partial \tilde{P}^+(\hat{k},0)}{\partial \varphi}}{\partial \hat{k}^2}, \tag{C.74}$$

$$\frac{\partial \tilde{P}^+(\hat{k}^+,0)}{\partial \varphi} = 0, \tag{C.75}$$

$$\frac{\partial \tilde{P}^+(\hat{k}^-,0)}{\partial \varphi} = 0. \tag{C.76}$$

From Auxiliary Theorem A.3, we convert the HJB and borders into the first of two formulations:

$$\frac{\partial \tilde{P}^+(\hat{k},0)}{\partial \varphi} = \mathbb{E}\left[\int_0^\tau \mathbb{P}^+(\hat{k}_t)\, \mathrm{d}t \middle| \hat{k}_0 = \hat{k} \right]. \tag{C.77}$$

To obtain the second formulation, note that by definition

$$\frac{\partial \tilde{P}^+(\hat{k},\varphi)}{\partial \varphi} = \mathbb{E}\left[\tau e^{\tau\varphi} \mathbb{I}[\hat{k}_\tau \geq \hat{k}^{*+}] \middle| \hat{k}_0 = \hat{k} \right], \tag{C.78}$$

evaluating at zero

$$\frac{\partial \tilde{P}^+(\hat{k},0)}{\partial \varphi} = \mathbb{E}\left[\tau \mathbb{I}[\hat{k}_\tau \geq \hat{k}^{*+}] \middle| \hat{k}_0 = \hat{k} \right]. \tag{C.79}$$

Using relations (C.77) and (C.79), we have that

$$\mathbb{E}\left[\int_0^\tau \mathbb{P}^+(\hat{k}_t)\, \mathrm{d}t \middle| \hat{k}_0 = \hat{k} \right] = \mathbb{E}\left[\tau \mathbb{I}[\hat{k}_\tau \geq \hat{k}^{*+}] \middle| \hat{k}_0 = \hat{k} \right] \tag{C.80}$$

Evaluating in $\hat{k}^{*\pm}$ and operating

$$\frac{r^- \mathbb{E}^-[\tau]}{\mathbb{E}[\tau]} \mathbb{E}^-[\mathbb{P}^+(\hat{k})] = r^- \frac{\bar{\mathbb{E}}^- \left[\tau \mathbb{I}[\hat{k}_\tau \geq \hat{k}^{*+}] \right]}{\mathbb{E}[\tau]}, \tag{C.81}$$

$$\frac{r^+ \mathbb{E}^+[\tau]}{\mathbb{E}[\tau]} \mathbb{E}^+[\mathbb{P}^+(\hat{k})] = r^+ \frac{\bar{\mathbb{E}}^+ \left[\tau \mathbb{I}[\hat{k}_\tau \geq \hat{k}^{*+}] \right]}{\mathbb{E}[\tau]}. \tag{C.82}$$

Suming the two equations,

$$\mathbb{E}[\mathbb{P}^+(\hat{k})] = \frac{\bar{\mathbb{E}} \left[\tau \mathbb{I}[\hat{k}_\tau \geq \hat{k}^{*+}] \right]}{\mathbb{E}[\tau]}. \tag{C.83}$$

C.3 Proof of Proposition 4

Proposition 4. ***(Sufficient statistics)*** *Up to the first order, the CIR of average capital-productivity ratios equals the sum of three steady-state cross-sectional moments:*

$$\frac{CIR(\delta)}{\delta} = \underbrace{\frac{\mathbb{V}ar[\hat{k}]}{\sigma^2} + \frac{\nu\mathbb{C}ov[\hat{k},a]}{\sigma^2}}_{\text{up to first adjustment}} + \underbrace{\frac{1}{\sigma^2}\,\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s[\mathrm{d}(\hat{k}_s\mathcal{M}(\hat{k}_s))]\right]}_{\text{subsequent adjustments}} + o(\delta). \tag{51}$$

Proof's strategy We prove the proposition without and with irreversibility. In each case, we construct the master equation that combines the HJB for deviations $m(\hat{k})$ and the KFE describing the distribution of $\hat{k}$. The trick is substituting the hazard $\Lambda(\hat{k})$ from the KFE into the HJB. Then, we multiply by $\hat{k}$ and compute the cross-sectional average. Depending on the case, we get three or four terms T_j that we compute using integration by parts, exploiting the border conditions of m and g. We prove the results for any continuous function $f(\hat{k})$.

C.3.1 Without irreversibility

We will show that

$$\frac{\text{CIR}(f,\delta)}{\delta} = \mathbb{C}ov\left[f(\hat{k}),\frac{\hat{k}+\nu a}{\sigma^2}\right] = \frac{\mathbb{C}ov\left[f(\hat{k}),\hat{k}\right]}{\sigma^2} + \frac{\nu\mathbb{C}ov\left[f(\hat{k}),a\right]}{\sigma^2}. \tag{C.84}$$

Rearranging the HJB for $m(\hat{k})$ in (C.39), we get:

$$\Lambda(\hat{k})m(\hat{k}) = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu m'(\hat{k}) + \frac{\sigma^2}{2}m''(\hat{k}) + \Lambda(\hat{k})m(\hat{k}^*)$$

Solve for $\Lambda(k) = \frac{\nu g'(\hat{k})+\frac{\sigma^2}{2}g''(\hat{k})}{g(\hat{k})}$ from the KFE and substitute it into (B.13) to obtain

$$\left[\frac{\nu g'(\hat{k})+\frac{\sigma^2}{2}g''(\hat{k})}{g(\hat{k})}\right]m(\hat{k}) = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu m'(\hat{k}) + \frac{\sigma^2}{2}m''(\hat{k}) + \left[\frac{\nu g'(\hat{k})+\frac{\sigma^2}{2}g''(\hat{k})}{g(\hat{k})}\right]m(\hat{k}^*). \tag{C.85}$$

Multiplying both sides by $g(\hat{k})\hat{k}$ and taking the definite integral between $\hat{k}^-$ and $\hat{k}^+$ (effectively, we compute the cross-sectional average) we obtain the following expression:

$$0 = \mathbb{E}\left[f(\hat{k})\hat{k}\right] - \mathbb{E}[f(\hat{k})]\mathbb{E}[\hat{k}] - \nu T_1 + \frac{\sigma^2}{2}T_2 + m(\hat{k}^*)T_3. \tag{C.86}$$

where we define the following three terms, characterized next:

$$T_1 \equiv \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left[m'(\hat{k})g(\hat{k}) + m(\hat{k})g'(\hat{k})\right]\mathrm{d}\hat{k} \tag{C.87}$$

$$T_2 \equiv \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left[m''(\hat{k})g(\hat{k}) - m(\hat{k})g''(\hat{k})\right]\mathrm{d}\hat{k} \tag{C.88}$$

$$T_3 \equiv \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left(\nu g'(\hat{k}) + \frac{\sigma^2}{2}g''(\hat{k})\right)\mathrm{d}\hat{k}. \tag{C.89}$$

We will now rearrange T_1, T_2 and T_3. We will often use the product rule, integration by parts, and the continuity and border conditions of g, namely $g(\hat{k}^+) = g(\hat{k}^-) = 0$, the border conditions of m, namely $m(\hat{k}^+) = m(\hat{k}^-) = m(\hat{k}^*)$, and the continuity of $m(\cdot)$ and $g(\cdot)$ around k^*.

(i) We re-write T_1 by the following steps: in step (1) we split the integral, in step (2) the product rule – $m'(\hat{k})g(\hat{k})+m(\hat{k})g'(\hat{k}) = \frac{\mathrm{d}[m(\hat{k})g(\hat{k})]}{\mathrm{d}\hat{k}}$, in step (3) we use integration by parts, in step (4) we rely on the border conditions and continuity of $m(\cdot)$ and $g(\cdot)$ around k^*, in step (5) we join the integral, and in step (6) we use that the cross-sectional mean of $m(\cdot)$ is zero:

$$
\begin{aligned}
\text{(C.90)} \qquad T_1 &= \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left[m'(\hat{k})g(\hat{k})+m(\hat{k})g'(\hat{k})\right]\mathrm{d}\hat{k} \\
&=^{(1)} \int_{\hat{k}^-}^{\hat{k}^*} \hat{k}\left[m'(\hat{k})g(\hat{k})+m(\hat{k})g'(\hat{k})\right]\mathrm{d}\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} \hat{k}\left[m'(\hat{k})g(\hat{k})+m(\hat{k})g'(\hat{k})\right]\mathrm{d}\hat{k} \\
&=^{(2)} \int_{\hat{k}^-}^{\hat{k}^*} \hat{k}(m(\hat{k})g(\hat{k}))'\,\mathrm{d}\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} \hat{k}(m(\hat{k})g(\hat{k}))'\,\mathrm{d}\hat{k} \\
&=^{(3)} \hat{k}m(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}m(\hat{k})g(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+} - \left[\int_{\hat{k}^-}^{\hat{k}^*} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k}\right] \\
&=^{(4)} 0+0-\left[\int_{\hat{k}^-}^{\hat{k}^*} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k}\right] =^{(5)} -\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} =^{(6)} 0
\end{aligned}
$$

(ii) To characterize T_2, we do the following steps. In step (1), we split the integral; in step (2), we use the equality $m''(\hat{k})g(\hat{k}) - m(\hat{k})g''(\hat{k}) = (m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k}))'$ and integration by parts; in step (3), we use continuity of $m'(\hat{k})$ and $g(\hat{k})$ around $\hat{k}^*$ and the border condition $g(\hat{k}^+) = g(\hat{k}^-) = 0$ for $\hat{k}m'(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^+} = 0$; in step (4), we use the border conditions of m; in step (5) we apply integration by parts to t$\int_{\hat{k}^-}^{\hat{k}^+} m'(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k}$; and step (6) groups common terms and relies on the continuity and border conditions of $m(\cdot)$ and $g(\cdot)$:

(C.91)

$$
\begin{aligned}
T_2 &= \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left[m''(\hat{k})g(\hat{k}) - m(\hat{k})g''(\hat{k})\right]\mathrm{d}\hat{k} \\
&= ^{(1)} \int_{\hat{k}^-}^{\hat{k}^*} \hat{k}\left[m''(\hat{k})g(\hat{k}) - m(\hat{k})g''(\hat{k})\right]\mathrm{d}\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} \hat{k}\left[m''(\hat{k})g(\hat{k}) - m(\hat{k})g''(\hat{k})\right]\mathrm{d}\hat{k} \\
&= ^{(2)} \hat{k}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\Big|_{\hat{k}^*}^{\hat{k}^+} \\
&\cdots - \left[\int_{\hat{k}^-}^{\hat{k}^*} \left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\mathrm{d}\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} \left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\mathrm{d}\hat{k}\right] \\
&= ^{(3)} \underbrace{\hat{k}m'(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^+}}_{=0} - m(\hat{k}^*)\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right] \\
&\cdots - \left[\int_{\hat{k}^-}^{\hat{k}^*} \left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\mathrm{d}\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} \left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\mathrm{d}\hat{k}\right] \\
&= ^{(4)} - m(\hat{k}^*)\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right] - \int_{\hat{k}^-}^{\hat{k}^+} m'(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} + \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} \\
&= ^{(5)} - m(\hat{k}^*)\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right] - \left[\underbrace{m(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^+}}_{=0} - \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k}\right] + \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} \\
&= ^{(6)} - m(\hat{k}^*)\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right] + 2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k}.
\end{aligned}
$$

(iii) To characterize T_3 we perform the following steps: In step (1) we split the integral, in step (2) we use integration by parts, in step (3) we use the border conditions of $g(\cdot)$, the definition of a density function and solve the integral $\int_{\hat{k}^-}^{\hat{k}^+} g'(\hat{k})\,\mathrm{d}\hat{k}$, in step (4) we use the border conditions of $g(\cdot)$:

$$
\begin{aligned}
\text{(C.92)} \qquad T_3 &= \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left(\nu g'(\hat{k}) + \frac{\sigma^2}{2} g''(\hat{k})\right)\mathrm{d}\hat{k} \\
&=^{(1)} \nu\left[\int_{\hat{k}^-}^{\hat{k}^*} \hat{k}g'(\hat{k})\,\mathrm{d}\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} \hat{k}g'(\hat{k})\,\mathrm{d}\hat{k}\right] + \frac{\sigma^2}{2}\left[\int_{\hat{k}^-}^{\hat{k}^*} \hat{k}g''(\hat{k})\,\mathrm{d}\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} \hat{k}g''(\hat{k})\,\mathrm{d}\hat{k}\right] \\
&=^{(2)} \nu\left[\underbrace{\hat{k}g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}}_{=0} - \underbrace{\int_{\hat{k}^-}^{\hat{k}^+} g(\hat{k})\,\mathrm{d}\hat{k}}_{=1}\right] + \frac{\sigma^2}{2}\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+} - \int_{\hat{k}^-}^{\hat{k}^+} g'(\hat{k})\,\mathrm{d}\hat{k}\right] \\
&=^{(3)} -\nu + \frac{\sigma^2}{2}\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+} - \underbrace{g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^+}}_{=0}\right] \\
&=^{(4)} -\nu + \frac{\sigma^2}{2}\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right]
\end{aligned}
$$

Substituting back the expressions for T_1, T_2, and T_3 from equations (C.90) to (C.92) into (C.86)

(C.93)

$$
\begin{aligned}
0 &= \mathbb{E}\Big[f(\hat{k})\hat{k}\Big] - \mathbb{E}\Big[\hat{k}\Big]\,\mathbb{E}\Big[f(\hat{k})\Big] - \nu T_1 + \frac{\sigma^2}{2}T_2 + m(\hat{k}^*)T_3 \\
&= \mathbb{E}\Big[f(\hat{k})\hat{k}\Big] - \mathbb{E}\Big[\hat{k}\Big]\,\mathbb{E}\Big[f(\hat{k})\Big] - \nu 0 + \frac{\sigma^2}{2}\left[-m(\hat{k}^*)\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right] + 2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k}\right] \\
&\cdots + m(\hat{k}^*)\left[-\nu + \frac{\sigma^2}{2}\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right]\right] \\
&= \mathbb{E}\Big[f(\hat{k})\hat{k}\Big] - \mathbb{E}\Big[\hat{k}\Big]\,\mathbb{E}\Big[f(\hat{k})\Big] + \sigma^2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} + \nu\mathbb{E}\left[a\left(f(\hat{k}) - \mathbb{E}\left[f(\hat{k})\right]\right)\right] \\
&= \mathbb{C}ov\left[f(\hat{k}), \hat{k} + \nu a\right] + \sigma^2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k}. \iff -\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k}) = \mathbb{C}ov\left[f(\hat{k}), \frac{\hat{k}+\nu a}{\sigma^2}\right]
\end{aligned}
$$

Combining this result with the CIR's first-order approximation in Proposition 2 yields:

$$
\text{(C.94)} \qquad \frac{\text{CIR}(f,\delta)}{\delta} = -\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} = \mathbb{C}ov\left[f(\hat{k}), \frac{\hat{k}+\nu a}{\sigma^2}\right]
$$

C.3.2 With irreversibility

From Propositions 2 and 3, we know that

$$0 = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu m'(\hat{k}) + \frac{\sigma^2}{2} m''(\hat{k}) + \Lambda(\hat{k}) \left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}), a] - m(\hat{k}) \right), \tag{C.95}$$

$$0 = \mathcal{M}(\hat{k}^{\pm}) - \mathbb{C}ov[f(\hat{k}), a] - m(\hat{k}^{\pm}), \tag{C.96}$$

$$0 = \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k}) g(\hat{k}) \, \mathrm{d}\hat{k}. \tag{C.97}$$

where $\mathcal{M}(\hat{k})$ is defined in (C.98)

$$\mathcal{M}(\hat{k}) \equiv \begin{cases} m(\hat{k}^{*-}) + \mathbb{C}ov(\hat{k}, a) < 0 & \text{if } \hat{k} \in [\hat{k}^-, \hat{k}^{*-}] \\ m(\hat{k}^{*+}) + \mathbb{C}ov(\hat{k}, a) > 0 & \text{if } \hat{k} \in [\hat{k}^{*+}, \hat{k}^+]. \end{cases} \tag{C.98}$$

From the KFE in (B.17) we solve for the adjustment hazard $\Lambda(k) = \frac{\nu g'(\hat{k}) + \frac{\sigma^2}{2} g''(\hat{k})}{g(\hat{k})}$ and using equation (C.41)

$$\frac{\nu g'(\hat{k}) + \frac{\sigma^2}{2} g''(\hat{k})}{g(\hat{k})} m(\hat{k}) = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu m'(\hat{k}) + \frac{\sigma^2}{2} m''(\hat{k}) + \frac{\nu g'(\hat{k}) + \frac{\sigma^2}{2} g''(\hat{k})}{g(\hat{k})} \left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}), a] \right) \tag{C.99}$$

Multiplying by $g(\hat{k})\hat{k}$ and taking the integral between $\hat{k}^-$ and $\hat{k}^+$:

$$0 = \mathbb{E}\left[f(\hat{k})\hat{k} \right] - \mathbb{E}\left[\hat{k} \right] \mathbb{E}\left[f(\hat{k}) \right] - \nu T_1 + \frac{\sigma^2}{2} T_2 + \nu T_3 + \frac{\sigma^2}{2} T_4 \tag{C.100}$$

where we define the following four terms, characterized next:

$$T_1 = \int_{\hat{k}^-}^{\hat{k}^+} \hat{k} \left[m'(\hat{k}) g(\hat{k}) + m(\hat{k}) g'(\hat{k}) \right] \mathrm{d}\hat{k} \tag{C.101}$$

$$T_2 = \int_{\hat{k}^-}^{\hat{k}^+} \hat{k} \left[m''(\hat{k}) g(\hat{k}) - m(\hat{k}) g''(\hat{k}) \right] \mathrm{d}\hat{k} \tag{C.102}$$

$$T_3 = \int_{\hat{k}^-}^{\hat{k}^+} (\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}), a]) \hat{k} g'(\hat{k}) \, \mathrm{d}\hat{k} \tag{C.103}$$

$$T_4 = \int_{\hat{k}^-}^{\hat{k}^+} (\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}), a]) \hat{k} g''(\hat{k}) \, \mathrm{d}\hat{k}. \tag{C.104}$$

(i) To characterize T_1, we use the following: (1) the product rule – $m'(\hat{k})g(\hat{k}) + m(\hat{k})g'(\hat{k}) = \frac{\mathrm{d}[m(\hat{k})g(\hat{k})]}{\mathrm{d}\hat{k}}$, (2) integration by parts, border and continuity conditions of $m(\cdot)$ and $g(\cdot)$, and the zero expectation of $m(\cdot)$:

$$\begin{aligned} T_1 &= \int_{\hat{k}^-}^{\hat{k}^+} \hat{k} \left[m'(\hat{k}) g(\hat{k}) + m(\hat{k}) g'(\hat{k}) \right] \mathrm{d}\hat{k} \\ &=^{(1)} \int_{\hat{k}^-}^{\hat{k}^+} \hat{k} \left(m(\hat{k}) g(\hat{k}) \right)' \mathrm{d}\hat{k} \\ &=^{(2)} \underbrace{\hat{k} m(\hat{k}) g(\hat{k}) \Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \hat{k} m(\hat{k}) g(\hat{k}) \Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \hat{k} m(\hat{k}) g(\hat{k}) \Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{=0} - \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k}) g(\hat{k}) \, \mathrm{d}\hat{k} = 0 \end{aligned} \tag{C.105}$$

(ii) To rewrite T_2 we carry out the following steps: (1) substitutes $m''(\hat{k})g(\hat{k}) - m(\hat{k})g''(\hat{k})$ with $\frac{\mathrm{d}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]}{\mathrm{d}\hat{k}}$ using the product rule and splits the integral; (2) applies integration by parts; (3) rearranges terms; (4) uses the continuity and border conditions of $m(\cdot)$ and $g(\cdot)$; (5) uses integration by parts; (6) uses the continuity and border conditions of $m(\cdot)$ and $g(\cdot)$ and joins common terms; (7) uses $m(\hat{k}) = \mathcal{M}(\hat{k}) + \mathbb{C}ov[f(\hat{k}), a]$:

(C.106)

$$
\begin{aligned}
T_2 \;&=\; \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left[m''(\hat{k})g(\hat{k}) - m(\hat{k})g''(\hat{k})\right]\mathrm{d}\hat{k} \\
&=^{(1)} \int_{\hat{k}^-}^{\hat{k}^{*-}} \hat{k}\frac{\mathrm{d}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]}{\mathrm{d}\hat{k}}\mathrm{d}\hat{k} + \int_{\hat{k}^{*-}}^{\hat{k}^{*+}} \hat{k}\frac{d\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]}{\mathrm{d}\hat{k}}d\hat{k} + \int_{\hat{k}^{*+}}^{\hat{k}^+} \hat{k}\frac{\mathrm{d}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]}{\mathrm{d}\hat{k}} \\
&=^{(2)} \hat{k}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \hat{k}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \hat{k}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\Big|_{\hat{k}^{*+}}^{\hat{k}^+} \\
&\cdots - \left[\int_{\hat{k}^-}^{\hat{k}^{*-}}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\mathrm{d}\,\hat{k} + \int_{\hat{k}^{*-}}^{\hat{k}^{*+}}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\mathrm{d}\,\hat{k} + \int_{\hat{k}^{*+}}^{\hat{k}^+}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\mathrm{d}\,\hat{k}\right] \\
&=^{(3)} \underbrace{\hat{k}m'(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^+} + \hat{k}m'(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \hat{k}m'(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{=0} - \left[m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^-}\right] \\
&\cdots - \left[\int_{\hat{k}^-}^{\hat{k}^+} m'(\hat{k})g(\hat{k})\,\mathrm{d}\,\hat{k} - \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k}\right] \\
&=^{(4)} -\left[m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}\right] - \int_{\hat{k}^-}^{\hat{k}^+} m'(\hat{k})g(\hat{k})\,\mathrm{d}\,\hat{k} + \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k} \\
&=^{(5)} -\left[m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}\right] + \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} \\
&- \left[\underbrace{m(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + m(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + m(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{=0} - \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k}\right] \\
&=^{(6)} -\left[m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}\right] + 2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k}. \\
&=^{(7)} -\left[\Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}), a]\Big)\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}), a]\Big)\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}}\right] \\
&- \left[\Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}), a]\Big)\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}\right] + 2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k}
\end{aligned}
$$

(iii) For T_3, step (1) divides the integration domain into the discontinuity points; step (2) uses continuity of $\mathcal{M}(\hat{k})$ and $g(\hat{k})$, together with the boundaries conditions of $g(\hat{k}^{\pm}) = 0$; step (3) re-writes the integral:

(C.107)

$$
\begin{aligned}
T_3 &= \int_{\hat{k}^-}^{\hat{k}^+} \Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\Big)\,\hat{k}g'(\hat{k})\,\mathrm{d}\hat{k} \\
&=^{(1)} \int_{\hat{k}^-}^{\hat{k}^{*-}} \hat{k}\Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\Big)\,g'(\hat{k})\,\mathrm{d}\hat{k} + \int_{\hat{k}^{*-}}^{\hat{k}^{*+}} \hat{k}\Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\Big)\,g'(\hat{k})\,\mathrm{d}\hat{k} \\
&+ \int_{\hat{k}^{*+}}^{\hat{k}^+} \hat{k}\Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\Big)\,g'(\hat{k})\,\mathrm{d}\hat{k} \\
&=^{(2)} \underbrace{\Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\Big)\,\hat{k}g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\Big)\,\hat{k}g(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\Big)\,\hat{k}g(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{=0} \\
&- \int_{\hat{k}^-}^{\hat{k}^+} \Big[\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a] + \hat{k}\mathcal{M}'(\hat{k})\Big]\,g(\hat{k})\,\mathrm{d}\hat{k} \\
&=^{(3)} \mathbb{C}ov[f(\hat{k}),a] - \mathbb{E}\Big[\mathcal{M}(\hat{k}) + \hat{k}\mathcal{M}'(\hat{k})\Big]
\end{aligned}
$$

(iv) For T_4, step (1) breaks the integral, step (2) uses integration by parts, step (3) uses integration by parts, step (4) uses the border conditions for $g(\cdot)$:

(C.108)

$$
\begin{aligned}
T_4 &= \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\Big)\,g''(\hat{k})\,\mathrm{d}\hat{k} \\
&=^{(1)} \int_{\hat{k}^-}^{\hat{k}^{*-}} \hat{k}\Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\Big)\,g''(\hat{k})\,\mathrm{d}\hat{k} + \int_{\hat{k}^{*-}}^{\hat{k}^{*+}} \hat{k}\Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\Big)\,g''(\hat{k})\,\mathrm{d}\hat{k} \\
&+ \int_{\hat{k}^{*+}}^{\hat{k}^+} \hat{k}\Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\Big)\,g''(\hat{k})\,\mathrm{d}\hat{k} \\
&=^{(2)} \Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\Big)\,\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\Big)\,\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} \\
&+ \Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\Big)\,\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+} - \int_{\hat{k}^-}^{\hat{k}^+} \Big[\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a] + \hat{k}\mathcal{M}'(\hat{k})\Big]\,g'(\hat{k})\,\mathrm{d}\hat{k} \\
&=^{(3)} \Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\Big)\,\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\Big)\,\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\Big)\,\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+} \\
&\cdots - \underbrace{\Big[\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a] + \hat{k}\mathcal{M}'(\hat{k})\Big]\,g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^+}}_{=0} + \int_{\hat{k}^-}^{\hat{k}^+} \Big[2\mathcal{M}'(\hat{k}) + \hat{k}\mathcal{M}''(\hat{k})\Big]\,g(\hat{k})\,\mathrm{d}\hat{k} \\
&= \Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\Big)\,\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\Big)\,\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\Big)\,\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*+}}^{\hat{k}^+} \\
&+ \mathbb{E}\Big[2\mathcal{M}'(\hat{k}) + \hat{k}\mathcal{M}''(\hat{k})\Big]
\end{aligned}
$$

From equations (C.104) to (C.109)

(C.109)

$$0 \;=\; \mathbb{E}\left[f(\hat{k})\hat{k}\right] - \mathbb{E}\left[\hat{k}\right]\mathbb{E}\left[f(\hat{k})\right] - \nu T_1 + \frac{\sigma^2}{2}T_2 + \nu T_3 + \frac{\sigma^2}{2}T_4$$

(C.110)

$$= \mathbb{E}\left[f(\hat{k})\hat{k}\right] - \mathbb{E}\left[\hat{k}\right]\mathbb{E}\left[f(\hat{k})\right] - \nu 0 + \sigma^2 \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k} + \nu\left[\mathbb{C}ov[f(\hat{k}),a] - \mathbb{E}\left[\mathcal{M}(\hat{k}) + \hat{k}\mathcal{M}'(\hat{k})\right]\right]$$

(C.111)

$$\pm\left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right)\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right)\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right)\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Big|_{\hat{k}^{*+}}^{\hat{k}^+}$$

$$+\frac{\sigma^2}{2}\mathbb{E}\left[2\mathcal{M}'(\hat{k}) + \hat{k}\mathcal{M}''(\hat{k})\right]$$

$$= \mathbb{C}ov\left[f(\hat{k}),\hat{k}\right] + \sigma^2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k} + \nu\mathbb{C}ov[f(\hat{k}),a] - \nu\mathbb{E}\left[\mathcal{M}(\hat{k}) + \hat{k}\mathcal{M}'(\hat{k})\right] + \frac{\sigma^2}{2}\mathbb{E}\left[2\mathcal{M}'(\hat{k}) + \hat{k}\mathcal{M}''(\hat{k})\right]$$

Recalling that $\frac{CIR(f,\delta)}{\delta} = \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k}$ and rearranging the terms we obtain:

$$\text{(C.112)} \qquad \frac{\mathrm{CIR}(f,\delta)}{\delta} \;=\; \mathbb{C}ov\left[f(\hat{k}), \frac{\hat{k}+\nu a}{\sigma^2}\right] - \frac{\nu}{\sigma^2}\mathbb{E}\left[\mathcal{M}(\hat{k}) + \hat{k}\mathcal{M}'(\hat{k})\right] + \frac{1}{2}\mathbb{E}\left[2\mathcal{M}'(\hat{k}) + \hat{k}\mathcal{M}''(\hat{k})\right] + o(\delta)$$

Finally, if we apply Ito's lemma to $\hat{k}\mathcal{M}(\hat{k})$, we have that

$$\text{(C.113)} \qquad \mathbb{E}_s[\mathrm{d}(\hat{k}_s\mathcal{M}(\hat{k}_s))|\hat{k}_s = \hat{k}] = \left[-\nu\left[\mathcal{M}(\hat{k}) + \hat{k}\mathcal{M}'(\hat{k})\right] + \frac{\sigma^2}{2}\mathbb{E}\left[2\mathcal{M}'(\hat{k}) + \hat{k}\mathcal{M}''(\hat{k}_s)\right]\right]\mathrm{d}s$$

such that:

$$\text{(C.114)} \qquad \frac{\mathrm{CIR}(f,\delta)}{\delta} = \mathbb{C}ov\left[f(\hat{k}), \frac{\hat{k}+\nu a}{\sigma^2}\right] + \frac{1}{\sigma^2}\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s[\mathrm{d}(\mathcal{M}(\hat{k}_s)\hat{k}_s)]\right]$$

This concludes the proof.

C.4 Proof of Proposition 5

Proposition 5. ***(Extreme cases)*** *Up to the first order, the CIR's sufficient statistics as a function of investment frictions are as follows.*

(i) No drift and only fixed cost: If $\nu = \omega = 0$ and $\theta > 0$, then

$$\text{(53)} \qquad \frac{CIR(\delta)}{\delta} = \frac{\mathbb{V}ar[\hat{k}]}{\sigma^2} = \left(\frac{12\tilde{\theta}}{(1-\alpha)\sigma^6}\right)^{1/4}, \quad \textit{where} \quad \tilde{\theta} = \frac{\theta}{\alpha}\left(\frac{p\mathcal{U}}{\alpha}\right)^{\frac{\alpha}{1-\alpha}}.$$

(ii) No drift and only partial irreversibility: If $\nu = \theta = 0$ and $\omega > 0$, then

$$\text{(54)} \qquad \frac{CIR(\delta)}{\delta} = 2 \times \frac{\mathbb{V}ar[\hat{k}]}{\sigma^2} = \left(\frac{12\tilde{\omega}}{(1-\alpha)\sigma^4}\right)^{1/3}, \quad \textit{where} \quad \tilde{w} = \frac{\omega/2}{\mathcal{U}(1-\omega/2)}.$$

(iii) Large drift: If $\sigma^2 > 0$ and $\nu \to \infty$, then the price wedge is irrelevant and

$$\text{(55)} \qquad \mathbb{E}\left[\frac{\mathbb{E}_s[\mathrm{d}(\hat{k}_s \mathcal{M}(\hat{k}_s))]}{\mathrm{d}s}\right] = 0, \qquad \nu\mathbb{C}ov[\hat{k}, a] = -\mathbb{V}ar[\hat{k}], \qquad \frac{CIR(\delta)}{\delta} = 0.$$

Proof's strategy We prove Proposition 5 in a sequence lemmas by departing from Proposition 1. First, Lemma C.2 shows that the investment policy can be separated into a static frictionless component and a dynamic frictional component, where we characterize the latter, introducing the notion of effective investment frictions. From a firm's perspective, what matters for investment decisions is the fixed adjustment cost relative to frictionless profits and the price wedge relative to the frictionless profits-capital ratio, respectively. The first lemma uses the dynamic frictional component to characterize CIR's sufficient statistics. Then, we divide the proof into two major cases: $\nu \to 0$ and $\nu \to \infty$. Within the first case, we consider $\theta = 0$ and $\omega = 0$.

C.4.1 Static and dynamic investment policies

Lemma C.2. *Let $\mathcal{K} \equiv \{\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+\}$ denote the firm's optimal investment policy. Fix any investment price $\hat{p}$. The optimal investment policy can be decomposed as the sum of a static and a dynamic component $\mathcal{K} = \hat{k}^{ss} + \mathcal{X}$, where $\hat{k}^{ss}$ is the static log capital-productivity ratio that firms would set in the absence of frictions under the investment price $\hat{p}$*

$$\text{(C.115)} \qquad \hat{k}^{ss} = \frac{1}{1-\alpha}\log\left(\frac{\alpha}{\hat{p}\mathcal{U}}\right)$$

and $\mathcal{X} \equiv \{x^-, x^{*-}, x^{*+}, x^+\}$ is the dynamic component that solves the following stopping-time problem for the normalized capital-productivity ratio $x := \hat{k} - \hat{k}^{ss}$

$$\text{(C.116)} \qquad \tilde{q}(x) = \mathbb{E}\left[\int_0^\tau e^{-\mathcal{U}s}\left(e^{(\alpha-1)x_s} - 1\right)\mathrm{d}s + e^{-\mathcal{U}s}\left(\tilde{q}(x_\tau + \Delta x) - \tilde{p}(\Delta x)\right)\bigg| x_0 = x\right],$$

$$\text{(C.117)} \qquad \mathrm{d}x_t = -\nu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t,$$

with the additional restriction

$$\tilde{\theta} = \int_{x^-}^{x^{*-}} e^x \left(\tilde{q}(x) - \tilde{p}^{buy}\right) \mathrm{d}x, \tag{C.118}$$

$$\tilde{\theta} = \int_{x^{*+}}^{x^+} e^x \left(\tilde{p}^{sell} - \tilde{q}(x)\right) \mathrm{d}x. \tag{C.119}$$

The effective fixed cost $\tilde{\theta}$ and the effective price wedge $\tilde{\omega}$ are define as

$$\tilde{\theta} \;=\; \frac{\theta}{\alpha e^{\alpha \hat{k}^{ss}}} \;=\; \frac{\theta}{\alpha}\left(\frac{\hat{p}\mathcal{U}}{\alpha}\right)^{\frac{\alpha}{1-\alpha}} \tag{C.120}$$

$$\tilde{p}^{buy} \;=\; \frac{p - \hat{p}}{\alpha e^{(\alpha-1)\hat{k}^{ss}}} \;=\; \frac{p - \hat{p}}{\mathcal{U}\hat{p}} \tag{C.121}$$

$$\tilde{p}^{sell} \;=\; \frac{p(1-\omega) - \hat{p}}{\alpha e^{(\alpha-1)\hat{k}^{ss}}} \;=\; \frac{p(1-\omega) - \hat{p}}{\mathcal{U}\hat{p}} \tag{C.122}$$

The static optimal policy $\hat{k}^{ss}$ in (C.115) sets the capital-productivity ratio to a constant, and its value reflects profitability α, the average user cost of capital $\mathcal{U}$, and the investment price p. By definition, investment frictions do not affect the static choice $\hat{k}^{ss}$. In contrast, the dynamic policy $\mathcal{X}$ characterized by (C.116) and (C.117) takes into account the fixed cost and the price wedge, scaled by static profits or the profit-capital ratio, respectively. The flow payoff in the dynamic problem $e^{(\alpha-1)x_s} - 1$ only depends on the curvature of the profit function α, and thus is invariant to frictions. Finally, any price can be used to construct $\hat{k}^{ss}$, because $\mathcal{X}$ moves accordingly so that $\mathcal{K}$ is invariant to the price. We use this property below to obtain symmetry in the problem.

Proof. The equilibrium conditions for the Tobin's q are given by:

$$\mathcal{U}q(\hat{k}) = \frac{\alpha e^{(\alpha-1)\hat{k}}}{p} \;-\; \nu q'(\hat{k}) \;+\; \frac{\sigma^2}{2}q''(\hat{k}), \quad \forall \;\; \hat{k} \in (\hat{k}^-, \hat{k}^+). \tag{C.123}$$

$$\frac{\theta}{p} = \int_{\hat{k}^-}^{\hat{k}^{*-}} e^{\hat{k}} \left(q(\hat{k}) - 1\right) \mathrm{d}\hat{k}, \tag{C.124}$$

$$\frac{\theta}{p} = \int_{\hat{k}^{*+}}^{\hat{k}^+} e^{\hat{k}} \left((1-\omega) - q(\hat{k})\right) \mathrm{d}\hat{k}, \tag{C.125}$$

$$q(\hat{k}) = 1, \qquad \hat{k} \in \left\{\hat{k}^-, \hat{k}^{*-}\right\} \tag{C.126}$$

$$q(\hat{k}) = (1-\omega), \quad \hat{k} \in \left\{\hat{k}^{*+}, \hat{k}^+\right\}. \tag{C.127}$$

Normalized $q^\star$ Define the normalized Tobins' q as

$$q^\star(x) \equiv \frac{q(x + \hat{k}^{ss}) - \hat{p}/p}{\alpha e^{(\alpha-1)\hat{k}^{ss}}}, \tag{C.128}$$

which satisfies the following properties:

$$q(\hat{k}) \;=\; q^\star(\hat{k} - \hat{k}^{ss})\alpha e^{(\alpha-1)\hat{k}^{ss}} + \hat{p}/p \tag{C.129}$$

$$q'(\hat{k}) \;=\; {q^\star}'(\hat{k} - \hat{k}^{ss})\alpha e^{(\alpha-1)\hat{k}^{ss}} \tag{C.130}$$

$$q''(\hat{k}) \;=\; {q^\star}''(\hat{k} - \hat{k}^{ss})\alpha e^{(\alpha-1)\hat{k}^{ss}}. \tag{C.131}$$

From the HJB, we have that for all $\hat{k} \in (\hat{k}^-, \hat{k}^+)$

$$\text{(C.132)}\qquad \begin{aligned} \mathcal{U}q(\hat{k}) &= \frac{\alpha e^{(\alpha-1)\hat{k}}}{p} - \nu q'(\hat{k}) + \frac{\sigma^2}{2}q''(\hat{k}), \iff \\ \mathcal{U}(q^\star(\hat{k}-\hat{k}^{ss})\alpha e^{(\alpha-1)\hat{k}^{ss}} + \hat{p}/p) &= \frac{\alpha e^{(\alpha-1)\hat{k}}}{p} - \nu q^{\star\prime}(\hat{k}-\hat{k}^{ss})\alpha e^{(\alpha-1)\hat{k}^{ss}} + \frac{\sigma^2}{2}q^{\star\prime\prime}(\hat{k}-\hat{k}^{ss})\alpha e^{(\alpha-1)\hat{k}^{ss}}, \iff \\ \mathcal{U}q^\star(x) &= \frac{e^{(\alpha-1)x}-1}{p} - \nu q^{\star\prime}(x) + \frac{\sigma^2}{2}q^{\star\prime\prime}(x), \end{aligned}$$

where the last equation holds for all $x \in (x^-, x^+)$. From the optimality condition

$$\text{(C.133)}\qquad q(\hat{k}) = 1, \qquad \hat{k} \in \left\{\hat{k}^-, \hat{k}^{*-}\right\} \iff q^\star(x) = \frac{p-\hat{p}}{p\alpha e^{(\alpha-1)\hat{k}^{ss}}}, \qquad x \in \left\{x^-, x^{*-}\right\}$$

$$\text{(C.134)}\qquad q(\hat{k}) = (1-\omega), \quad \hat{k} \in \left\{\hat{k}^{*+}, \hat{k}^+\right\} \iff q^\star(x) = \frac{p(1-\omega)-\hat{p}}{p\alpha e^{(\alpha-1)\hat{k}^{ss}}}, \qquad x \in \left\{x^+, x^{*+}\right\}$$

From the value-matching condition with $\hat{k}^-$, we have that

$$\text{(C.135)}\qquad \begin{aligned} \frac{\theta}{p} &= \int_{\hat{k}^-}^{\hat{k}^{*-}} e^{\hat{k}}\left(q(\hat{k})-1\right)\mathrm{d}\hat{k} \iff \\ \frac{\theta}{p\alpha e^{(\alpha-1)\hat{k}^{ss}}} &= \int_{\hat{k}^-}^{\hat{k}^{*-}} e^{\hat{k}}\left(q^\star(\hat{k}-\hat{k}^{ss}) - \frac{p-\hat{p}}{p\alpha e^{(\alpha-1)\hat{k}^{ss}}}\right)\mathrm{d}\hat{k} \iff \\ \frac{\theta}{p\alpha e^{(\alpha-1)\hat{k}^{ss}}} &= \int_{\hat{k}^-}^{\hat{k}^{*-}} e^{\hat{k}-\hat{k}^{ss}+\hat{k}^{ss}}\left(q^\star(\hat{k}-\hat{k}^{ss}) - \frac{p-\hat{p}}{p\alpha e^{(\alpha-1)\hat{k}^{ss}}}\right)\mathrm{d}\hat{k} \iff \\ \frac{\theta}{p\alpha e^{\alpha\hat{k}^{ss}}} &= \int_{x^-}^{x^{*-}} e^{x}\left(q^\star(x) - \frac{p-\hat{p}}{p\alpha e^{(\alpha-1)\hat{k}^{ss}}}\right)\mathrm{d}x \iff \\ \frac{\theta}{\alpha e^{\alpha\hat{k}^{ss}}} &= \int_{x^-}^{x^{*-}} e^{x}\left(\tilde{q}(x) - \frac{p-\hat{p}}{\alpha e^{(\alpha-1)\hat{k}^{ss}}}\right)\mathrm{d}x. \end{aligned}$$

where we define $\tilde{q}(x) \equiv pq^\star(x)$,. Similar steps apply to the value matching condition for $\hat{k}^+$.

□

C.4.2 Proof for $\nu = 0$

Lemma C.3. *Let $\nu = 0$ and set $\hat{p} = p(1-\omega/2)$. Consider a first-order approximation of the flow profits*

$$\text{(C.136)}\qquad e^{(\alpha-1)x} - 1 \approx -(1-\alpha)x$$

and assume the unweighted boundary conditions are a good approximation of the weighted conditions:

$$\text{(C.137)}\qquad \int_{x^-}^{x^{*-}} e^x\left(\tilde{q}(x) - \tilde{p}^{buy}\right)\mathrm{d}x \approx \int_{x^-}^{x^{*-}}\left(\tilde{q}(x) - \tilde{p}^{buy}\right)\mathrm{d}x$$

$$\text{(C.138)}\qquad \int_{x^{*+}}^{x^{+}} e^x\left(\tilde{p}^{sell} - \tilde{q}(x)\right)\mathrm{d}x \approx \int_{x^{*+}}^{x^{+}}\left(\tilde{p}^{sell} - \tilde{q}(x)\right)\mathrm{d}x.$$

Then $\tilde{q}(x)$ is anti-symmetric, the policies are $x^- = -x^+$ and $x^- = -x^+$, and satisfy:

$$\mathcal{U}\tilde{q}(x) = -(1-\alpha)x \ + \ \frac{\sigma^2}{2}\tilde{q}''(x), \qquad \forall \ \ x \in (0, x^+), \tag{C.139}$$

$$\tilde{q}(x) = -\tilde{\omega}, \ x \in \left\{x^+, x^{*+}\right\}, \qquad \tilde{q}(0) = 0 \tag{C.140}$$

$$-\tilde{\theta} = \int_{x^{*+}}^{x^+} (\tilde{q}(x) + \tilde{\omega}) \, \mathrm{d}x. \tag{C.141}$$

Moreover, x^+ and x^{*+} satisfy the non-linear system of equations

$$\sqrt{\frac{2\mathcal{U}}{\sigma^2}} \frac{-\tilde{\theta}\mathcal{U} + (1-\alpha)\frac{(x^+)^2-(x^{*+})^2}{2} - \mathcal{U}\tilde{\omega}(x^+ - x^{*+})}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}}} (e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}}) - (1-\alpha)x^{*+} = -\omega\mathcal{U} \tag{C.142}$$

$$\sqrt{\frac{2\mathcal{U}}{\sigma^2}} \frac{-\tilde{\theta}\mathcal{U} + (1-\alpha)\frac{(x^+)^2-(x^{*+})^2}{2} - \mathcal{U}\tilde{\omega}(x^+ - x^{*+})}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}}} (e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}) - (1-\alpha)x^+ = -\omega\mathcal{U} \tag{C.143}$$

- *If $\tilde{\omega} = 0$, then*

$$x^{*+} = 0; \qquad x^+ = \left(\frac{12\tilde{\theta}\sigma^2}{1-\alpha}\right)^{1/4}. \tag{C.144}$$

- *If $\tilde{\theta} = 0$, then*

$$x^{*+} = x^+ = \left(\frac{3\tilde{\omega}\sigma^2}{2(1-\alpha)}\right)^{1/3}. \tag{C.145}$$

Proof. We show that $q(x) = -q(-x)$, $x^- = x^+$, and $x^{*-} = -x^{*+}$ using a guess and verify strategy in the equilibrium conditions.

Step 1: q is antisymmetric. Observe that if $q(x) = -q(-x)$, then $q''(x) = -q''(-x)$. Assume that for all $x \in (x^-, 0]$

$$\mathcal{U}\tilde{q}(x) = -(1-\alpha)x + \nu q'(x) \ + \ \frac{\sigma^2}{2}q''(x), \qquad \forall \ \ \hat{k} \in (x^-, x^+) \tag{C.146}$$

Multiplying by -1 both sizes of the equality

$$\mathcal{U}\tilde{q}(x) = -(1-\alpha)x \ + \ \frac{\sigma^2}{2}q''(x), \qquad \forall \ \ x \in (x^-, 0] \tag{C.147}$$

$$\mathcal{U}(-\tilde{q}(x)) = -(1-\alpha)(-x) \ + \ \frac{\sigma^2}{2}(-q''(x)), \qquad \forall \ \ x \in (x^-, 0] \tag{C.148}$$

$$\mathcal{U}(\tilde{q}(-x)) = -(1-\alpha)(-x) \ + \ \frac{\sigma^2}{2}(q''(-x)), \qquad \forall \ \ x \in (x^-, 0] \tag{C.149}$$

$$\mathcal{U}\tilde{q}(x) = -(1-\alpha)x \ + \ \frac{\sigma^2}{2}q''(x), \qquad \forall \ \ x \in [0, x^+). \tag{C.150}$$

Observe that under $\hat{p} = p(1-\omega)$

$$\tilde{p}^{buy} = \frac{p - p(1-\omega/2)}{\mathcal{U}p(1-\omega/2)} = \frac{\omega/2}{\mathcal{U}(1-\omega/2)} =: \tilde{\omega} \tag{C.151}$$

$$\tilde{p}^{sell} = \frac{p(1-\omega) - \hat{p}}{\mathcal{U}\hat{p}} = \frac{\omega/2}{\mathcal{U}(1-\omega/2)} =: -\tilde{\omega} \tag{C.152}$$

Then

$$\tilde{q}(x) = \tilde{\omega},\ x \in \left\{x^-, x^{*-}\right\} \iff -\tilde{q}(x) = -\tilde{\omega},\ x \in \left\{x^-, x^{*-}\right\} \iff \tilde{q}(x) = -\tilde{\omega},\ x \in \left\{x^+, x^{*+}\right\}. \tag{C.153}$$

Finally, using the unweighted boundary condition, using a change of variables $s = -x$ and $\mathrm{d}x = -\,\mathrm{d}s$

$$\tilde{\theta} = \int_{x^-}^{x^{*-}} (\tilde{q}(x) - \tilde{\omega})\,\mathrm{d}x \iff -\tilde{\theta} = \int_{x^-}^{x^{*-}} (-\tilde{q}(x) + \tilde{\omega})\,\mathrm{d}x \iff -\tilde{\theta} = \int_{x^-}^{x^{*-}} (\tilde{q}(-x) + \tilde{\omega})\,\mathrm{d}x \iff \tag{C.154}$$

$$-\tilde{\theta} = -\int_{-x^-}^{-x^{*-}} (\tilde{q}(s) + \tilde{\omega})\,\mathrm{d}s \iff -\tilde{\theta} = \int_{-x^{*+}}^{x^+} (\tilde{q}(s) + \tilde{\omega})\,\mathrm{d}s. \tag{C.155}$$

Thus, we have shown that $\tilde{q}(x)$ is anti-symmetric.

Step 2: q equilibrium conditions. Since $\tilde{q}(x) = -\tilde{q}(-x)$, we have that $\tilde{q}(0) = -\tilde{q}(0)$ if and only if $\tilde{q}(0)$. Thus, the equilibrium conditions for $\tilde{q}(x)$ and $\{x^{*+}, x^+\}$ are given by

$$\mathcal{U}\tilde{q}(x) = -(1-\alpha)x \;+\; \frac{\sigma^2}{2}\tilde{q}''(x), \qquad \forall\;\; x \in (0, x^+), \tag{C.156}$$

$$\tilde{q}(x) = -\tilde{\omega},\ x \in \left\{x^+, x^{*+}\right\}, \qquad \tilde{q}(0) = 0 \tag{C.157}$$

$$-\tilde{\theta} = \int_{x^{*+}}^{x^+} (\tilde{q}(x) + \tilde{\omega})\,\mathrm{d}x. \tag{C.158}$$

The solution to the HJB in (C.156) is given by

$$\tilde{q}(x) = \frac{Ae^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x} + Be^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x} - (1-\alpha)x}{\mathcal{U}} \tag{C.159}$$

Since $\tilde{q}(0) = 0$, we find that $A = -B$ and thus

$$\tilde{q}(x) = \frac{A(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x}) - (1-\alpha)x}{\mathcal{U}} \tag{C.160}$$

To find A, we use the border condition (C.158)

$$\begin{aligned}
-\tilde{\theta} &= \int_{x^{*+}}^{x^+} \left(\frac{A}{\mathcal{U}}(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x}) - \frac{(1-\alpha)x}{\mathcal{U}} + \tilde{\omega}\right)\mathrm{d}x \\
&= \frac{A}{\mathcal{U}}\left(\sqrt{\frac{2\mathcal{U}}{\sigma^2}}\right)^{-1}\left(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x}\right)\Bigg|_{x=x^{*+}}^{x^+} - (1-\alpha)\frac{(x^+)^2 - (x^{*+})^2}{2\mathcal{U}} + \tilde{\omega}(x^+ - x^{*+}) \\
&= \frac{A}{\mathcal{U}}\left(\sqrt{\frac{2\mathcal{U}}{\sigma^2}}\right)^{-1}\left(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}}\right) - (1-\alpha)\frac{(x^+)^2 - (x^{*+})^2}{2\mathcal{U}} + \tilde{\omega}(x^+ - x^{*+})
\end{aligned}$$

Solving for A we get:

$$A = \sqrt{\frac{2\mathcal{U}}{\sigma^2}} \frac{-\tilde{\theta}\mathcal{U} + \frac{1-\alpha}{2}\left((x^+)^2 - (x^{*+})^2\right) - \tilde{\omega}\mathcal{U}(x^+ - x^{*+})}{\left(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}}.\right)} \tag{C.161}$$

The equilibrium policy satisfies the following system of equations

$$\sqrt{\frac{2\mathcal{U}}{\sigma^2}} \frac{-\tilde{\theta}\mathcal{U} + (1-\alpha)\frac{(x^+)^2-(x^{*+})^2}{2} - \mathcal{U}\tilde{\omega}(x^+ - x^{*+})}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}}}\left(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}}\right) - (1-\alpha)x^{*+} = -\omega\mathcal{U} \tag{C.162}$$

$$\sqrt{\frac{2\mathcal{U}}{\sigma^2}} \frac{-\tilde{\theta}\mathcal{U} + (1-\alpha)\frac{(x^+)^2-(x^{*+})^2}{2} - \mathcal{U}\tilde{\omega}(x^+ - x^{*+})}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}}}\left(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}\right) - (1-\alpha)x^+ = -\omega\mathcal{U} \tag{C.163}$$

C.4.3 Proof for $\omega = 0$

If $\tilde{\omega} = 0$, then $x^{*+} = 0$ we have that

$$\sqrt{\frac{2\mathcal{U}}{\sigma^2}}\left((1-\alpha)\frac{(x^+)^2}{2} - \tilde{\theta}\mathcal{U}\right)\frac{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - 2} - (1-\alpha)x^+ = 0 \tag{C.164}$$

We can operate over the previous equation, and we have

$$(1-\alpha)x^+ \underbrace{\left(-1 + \frac{x^+}{2}\sqrt{\frac{2\mathcal{U}}{\sigma^2}}\frac{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - 2}\right)}_{=(2)} = \sqrt{\frac{2\mathcal{U}}{\sigma^2}}\tilde{\theta}\mathcal{U}\underbrace{\frac{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - 2}}_{=(1)} \tag{C.165}$$

First, we approximate the term (1), $e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} / \left(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - 2\right)$ for low value of x^+. Observe that when $x^+ \downarrow 0$, $e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} \downarrow 0$ and $\left(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - 2\right) \downarrow 0$. Thus, we use a Taylor approximation to approximate the ratio—keeping the lowest order to determine the sign of the denominator and numerator. For the denominator and numerator, we have that

$$e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} = \underbrace{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}0} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}0}}_{=0} + \underbrace{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}\left(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}0} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}0}\right)(x^+ - 0)}_{=\sqrt{\frac{2\mathcal{U}}{\sigma^2}}2x^+}$$

$$\begin{aligned} e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - 2 &= \underbrace{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}0} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}0} - 2}_{=0} + \underbrace{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}\left(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}0} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}0}\right)(x^+ - 0)}_{=0x^+} \\ &+ \frac{1}{2}\underbrace{\left(\frac{2\mathcal{U}}{\sigma^2}\right)\left(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}0} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}0}\right)(x^+ - 0)^2}_{=\frac{2\mathcal{U}}{\sigma^2}(x^+)^2} \end{aligned}$$

Using this approximation

$$\sqrt{\frac{2\mathcal{U}}{\sigma^2}}\tilde{\theta}\mathcal{U}\frac{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - 2} \approx \sqrt{\frac{2\mathcal{U}}{\sigma^2}}\tilde{\theta}\mathcal{U}\frac{2}{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} = \frac{2\tilde{\theta}\mathcal{U}}{x^+} \tag{C.166}$$

Now, we approximate the term (2),

$$(2) = -1 + \frac{x^+}{2}\sqrt{\frac{2\mathcal{U}}{\sigma^2}}\frac{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - 2} \tag{C.167}$$

for low value of x^+. Doing a third-order Taylor approximation over x near 0 in the numerator and a second in the denominator the denominator

$$(2) = -1 + \frac{x^+}{2}\sqrt{\frac{2\mathcal{U}}{\sigma^2}}\frac{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - 2}, \tag{C.168}$$

$$= -1 + \frac{x^+}{2}\sqrt{\frac{2\mathcal{U}}{\sigma^2}}\frac{2\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+ + \frac{2}{3!}\left(\frac{2\mathcal{U}}{\sigma^2}\right)^{3/2}(x^+)^3}{\frac{2\mathcal{U}}{\sigma^2}(x^+)^2}, \tag{C.169}$$

$$= -1 + 1 + \left(\frac{2\mathcal{U}}{\sigma^2}\right)\frac{(x^+)^2}{3!}. \tag{C.170}$$

Using this approximation

$$(1-\alpha)x^+\left(-1 + \frac{x^+}{2}\sqrt{\frac{2\mathcal{U}}{\sigma^2}}\frac{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - 2}\right) = (1-\alpha)\left(\frac{2\mathcal{U}}{\sigma^2}\right)\frac{(x^+)^3}{3!}. \tag{C.171}$$

Thus,

$$\frac{2\tilde{\theta}\mathcal{U}}{x^+} = (1-\alpha)\left(\frac{2\mathcal{U}}{\sigma^2}\right)\frac{(x^+)^3}{3!} \iff x^+ = \left(\frac{12\tilde{\theta}\sigma^2}{1-\alpha}\right)^{1/4}. \tag{C.172}$$

Since $\omega = 0$, we have that $\tilde{\theta} = \frac{\theta}{\alpha}\left(\frac{p\mathcal{U}}{\alpha}\right)^{\frac{\alpha}{1-\alpha}} = \frac{\theta}{\alpha}\left(\frac{p(1-\omega/2)\mathcal{U}}{\alpha}\right)^{\frac{\alpha}{1-\alpha}} = \frac{\theta}{\alpha}\left(\frac{p\mathcal{U}}{\alpha}\right)^{\frac{\alpha}{1-\alpha}}$ and

$$x^+ = \left(\frac{12\tilde{\theta}\sigma^2}{1-\alpha}\right)^{1/4}, \text{ with } \tilde{\theta} = \frac{\theta}{\alpha}\left(\frac{p\mathcal{U}}{\alpha}\right)^{\frac{\alpha}{1-\alpha}} \tag{C.173}$$

C.4.4 Proof for $\theta = 0$

Since $q(x) = -\tilde{\omega}$ has two roots for $x \geq 0$ with $q(0) = 0$, it is easy to see that when $\tilde{\theta} \downarrow 0$, then $x^{*+} \to x^+$ with $q'(x^+) = 0$. Thus, we replace $-\tilde{\theta} = \int_{x^{*+}}^{x^+}(\tilde{q}(x) + \tilde{\omega})\,\mathrm{d}x$, by the reflecting barrier condition $q'(x^+) = 0$.

$$\mathcal{U}\tilde{q}(x) = -(1-\alpha)x + \frac{\sigma^2}{2}\tilde{q}''(x), \quad \forall\ x \in (0, x^+), \tag{C.174}$$

$$\tilde{q}(x^+) = -\tilde{\omega}, \qquad \tilde{q}(0) = 0, \qquad \tilde{q}'(x^+) = 0. \tag{C.175}$$

Given the solution

$$\tilde{q}(x) = \frac{A(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x}) - (1-\alpha)x}{\mathcal{U}} \tag{C.176}$$

The border condition $\tilde{q}'(x^+) = 0$ implies

$$\sqrt{\frac{2\mathcal{U}}{\sigma^2}}A(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}) - (1-\alpha) = 0 \iff A = \sqrt{\frac{\sigma^2}{2\mathcal{U}}}\frac{1-\alpha}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}} \tag{C.177}$$

Thus, x^+ satisfies

$$(1-\alpha)\left(\sqrt{\frac{\sigma^2}{2\mathcal{U}}}\frac{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}-e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}+e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}-x^+\right)=-\tilde{w}\mathcal{U} \tag{C.178}$$

Applying similar steps as before

$$\sqrt{\frac{\sigma^2}{2\mathcal{U}}}\frac{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}-e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}+e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}-x^+ \tag{C.179}$$

$$\approx\sqrt{\frac{\sigma^2}{2\mathcal{U}}}\frac{2\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^++\frac{2}{3!}\left(\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+\right)^3}{2+\frac{2\mathcal{U}}{\sigma^2}(x^+)^2}-x^+ \tag{C.180}$$

$$=\frac{2x^++\frac{1}{3}\left(\frac{2\mathcal{U}}{\sigma^2}\right)(x^+)^3-2x^+-\frac{2\mathcal{U}}{\sigma^2}(x^+)^3}{2+\frac{2\mathcal{U}}{\sigma^2}(x^+)^2} \tag{C.181}$$

$$=\frac{2\mathcal{U}}{2\sigma^2}(x^+)^3\frac{1/3-1}{1+\frac{\mathcal{U}}{\sigma^2}(x^+)^2} \tag{C.182}$$

$$=-\frac{\mathcal{U}}{\sigma^2}(x^+)^3\frac{2}{3} \tag{C.183}$$

Therefore, we get the result:

$$x^+=\left(\frac{3\tilde{\omega}\sigma^2}{2(1-\alpha)}\right)^{1/3}, \text{ with } \tilde{w}=\frac{\omega/2}{\mathcal{U}(1-\omega/2)} \tag{C.184}$$

□

C.4.5 Sufficient statistics

We finish the proof with the investment statistics and the CIR′(0) components. To simplify the exposition, we work in the space of normalized capital-productivity ratios $x \equiv \hat{k}-\hat{k}^{ss}$

Lemma C.4. *The variance is given by*

$$\mathbb{V}ar[x]=\frac{(\bar{x}^+)^2+(x^{*+})^2}{6}. \tag{C.185}$$

The covariance is given by

$$\mathbb{C}ov[a,x]=0. \tag{C.186}$$

The irreversibility term is given by

$$\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s[\mathrm{d}(\mathcal{M}(x_s)x_s)]\right]=\frac{x^{*+}x^+}{3}. \tag{C.187}$$

The CIR is given by

$$\frac{CIR(\delta)}{\delta} = \frac{(\bar{x}^+)^2+(x^{*+})^2}{6\sigma^2} + \frac{x^{*+}x^+}{3\sigma^2} + o(\delta). \tag{C.188}$$

If $\theta = 0$, *then*

$$\frac{CIR(\delta)}{\delta} = 2\frac{\mathbb{V}ar[x]}{\sigma^2} = \left(\frac{12\tilde{\omega}}{(1-\alpha)\sigma^4}\right)^{1/3} \tag{C.189}$$

If $\omega = 0$, *then*

$$\frac{CIR(\delta)}{\delta} = \frac{\mathbb{V}ar[x]}{\sigma^2} = \left(\frac{12\tilde{\theta}}{\sigma^6(1-\alpha)}\right)^{1/4} \tag{C.190}$$

Proof. We exploit symmetry in x to compute all the inputs for the CIR.

Cross-sectional distribution. The stationary density $g(x)$ solves the KFE with border, continuity, and reinjection (exit mass equals entry mass) conditions:

$$0 = \frac{\sigma^2}{2} g''(x), \tag{C.191}$$

$$g(\overline{x}) = g(-\overline{x}) = 0, \tag{C.192}$$

$$\int_{-\overline{x}}^{\overline{x}} g(x)\,\mathrm{d}x = 1, \tag{C.193}$$

$$\lim_{x \downarrow -x^*} g(x) = \lim_{x \uparrow -x^*} g(x), \qquad \lim_{x \downarrow x^*} g(x) = \lim_{x \uparrow x^*} g(x), \tag{C.194}$$

$$\lim_{x \downarrow -\overline{x}} g'(x) = \lim_{x \uparrow -x^*} g'(x) - \lim_{x \downarrow -x^*} g'(x), \qquad \lim_{x \uparrow \overline{x}} g'(x) = \lim_{x \downarrow x^*} g'(x) - \lim_{x \uparrow x^*} g'(x). \tag{C.195}$$

Solving for $g(x)$, we obtain a linear function:

$$g''(x) = 0, \quad g'(x) = A, \quad g(x) = Ax + B. \tag{C.196}$$

We split the state-space into three segments $[-\overline{x}, -x^*] \cup [-x^*, x^*] \cup [x^*, \overline{x}]$ and consider three different functions $g_k(x) = A_k x + B_k$ for $j = 1, 2, 3$, one for each segment. Evaluating at the border conditions, we obtain relationships for (A_1, B_1) and (A_3, B_3):

$$\left.\begin{aligned} -A_1\overline{x} + B_1 &= 0 \\ A_3\overline{x} + B_3 &= 0 \end{aligned}\right\} \implies \overline{x} = B_1/A_1 = -B_3/A_3. \tag{C.197}$$

Evaluating at the reinjection conditions, we obtain A_2:

$$\left.\begin{aligned} A_1 &= A_1 - A_2 \\ A_3 &= A_3 - A_2 \end{aligned}\right\} \implies A_2 = 0. \tag{C.198}$$

Evaluating at the continuity conditions, using $A_2 = 0$ we obtain for (A_1, B_1) and (A_3, B_3):

$$\left.\begin{aligned} B_2 &= -A_1 x^* + B_1 \\ B_2 &= A_3 x^* + B_3 \end{aligned}\right\} \implies x^* = \frac{B_1 - B_2}{A_1} = \frac{B_2 - B_3}{A_3}. \tag{C.199}$$

Finally, we use the fact that the density integrates to one:

$$
\begin{aligned}
1 &= \int_{-\overline{x}}^{-x^*} (A_1 x + B_1)\,\mathrm{d}x + \int_{-x^*}^{x^*} B_2\,\mathrm{d}x + \int_{x^*}^{\overline{x}} (A_3 x + B_3)\,\mathrm{d}x \\
&= \left(A_1 \frac{x^2}{2} + B_1 x\right)\Big|_{-\overline{x}}^{-x^*} + B_2 x\Big|_{-x^*}^{x^*} + \left(A_3 \frac{x^2}{2} + B_3 x\right)\Big|_{x^*}^{\overline{x}} \\
&= A_1 \left(\frac{x^{*2} - \overline{x}^2}{2}\right) + B_1 (\overline{x} - x^*) + 2B_2 x^* + A_3 \left(\frac{\overline{x}^2 - x^{*2}}{2}\right) + B_3(\overline{x} - x^*) \\
&= (A_3 - A_1) \left(\frac{\overline{x}^2 - x^{*2}}{2}\right) + 2B_2 x^* + (B_1 + B_3)(\overline{x} - x^*).
\end{aligned} \tag{C.200}
$$

Substituting $B_1 = \overline{x}A_1$ and $B_3 = -\overline{x}A_3$ from (C.197) into the previous expression:

$$
\begin{aligned}
1 &= (A_3 - A_1) \left(\frac{\overline{x}^2 - x^{*2}}{2}\right) + 2B_2 x^* - \overline{x}\,(A_3 - A_1)\,(\overline{x} - x^*) \\
&= (A_3 - A_1)\,(\overline{x} - x^*) \left[\frac{\overline{x} + x^*}{2} - \overline{x}\right] + 2B_2 x^* \\
&= (A_3 - A_1)\,\frac{(\overline{x} - x^*)^2}{2} + 2B_2 x^*.
\end{aligned} \tag{C.201}
$$

Therefore, the cross-sectional density is equal to:

$$
g(x) = \frac{1}{\overline{x}^2 - x^{*2}} \begin{cases} \overline{x} + x & \text{for} \quad x \in [-\overline{x}, -x^*] \\ \overline{x} - x^* & \text{for} \quad x \in [-x^*, x^*] \\ \overline{x} - x & \text{for} \quad x \in [x^*, \overline{x}]. \end{cases} \tag{C.202}
$$

Renewal probabilities and relative shares. The renewal probabilities (the mass of adjusters from each reset point) are equal to:

$$
\mathcal{N}^- = \frac{\sigma^2}{2} \lim_{x \downarrow -\overline{x}} g'(x) = \frac{\sigma^2}{2} A_1 = \frac{\sigma^2}{2} \frac{1}{(\overline{x}^2 - x^{*2})} \tag{C.203}
$$

$$
\mathcal{N}^+ = -\frac{\sigma^2}{2} \lim_{x \uparrow \overline{x}} g'(x) = -\frac{\sigma^2}{2} A_3 = \frac{\sigma^2}{2} \frac{1}{(\overline{x}^2 - x^{*2})}. \tag{C.204}
$$

The shares of total, upward, and downward adjustment are:

$$
\mathcal{N} = \mathcal{N}^- + \mathcal{N}^+ = \frac{\sigma^2}{(\overline{x}^2 - x^{*2})} \tag{C.205}
$$

$$
\frac{\mathcal{N}^-}{\mathcal{N}} = \frac{1}{2}; \qquad \frac{\mathcal{N}^+}{\mathcal{N}} = \frac{1}{2}. \tag{C.206}
$$

Probability of negative adjustment. Let $\mathbb{P}^+(x) \equiv \Pr[\Delta x < 0 | x]$ denote the probability of doing a negative adjustment (after hitting the upper bound) conditional on the state x. It solves the HJB with border conditions:

$$
0 = \mathbb{P}^{+\prime\prime}(x); \quad \mathbb{P}^+(\overline{x}) = 1; \quad \mathbb{P}^+(-\overline{x}) = 0 \tag{C.207}
$$

Solving for $\mathbb{P}^+(x) = Ax + B$ and evaluating at the border conditions:

$$
\left.\begin{aligned} A\overline{x} + B = 1 \\ -A\overline{x} + B = 0 \end{aligned}\right\} \implies \left.\begin{aligned} A = 1/2\overline{x} \\ B = 1/2 \end{aligned}\right\} \implies \mathbb{P}^+(x) = \frac{\overline{x} + x}{2\overline{x}} = \frac{1}{2} + \frac{x}{2\overline{x}}. \tag{C.208}
$$

The unconditional probability of a negative adjustment is:

$$\mathbb{E}[\mathbb{P}^{+}] \;=\; \frac{1}{2}+\frac{1}{2\overline{x}}\mathbb{E}[x] \;=\; \frac{1}{2}. \tag{C.209}$$

The probability of a negative adjustment conditional on the last adjusting being positive (a switch in adjustment sign) equals:

$$\mathbb{P}^{+}(-x^{*}) \equiv \Pr[\Delta x<0|-x^{*}] \;=\; \frac{\overline{x}-x^{*}}{2\overline{x}}. \tag{C.210}$$

Probability of positive adjustment. Let $\mathbb{P}^{-}(x) \equiv \Pr[\Delta x > 0|x]$ denote the probability of doing a positive adjustment (after hitting the lower bound) conditional on the state x. It solves the HJB with border conditions:

$$0=\mathbb{P}^{-\prime\prime}(x); \quad \mathbb{P}^{-}(-\overline{x})=1; \quad \mathbb{P}^{-}(\overline{x})=0. \tag{C.211}$$

Solving for $\mathbb{P}^{-}(x)=Ax+B$ and evaluating at the border conditions:

$$\left.\begin{array}{r}-A\overline{x}+B=1\\ A\overline{x}+B=0\end{array}\right\} \quad \left.\begin{array}{r}A=-1/2\overline{x}\\ B=1/2\end{array}\right\} \quad \mathbb{P}^{-}(x)=\frac{\overline{x}-x}{2\overline{x}} \;=\; \frac{1}{2}-\frac{x}{2\overline{x}}. \tag{C.212}$$

The unconditional probability of a positive adjustment is:

$$\mathbb{E}[\mathbb{P}^{-}] \;=\; \frac{1}{2}-\frac{1}{2\overline{x}}\mathbb{E}[x] \;=\; \frac{1}{2}. \tag{C.213}$$

The probability of a positive adjustment conditional on the last adjusting being negative (a switch in adjustment sign) equals:

$$\mathbb{P}^{-}(x^{*}) \equiv \Pr[\Delta x>0|x^{*}] \;=\; \frac{\overline{x}-x^{*}}{2\overline{x}}. \tag{C.214}$$

Expected duration of inaction. Let $T(x) \equiv \mathbb{E}[\tau|x]$. It solves the HJB with border conditions:

$$0 \;=\; 1+\frac{\sigma^2}{2}T''(x), \qquad T(\overline{x}) \;=\; T(-\overline{x}) \;=\; 0. \tag{C.215}$$

Solving for $T(x)$:

$$T''(x) \;=\; -\frac{2}{\sigma^2}, \quad T'(x) \;=\; -\frac{2}{\sigma^2}x+A, \quad T(x) \;=\; -\frac{x^2}{\sigma^2}+Ax+B. \tag{C.216}$$

Evaluating at the border conditions, we obtain values for A and B:

$$\left.\begin{array}{r}-\dfrac{\overline{x}^2}{\sigma^2}+A\overline{x}+B=0\\ -\dfrac{\overline{x}^2}{\sigma^2}-A\overline{x}+B=0\end{array}\right\} \implies \left.\begin{array}{r}2A\overline{x}=0\\ -\dfrac{2\overline{x}^2}{\sigma^2}+2B=0\end{array}\right\} \implies \left.\begin{array}{r}A=0\\ B=\dfrac{\overline{x}^2}{\sigma^2}\end{array}\right\} \implies T(x)=\frac{\overline{x}^2-x^2}{\sigma^2}. \tag{C.217}$$

The expected duration of inaction given the current state $\mathbb{E}[\tau|x]$, the expected duration of a complete inaction spell conditional on the last reset point $(\overline{\mathbb{E}}^{+}[\tau], \overline{\mathbb{E}}^{-}[\tau])$, and the unconditional expected duration of inaction $\overline{\mathbb{E}}[\tau]$

are given by:

$$\mathbb{E}[\tau|x] = \frac{\overline{x}^2 - x^2}{\sigma^2}, \tag{C.218}$$

$$\overline{\mathbb{E}}^+[\tau] = \overline{\mathbb{E}}^-[\tau] = \frac{\overline{x}^2 - x^{*2}}{\sigma^2}, \tag{C.219}$$

$$\overline{\mathbb{E}}[\tau] = \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\tau] + \frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\tau] = \frac{\overline{x}^2 - x^{*2}}{\sigma^2}, \tag{C.220}$$

where the shares of upward and downward adjustment are identical: $\mathcal{N}^+/\mathcal{N} = \mathcal{N}^-/\mathcal{N} = 1/2$.

Cross-sectional means Let $m(x) \equiv \mathbb{E}\left[\int_0^\tau x_s \, \mathrm{d}s | x_0 = x\right]$. It solves the HJB with border conditions:

$$0 = x + \frac{\sigma^2}{2} m''(x), \quad m(\overline{x}) = m(-\overline{x}) = 0. \tag{C.221}$$

Solving for $m(x)$:

$$m''(x) = -\frac{2}{\sigma^2}x, \qquad m'(x) = -\frac{x^2}{\sigma^2} + A, \qquad m(x) = -\frac{x^3}{3\sigma^2} + Ax + B. \tag{C.222}$$

Evaluating at the border conditions, we obtain values for A and B:

$$\left.\begin{aligned} -\frac{\overline{x}^3}{3\sigma^2} + A\overline{x} + B = 0 \\ \frac{\overline{x}^3}{3\sigma^2} - A\overline{x} + B = 0 \end{aligned}\right\} \implies \left.\begin{aligned} A = \frac{\overline{x}^2}{3\sigma^2} \\ B = 0 \end{aligned}\right\} \implies m(x) = \frac{\overline{x}^2 x - x^3}{3\sigma^2} = \frac{x}{3}\frac{\overline{x}^2 - x^2}{\sigma^2} = \frac{x}{3}\mathbb{E}[\tau|x] \tag{C.223}$$

Unconditional means. Using the occupancy measure, we obtain the means conditional on the last rest point:

$$\overline{\mathbb{E}}^-[x] = \frac{m(-x^*)}{\overline{\mathbb{E}}^-[\tau]} = -\frac{x^*}{3}; \qquad \overline{\mathbb{E}}^+[x] = \frac{m(x^*)}{\overline{\mathbb{E}}^+[\tau]} = \frac{x^*}{3}, \tag{C.224}$$

where $\overline{\mathbb{E}}^-[\tau] = \mathbb{E}[\tau| - x^*]$ and $\overline{\mathbb{E}}^+[\tau] = \mathbb{E}[\tau|x^*]$.

Conditional mean. By symmetry, $\mathbb{E}[x] = 0$. To show this formally, we use the conditional means and the renewal distribution:

$$\mathbb{E}[x] = \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[x] + \frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[x] = \frac{1}{2}\left(\frac{x^*}{3}\right) + \frac{1}{2}\left(\frac{-x^*}{3}\right) = 0. \tag{C.225}$$

Unconditional variance. Since $\mathbb{E}[x] = 0$, then $\mathbb{V}ar[x] = \mathbb{E}[x^2]$. Using the cross-sectional distribution, the second moment equals:

$$\text{(C.226)} \qquad \mathbb{V}ar[x] = \int_{-\overline{x}}^{\overline{x}} x^2 g(x)\,\mathrm{d}x$$

$$= \frac{1}{(\overline{x}^2 - x^{*2})}\left[\int_{-\overline{x}}^{-x^*} x^2(\overline{x}+x)\,\mathrm{d}x \;+\; (\overline{x}-x^*)\int_{-x^*}^{x^*} x^2\,\mathrm{d}x \;+\; \int_{x^*}^{\overline{x}} x^2(\overline{x}-x)\,\mathrm{d}x\right]$$

$$= \frac{1}{(\overline{x}^2 - x^{*2})}\left[\left(\frac{x^3\overline{x}}{3}+\frac{x^4}{4}\right)\Big|_{-\overline{x}}^{-x^*} \;+\; (\overline{x}-x^*)\frac{x^3}{3}\Big|_{-x^*}^{x^*} \;+\; \left(\frac{x^3\overline{x}}{3}-\frac{x^4}{4}\right)\Big|_{x^*}^{\overline{x}}\right]$$

$$= \frac{1}{(\overline{x}^2 - x^{*2})}\left[\frac{-x^{*3}\overline{x}}{3}+\frac{x^{*4}}{4}+\frac{\overline{x}^{*4}}{3}-\frac{\overline{x}^4}{4}+(\overline{x}-x^*)\frac{x^{*3}+x^{*3}}{3}+\frac{\overline{x}^4}{3}-\frac{\overline{x}^4}{4}-\frac{x^{*3}\overline{x}}{3}+\frac{x^{*4}}{4}\right]$$

$$= \frac{1}{(\overline{x}^2 - x^{*2})}\left(\frac{\overline{x}^4 - x^{*4}}{6}\right) = \frac{1}{(\overline{x}^2 - x^{*2})}\left(\frac{(\overline{x}^2 - x^{*2})(\overline{x}^2 + x^{*2})}{6}\right)$$

$$= \frac{\overline{x}^2 + x^{*2}}{6}$$

CIR. From (51) and (62), the CIR without drift equals:

$$\text{(C.227)} \qquad \frac{\text{CIR}(\delta)}{\delta} = \frac{\mathbb{V}ar[x]}{\sigma^2} - \frac{\overline{\mathbb{C}ov}[\Delta\hat{k}, \mathcal{M}(\Delta\hat{k})]}{\sigma^2\overline{\mathbb{E}}[\tau]} + o(\delta).$$

Cumulative deviations. Recall the values for the unconditional probabilities of a negative and a positive adjustment $\mathbb{E}[\mathbb{P}^+] = \mathbb{E}[\mathbb{P}^-] = 1/2$ in (C.209) and (C.213), and the conditional probabilities of switching adjustment sign $\mathbb{P}^+(-x^*) = \mathbb{P}^-(x^*) = (\overline{x} - x^*)/2\overline{x}$ in (C.210) and (C.214). Substituting these probabilities, the conditional means $\mathbb{E}^-[x] = -x^*/3$ and $\mathbb{E}^-[x] = x^*/3$ in (C.224), and the conditional durations $\mathbb{E}^-[\tau] = \mathbb{E}^+[\tau] = (\overline{x}^2 - x^{*2})/\sigma^2$ in (C.219) into the definition of cumulative deviations $m(\hat{k}^{*-})$ and $m(\hat{k}^{*+})$ yields:

$$\text{(C.228)} \quad m(\hat{k}^{*-}) = \mathbb{E}[\mathbb{P}^-]\frac{1}{\mathbb{P}^+(-x^*)}(\mathbb{E}^-[x] - \mathbb{E}[x])\mathbb{E}^-[\tau] = \frac{1}{2}\left(\frac{2\overline{x}}{\overline{x}-x^*}\right)\left(-\frac{x^*}{3}-0\right)\frac{\overline{x}^2 - x^{*2}}{\sigma^2} = -\frac{x^*\overline{x}(\overline{x}+x^*)}{3\sigma^2},$$

$$\text{(C.229)} \quad m(\hat{k}^{*+}) = \mathbb{E}[\mathbb{P}^+]\frac{1}{\mathbb{P}^-(x^*)}(\mathbb{E}^+[x] - \mathbb{E}[x])\mathbb{E}^+[\tau] = \frac{1}{2}\left(\frac{2\overline{x}}{\overline{x}-x^*}\right)\left(\frac{x^*}{3}-0\right)\frac{\overline{x}^2 - x^{*2}}{\sigma^2} = \frac{x^*\overline{x}(\overline{x}+x^*)}{3\sigma^2}.$$

Irreversibility term. The irreversibility term for the CIR equals the covariance of the adjustment size and the auxiliary capital-deviation deviation function $\mathcal{M}(\Delta x)$ defined in (**??**). Recall $x^* = \overline{x} - \Delta x$ for $\Delta x < 0$ and $-x^* = -\overline{x} - \Delta x$ for $\Delta x > 0$ and by symmetry $\overline{\mathbb{E}}[\Delta x] = 0$. The numerator of the irreversibility term equals:

$$\begin{aligned}
\overline{\mathbb{C}ov}[\Delta x, \mathcal{M}(\Delta x)] &= \overline{\mathbb{E}}[\Delta x \mathcal{M}(\Delta x)] - \overline{\mathbb{E}}[\Delta x]\,\overline{\mathbb{E}}[\mathcal{M}(\Delta x)] \\
&= \frac{1}{2}\left[\overline{\mathbb{E}}^-[\Delta x \mathcal{M}(\Delta x)] + \overline{\mathbb{E}}^+[\Delta x \mathcal{M}(\Delta x)]\right] \\
&= \frac{1}{2}\left[(\overline{x}-x^*)m(\hat{k}^{*-}) + (x^*-\overline{x})m(\hat{k}^{*+})\right] = (\overline{x}-x^*)m(\hat{k}^{*-}), \\
&= -(\overline{x}-x^*)\frac{x^*\overline{x}(\overline{x}+x^*)}{3\sigma^2} = -\frac{x^*\overline{x}}{3}\left(\frac{\overline{x}^2 - x^{*2}}{\sigma^2}\right) \\
&= -\frac{x^*\overline{x}}{3}\overline{\mathbb{E}}[\tau].
\end{aligned}$$

Therefore, the irreversibility term of the CIR equals:

$$\text{(C.230)} \qquad -\frac{\overline{\mathbb{C}ov}[\Delta x, \mathcal{M}(\Delta x)]}{\overline{\mathbb{E}}[\tau]} = \frac{x^*\overline{x}}{3} > 0.$$

Finally, substituting the expression for the cross-sectional variance in (C.226) and the irreversibility term in (C.230) into the CIR yields:

$$\frac{\text{CIR}(\delta)}{\delta} = \frac{\overline{x}^2 + x^{*2}}{6\sigma^2} + \frac{x^*\overline{x}}{3\sigma^2} + o(\delta). \tag{C.231}$$

In the benchmark cases:

$$\frac{\text{CIR}(\delta)}{\delta} = \begin{cases} \frac{\overline{x}^2}{6\sigma^2} + o(\delta) & \text{if } \tilde{p} = 0, \\ \\ \frac{2\overline{x}^{*2}}{3\sigma^2} + o(\delta) & \text{if } \tilde{\theta} = 0, \end{cases} \tag{C.232}$$

□

C.4.6 Proof for $\nu \to \infty$

Lemma C.5. *Let $\nu > 0$ and $\sigma^2 \to 0$ such that $\nu/\sigma^2 \to \infty$. The mean and variance of $\hat{k} - \hat{k}^{ss}$ satisfy the joint system*

$$\mathbb{E}[x]\sqrt{\mathbb{V}ar[x]} = -\frac{(\mathcal{U}-\nu)\tilde{\theta}}{\sqrt{12}(1-\alpha)} \quad ; \quad \mathbb{V}ar[x] = 2\left(\frac{\nu}{\mathcal{U}}\right)^2 \frac{e^{(\alpha-1)\mathbb{E}[x]} - 1}{1 - \left(1 - \frac{\nu}{\mathcal{U}}(1-\alpha)\right)^2 e^{(\alpha-1)\mathbb{E}[x]}} \tag{C.233}$$

The covariance is given by

$$\mathbb{C}ov[a, x] = -\nu\mathbb{V}ar[x]. \tag{C.234}$$

The irreversibility term is given by

$$\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s[\mathrm{d}(\mathcal{M}(x_s)x_s)]\right] = 0. \tag{C.235}$$

The CIR is given by

$$\frac{CIR(\delta)}{\delta} = 0 + o(\delta). \tag{C.236}$$

Proof. We depart from the equilibrium condition for $\tilde{q}(x)$ to characterize the policy in the case with $\sigma = 0$. Let $\hat{p} = p$. Then $\tilde{p}^{buy} = 0$ and since $\sigma = 0$, the domain in $x > x^{*-}$ is not operating. $\tilde{q}(x)$ satisfies

$$\mathcal{U}\tilde{q}(x) = e^{(\alpha-1)x} - 1 - \nu\tilde{q}(x), \tag{C.237}$$

$$\tilde{q}(x^-) = \tilde{q}(x^{*-}) = 0, \tag{C.238}$$

$$\tilde{\theta} = \int_{x^-}^{x^{*-}} e^x \tilde{q}(x)\,\mathrm{d}x. \tag{C.239}$$

Next, we obtain a system of two equations to characterize x^- and x^{*-}. Multiplying the HJB equation by $e^{\frac{\mathcal{U}}{\nu}x}$, we have that

$$e^{\frac{\mathcal{U}}{\nu}x}(\mathcal{U}\tilde{q}(x) + \nu\tilde{q}'(x)) = e^{\frac{\mathcal{U}}{\nu}x}(e^{(\alpha-1)x} - 1), \tag{C.240}$$

which is equivalent to

$$\nu \frac{\mathrm{d} e^{\frac{\mathcal{U}}{\nu} x} \tilde{q}(x)}{\mathrm{d}x} = e^{\frac{\mathcal{U}}{\nu} x} (e^{(\alpha-1)x} - 1). \tag{C.241}$$

Integrating from x^- to x

$$\nu (e^{\frac{\mathcal{U}}{\nu} x} \tilde{q}(x) - e^{\frac{\mathcal{U}}{\nu} x_-} \tilde{q}(x_-)) = \int_{x^-}^{x} e^{\frac{\mathcal{U}}{\nu} s} (e^{(\alpha-1)s} - 1) \, \mathrm{d}s \tag{C.242}$$

Using the optimality condition $q(x^{*-}) = 0$, we have that

$$\tilde{q}(x) = \frac{\int_{x^-}^{x} e^{\frac{\mathcal{U}}{\nu}(s-x)} (e^{(\alpha-1)s} - 1) \, \mathrm{d}s}{\nu}. \tag{C.243}$$

Evaluation at x^{*+}, we have the first equilibrium condition

$$0 = \int_{x^-}^{x^{*+}} e^{\frac{\mathcal{U}}{\nu}(s - x^{*+})} (e^{(\alpha-1)s} - 1) \, \mathrm{d}s \iff 0 = \int_{x^-}^{x^{*+}} e^{\frac{\mathcal{U}}{\nu} s} (e^{(\alpha-1)s} - 1) \, \mathrm{d}s. \tag{C.244}$$

To obtain the second optimality condition, we multiply the HJB equation by e^x, we have that

$$\mathcal{U} e^x \tilde{q}(x) = e^x (e^{(\alpha-1)x} - 1) - \nu e^x \tilde{q}'(x). \tag{C.245}$$

Integrating between x^- to x^{*-},

$$\mathcal{U} \int_{x^-}^{x^{*-}} e^x \tilde{q}(x) \, \mathrm{d}x = \int_{x^-}^{x^{*-}} e^x (e^{(\alpha-1)x} - 1) \, \mathrm{d}x - \nu \int_{x^-}^{x^{*-}} e^x \tilde{q}'(x) \, \mathrm{d}x. \tag{C.246}$$

Doing integration by part and using the boundary condition for $\tilde{q}(x)$,

$$\int_{x^-}^{x^{*-}} e^x \tilde{q}'(x) \, \mathrm{d}x = \underbrace{e^x \tilde{q}(x)|_{x^-}^{x^{*-}}}_{=0} - \int_{x^-}^{x^{*-}} e^x \tilde{q}(x) \, \mathrm{d}x. \tag{C.247}$$

Rearranging

$$(\mathcal{U} - \nu) \int_{x^-}^{x^{*-}} e^x \tilde{q}(x) \, \mathrm{d}x = \int_{x^-}^{x^{*-}} e^x (e^{(\alpha-1)x} - 1) \, \mathrm{d}x. \tag{C.248}$$

Finally, since $\tilde{\theta} = \int_{x^-}^{x^{*-}} e^x \tilde{q}(x) \, \mathrm{d}x$,

$$(\mathcal{U} + \nu) \tilde{\theta} = \int_{x^-}^{x^{*-}} (e^{\alpha x} - e^x) \, \mathrm{d}x. \tag{C.249}$$

In conclusion, the optimality conditions are given by

$$0 = \int_{x^-}^{x^{*+}} e^{\frac{\mathcal{U}}{\nu} x} (e^{(\alpha-1)x} - 1) \, \mathrm{d}x \tag{C.250}$$

$$(\mathcal{U} - \nu) \tilde{\theta} = \int_{x^-}^{x^{*-}} e^x (e^{(\alpha-1)x} - 1) \, \mathrm{d}x \tag{C.251}$$

Observe that the equilibrium conditions are similar to Sheshinski and Weiss (1977) with the objective function

$F(x) = \frac{x^\alpha}{\alpha} - x$ and discounting $\rho = \mathcal{U} - \nu$. Moreover, for this problem to be well-defined, $\mathcal{U} > \nu$ (if not, firms value is infinite). Now, we describe steady-moments as a function of the investment friction. Using a first-order Taylor approximation around zero for $e^x(e^{(\alpha-1)x} - 1)$, we have

$$e^x(e^{(\alpha-1)x} - 1) \approx e^0(e^{(\alpha-1)0} - 1) + (\alpha e^0 - e^0)(x - 0) = -(1-\alpha)x \tag{C.252}$$

Applying this approximation to the first optimality condition in (C.251) yields:

$$x^{*-2} - x^{-2} = -\frac{2(\mathcal{U} - \nu)\tilde{\theta}}{1-\alpha}. \tag{C.253}$$

Since the cross-sectional distribution is uniform in the range $[x^-, x^{*-}]$, it has the following moments:

$$\mathbb{V}ar[x] = \frac{(x^{*-} - x^-)^2}{12}; \qquad \mathbb{E}[x] = \frac{x^{*-} + x^-}{2}. \tag{C.254}$$

Thus, we can write the first optimality condition in (C.253) as:

$$\mathbb{E}[x]\sqrt{\mathbb{V}ar[x]} = -\frac{(\mathcal{U} - \nu)\tilde{\theta}}{\sqrt{12}(1-\alpha)}. \tag{C.255}$$

It is easy to see that $\mathbb{E}[x] < 0$. Firms compensate, but not undo, capital depresiation and productivity growth. Define $j(x) = e^{\frac{\mathcal{U}}{\nu}x}(e^{(\alpha-1)x} - 1)$. Observe that if we divide and multiply by $x^{*-} - x^-$, we can re-express as

$$0 = \int_{x^-}^{x^{*+}} e^{\frac{\mathcal{U}}{\nu}x}(e^{(\alpha-1)x} - 1)\,\mathrm{d}x = (x^{*-} - x^-)\int_{x^-}^{x^{*+}} j(x)\frac{1}{x^{*-} - x^-}\,\mathrm{d}x \iff 0 = \mathbb{E}[j(x)] \tag{C.256}$$

Thus, the expected discounted marginal product of capital relative to its cost is equal to zero. Doing a second-order Taylor approximation over $j(x)$ around the mean

$$0 = \mathbb{E}[j(x)] \approx \mathbb{E}[j(\mathbb{E}[x]) + j'(\mathbb{E}[x])(x - \mathbb{E}[x]) + \frac{j''(\mathbb{E}[x])}{2}(x - \mathbb{E}[x])^2] = j(\mathbb{E}[x]) + \frac{j''(\mathbb{E}[x])}{2}\mathbb{V}ar[x]. \tag{C.257}$$

Re-expressing the previous equation

$$\mathbb{V}ar[x] = 2\frac{j(\mathbb{E}[x])}{-j''(\mathbb{E}[x])} \tag{C.258}$$

Since $\mathbb{E}[x] < 0$, $j(\mathbb{E}[x]) > 0$ and since $j(x)$ is concave (because $\mathcal{U}/\nu > 1$ and $\alpha - 1 < 0$), $-j''(\mathbb{E}[x]) > 0$. Thus,

$$\mathbb{V}ar[x] = -2\frac{e^{(\frac{\mathcal{U}}{\nu}+\alpha-1)\mathbb{E}[x]} - e^{\frac{\mathcal{U}}{\nu}\mathbb{E}[x]}}{\left(\frac{\mathcal{U}}{\nu} + \alpha - 1\right)^2 e^{(\frac{\mathcal{U}}{\nu}+\alpha-1)\mathbb{E}[x]} - \left(\frac{\mathcal{U}}{\nu}\right)^2 e^{\frac{\mathcal{U}}{\nu}\mathbb{E}[x]}} \tag{C.259}$$

Thus, the equilibrium conditions are given by

$$\mathbb{E}[x]\sqrt{\mathbb{V}ar[x]} = -\frac{(\mathcal{U} - \nu)\tilde{\theta}}{\sqrt{12}(1-\alpha)} \quad ; \quad \mathbb{V}ar[x] = -2\frac{e^{(\frac{\mathcal{U}}{\nu}+\alpha-1)\mathbb{E}[x]} - e^{\frac{\mathcal{U}}{\nu}\mathbb{E}[x]}}{\left(\frac{\mathcal{U}}{\nu} + \alpha - 1\right)^2 e^{(\frac{\mathcal{U}}{\nu}+\alpha-1)\mathbb{E}[x]} - \left(\frac{\mathcal{U}}{\nu}\right)^2 e^{\frac{\mathcal{U}}{\nu}\mathbb{E}[x]}} \tag{C.260}$$

Rearranging the second equation

$$\mathbb{V}ar[x] = -2\frac{e^{(\frac{\mathcal{U}}{\nu}+\alpha-1)\mathbb{E}[x]} - e^{\frac{\mathcal{U}}{\nu}\mathbb{E}[x]}}{\left(\frac{\mathcal{U}}{\nu}+\alpha-1\right)^2 e^{(\frac{\mathcal{U}}{\nu}+\alpha-1)\mathbb{E}[x]} - \left(\frac{\mathcal{U}}{\nu}\right)^2 e^{\frac{\mathcal{U}}{\nu}\mathbb{E}[x]}} \tag{C.261}$$

$$= -2\frac{e^{(\alpha-1)\mathbb{E}[x]} - 1}{\left(\frac{\mathcal{U}}{\nu}+\alpha-1\right)^2 e^{(\alpha-1)\mathbb{E}[x]} - \left(\frac{\mathcal{U}}{\nu}\right)^2} \tag{C.262}$$

$$= 2\frac{e^{(\alpha-1)\mathbb{E}[x]} - 1}{\left(\frac{\mathcal{U}}{\nu}\right)^2 - \left(\frac{\mathcal{U}}{\nu}+\alpha-1\right)^2 e^{(\alpha-1)\mathbb{E}[x]}} \tag{C.263}$$

$$= 2\left(\frac{\nu}{\mathcal{U}}\right)^2 \frac{e^{(\alpha-1)\mathbb{E}[x]} - 1}{1 - \left(1 - \frac{\nu}{\mathcal{U}}(1-\alpha)\right)^2 e^{(\alpha-1)\mathbb{E}[x]}} \tag{C.264}$$

Thus, the equilibrium mean and variance of capital-to-productivity ratios is given by

$$\mathbb{E}[x]\sqrt{\mathbb{V}ar[x]} \;=\; -\frac{(\mathcal{U}-\nu)\tilde{\theta}}{\sqrt{12}(1-\alpha)} \quad ; \quad \mathbb{V}ar[x] = 2\left(\frac{\nu}{\mathcal{U}}\right)^2 \frac{e^{(\alpha-1)\mathbb{E}[x]} - 1}{1 - \left(1 - \frac{\nu}{\mathcal{U}}(1-\alpha)\right)^2 e^{(\alpha-1)\mathbb{E}[x]}} \tag{C.265}$$

The proof for the fact that $\mathbb{C}ov[x,a] = -\nu\mathbb{V}ar[x]$ is in Baley and Blanco (2021). □

C.5 Preliminaries for Proofs of Propositions 6, 7, 8 and 9

We review some properties of Markov chains in this model. Following the main text notation, let $\mathbb{P}^{++}$ and $\mathbb{P}^{--}$ be the transition probabilities. In a steady state, the probability of a current upsizing (downsizing) equals the ergodic probability of a subsequent upsizing (downsizing):

$$\frac{\mathcal{N}^-}{\mathcal{N}} \;=\; \frac{\mathcal{N}^-}{\mathcal{N}}\mathbb{P}^{--} \;+\; \frac{\mathcal{N}^+}{\mathcal{N}}\mathbb{P}^{+-} \tag{C.266}$$

$$\frac{\mathcal{N}^+}{\mathcal{N}} \;=\; \frac{\mathcal{N}^+}{\mathcal{N}}\mathbb{P}^{++} \;+\; \frac{\mathcal{N}^-}{\mathcal{N}}\mathbb{P}^{-+} \tag{C.267}$$

Ergodicity of stopping times We show that

$$\frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\tau'] \;+\; \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\tau'] \;=\; \overline{\mathbb{E}}[\tau]. \tag{C.268}$$

This relationship follows directly from the law of iterated expectations.

Ergodicity of reset points We show that the average reset capital conditional on an investment $\overline{\mathbb{E}}\left[\hat{k}^*(\Delta\hat{k})\right] = \mathcal{H}^-\hat{k}^{*-} + \mathcal{H}^+\hat{k}^{*+}$ is equal to the average reset capital in the next investment conditional on current investment $\overline{\mathbb{E}}\left[\overline{\mathbb{E}}\left[\hat{k}^*(\Delta\hat{k}')\right]\Big|\Delta k\right] = \overline{\mathbb{E}}\left[\hat{k}^*(\Delta\hat{k})\right]$.

$$\frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\hat{k}^{*'}] \;+\; \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\hat{k}^{*'}] \;=\; \overline{\mathbb{E}}[\hat{k}^*]. \tag{C.269}$$

The first term:

$$\frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\hat{k}^{*'}] = \frac{\mathcal{N}^-}{\mathcal{N}}\mathbb{P}^{--}\hat{k}^{*-} \;+\; \frac{\mathcal{N}^-}{\mathcal{N}}\mathbb{P}^{-+}\hat{k}^{*+} \tag{C.270}$$

The second term:

$$\frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\hat{k}^{*\prime}] = \frac{\mathcal{N}^+}{\mathcal{N}}\mathbb{P}^{++}\hat{k}^{*+} + \frac{\mathcal{N}^+}{\mathcal{N}}\mathbb{P}^{+-}\hat{k}^{*-} \tag{C.271}$$

Summing the two terms up and using the relationships in (C.266) and (C.267) we get:

$$\begin{aligned}
\frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\hat{k}^{*\prime}] + \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\hat{k}^{*\prime}] &= \frac{\mathcal{N}^-}{\mathcal{N}}\mathbb{P}^{--}\hat{k}^{*-} + \frac{\mathcal{N}^-}{\mathcal{N}}\mathbb{P}^{-+}\hat{k}^{*+} + \frac{\mathcal{N}^+}{\mathcal{N}}\mathbb{P}^{++}\hat{k}^{*+} + \frac{\mathcal{N}^+}{\mathcal{N}}\mathbb{P}^{+-}\hat{k}^{*-} \\
&= \left(\frac{\mathcal{N}^-}{\mathcal{N}}\mathbb{P}^{--} + \frac{\mathcal{N}^+}{\mathcal{N}}\mathbb{P}^{+-}\right)\hat{k}^{*-} + \left(\frac{\mathcal{N}^-}{\mathcal{N}}\mathbb{P}^{-+} + \frac{\mathcal{N}^+}{\mathcal{N}}\mathbb{P}^{++}\right)\hat{k}^{*+} \\
&= \frac{\mathcal{N}^-}{\mathcal{N}}\hat{k}^{*-} + \frac{\mathcal{N}^+}{\mathcal{N}}\hat{k}^{*+} = \overline{\mathbb{E}}[\hat{k}^*]
\end{aligned}$$

C.6 Proof of Proposition 6

Proposition 6. ***(Recovering parameters)*** *The drift ν and volatility σ^2 of capital-productivity ratios implied by investment microdata are recovered through the following mappings:*

$$\nu = \frac{\overline{\mathbb{E}}[\Delta\hat{k}]}{\overline{\mathbb{E}}[\tau]}, \tag{56}$$

$$\sigma^2 = \frac{\overline{\mathbb{E}}[(\hat{k}_{\tau'} + \nu\tau')^2 - (\hat{k}^*)^2]}{\overline{\mathbb{E}}[\tau]}. \tag{57}$$

C.6.1 Drift

Conditional on the previous reset point $\hat{k}^*$, the law of motion of capital-productivity ratios implies

$$\hat{k}_s + \nu s - \hat{k}^* = \sigma W_s. \tag{C.272}$$

Evaluate at a future stopping time $s = \tau'$ to get

$$\hat{k}_{\tau'} + \nu\tau' - \hat{k}^* = \sigma W_{\tau'}. \tag{C.273}$$

Since the expectation of the future stopped capital depends on the previous reset point, we take expectations conditional on the last adjustment:

$$\overline{\mathbb{E}}^{\pm}[\hat{k}_{\tau'}] + \nu\overline{\mathbb{E}}^{\pm}[\tau'] - \hat{k}^{*\pm} = 0. \tag{C.274}$$

Note that from these expressions, we can find mappings for the drift using conditional means:

$$\nu = \frac{\hat{k}^{*\pm} - \overline{\mathbb{E}}^{\pm}[\hat{k}_{\tau'}]}{\overline{\mathbb{E}}^{\pm}[\tau']}. \tag{C.275}$$

To derive mappings using the unconditional mean, we average the conditional expectations in (C.274) with the shares of upward and downward adjustments:

$$\frac{\mathcal{N}^-}{\mathcal{N}}\left(\overline{\mathbb{E}}^-[\hat{k}_{\tau'}] + \nu\overline{\mathbb{E}}^-[\tau'] - \hat{k}^{*-}\right) + \frac{\mathcal{N}^+}{\mathcal{N}}\left(\overline{\mathbb{E}}^+[\hat{k}_{\tau'}] + \nu\overline{\mathbb{E}}^+[\tau'] - \hat{k}^{*+}\right) = 0 \tag{C.276}$$

Join similar terms and use the ergodic property of stopping times in (C.268)

$$\text{(C.277)}\qquad \frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\hat{k}_{\tau'}] + \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\hat{k}_{\tau'}] + \nu \underbrace{\left(\frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\tau'] + \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\tau']\right)}_{\overline{\mathbb{E}}[\tau]} - \frac{\mathcal{N}^+}{\mathcal{N}}\hat{k}^{*-} - \frac{\mathcal{N}^+}{\mathcal{N}}\hat{k}^{*+} = 0.$$

Substitute the relationship $\hat{k}_{\tau'} = \hat{k}^{*'} - \Delta\hat{k}'$ and use the ergodic property of reset points in (C.269)

$$\text{(C.278)}\qquad \underbrace{\frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\hat{k}^{*'}] + \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\hat{k}^{*'}]}_{\overline{\mathbb{E}}[\hat{k}^*]} \underbrace{-\frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\Delta\hat{k}'] - \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\Delta\hat{k}']}_{-\overline{\mathbb{E}}[\Delta\hat{k}']} + \nu\overline{\mathbb{E}}[\tau'] \underbrace{-\frac{\mathcal{N}^-}{\mathcal{N}}\hat{k}^{*-} - \frac{\mathcal{N}^+}{\mathcal{N}}\hat{k}^{*+}}_{-\overline{\mathbb{E}}[\hat{k}^*]} \;=\; 0$$

Cancel terms and rearrange

$$\text{(C.279)}\qquad \cancel{\overline{\mathbb{E}}[\hat{k}^*]} \;-\; \overline{\mathbb{E}}[\Delta\hat{k}'] \;+\; \nu\overline{\mathbb{E}}[\tau'] \;-\; \cancel{\overline{\mathbb{E}}[\hat{k}^*]} \;=\; 0$$

to obtain the result:

$$\text{(C.280)}\qquad \nu \;=\; \frac{\overline{\mathbb{E}}[\Delta\hat{k}']}{\overline{\mathbb{E}}[\tau']}.$$

C.6.2 Idiosyncratic volatility

Let $Y_s \equiv (\hat{k}_s + \nu s)^2$. Applying Itō's lemma to Y_s,

$$\text{(C.281)}\qquad \mathrm{d}Y_s \;=\; 2(\hat{k}_s + \nu s)(\mathrm{d}\hat{k}_s + \nu\,\mathrm{d}s) \;+\; (\mathrm{d}\hat{k}_s)^2 \;=\; 2(\hat{k}_s + \nu s)\sigma\,\mathrm{d}W_s + \sigma^2\,\mathrm{d}s.$$

We integrate both sides from 0 to τ' and take expectations conditional on the previous reset point:

$$\text{(C.282)}\qquad \overline{\mathbb{E}}^\pm[Y_{\tau'}] - Y_0 \;=\; 2\sigma\overline{\mathbb{E}}^\pm\left[\int_0^{\tau'}(\hat{k}_s + \nu s)\,\mathrm{d}W_s\right] + \sigma^2\overline{\mathbb{E}}^\pm\left[\int_0^{\tau'} 1\,\mathrm{d}s\right]$$

We use the OST (Auxiliary Theorem in A.1) to set the martingale to zero, $\overline{\mathbb{E}}^\pm[\int_0^{\tau'}(\hat{k}_s + \nu s)\,\mathrm{d}W_s] = 0$, and obtain:

$$\text{(C.283)}\qquad \overline{\mathbb{E}}^\pm[Y_{\tau'}] - Y_0 \;=\; \sigma^2\overline{\mathbb{E}}^\pm[\tau'].$$

Substituting $Y_{\tau'} \equiv (\hat{k}_{\tau'} + \nu\tau')^2$ and $Y_0 \equiv (\hat{k}^{*\pm})^2$

$$\text{(C.284)}\qquad \overline{\mathbb{E}}^\pm\left[(\hat{k}_{\tau'} + \nu\tau')^2\right] - (\hat{k}^\pm)^2 \;=\; \sigma^2\overline{\mathbb{E}}^\pm[\tau']$$

We average the conditional expectations with the shares of upward and downward adjustments

$$\text{(C.285)}\qquad \underbrace{\frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-\left[(\hat{k}_{\tau'} + \nu\tau')^2\right] + \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+\left[(\hat{k}_{\tau'} + \nu\tau')^2\right]}_{\overline{\mathbb{E}}[(\hat{k}_{\tau'}+\nu\tau')^2]} - \underbrace{\left(\frac{\mathcal{N}^-}{\mathcal{N}}(\hat{k}^{*-})^2 + \frac{\mathcal{N}^+}{\mathcal{N}}(\hat{k}^{*+})^2\right)}_{\overline{\mathbb{E}}[(\hat{k}^*)^2]} = \sigma^2\underbrace{\left(\frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^+[\tau'] + \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^-[\tau']\right)}_{\overline{\mathbb{E}}[\tau]\text{ by (C.268)}}$$

Rearranging, we obtain the mapping from data to σ^2:

$$\text{(C.286)}\qquad \sigma^2 \;=\; \frac{\overline{\mathbb{E}}[(\hat{k}_{\tau'} + \nu\tau')^2] - \overline{\mathbb{E}}[(\hat{k}^*)^2]}{\overline{\mathbb{E}}[\tau]}.$$

C.7 Proof of Proposition 7

Proposition 7. ***(Recovering means)*** *Let $r^{\pm}$ be the adjusted shares in (25). The unconditional mean $\mathbb{E}[\hat{k}]$ and means conditional on the previous reset $\mathbb{E}^{\pm}[\hat{k}]$ are recovered as:*

$$\mathbb{E}[\hat{k}] \;=\; r^{-}\mathbb{E}^{-}[\hat{k}] \;+\; r^{+}\mathbb{E}^{+}[\hat{k}], \tag{58}$$

$$\mathbb{E}^{\pm}[\hat{k}] \;=\; \overline{\mathbb{E}}^{\pm}\left[\left(\frac{\hat{k}^{*\pm}+\hat{k}_{\tau'}}{2}\right)\left(\frac{\hat{k}^{*\pm}-\hat{k}_{\tau'}}{\overline{\mathbb{E}}^{\pm}[\hat{k}^{*\pm}-\hat{k}_{\tau'}]}\right)\right] \;+\; \frac{\sigma^2}{2\nu}. \tag{59}$$

Proof. This proposition expresses the cross-sectional moments of $\hat{k}$ as functions of the data. We derive mappings for any moment n of $\hat{k}$. We apply Itō's lemma to $\hat{k}_s^n$ for $n \geq 2$:

$$\mathrm{d}\hat{k}_s^{n+1} \;=\; -\nu(n+1)\hat{k}_s^n\,\mathrm{d}s \;+\; \sigma(n+1)\hat{k}_s^n\,\mathrm{d}Ws \;+\; \frac{\sigma^2 n(n+1)}{2}\hat{k}_s^{n-1}\,\mathrm{d}s. \tag{C.287}$$

We integrate this expression from 0 to τ' and take expectations conditional on the initial condition given by the previous reset point $\hat{k}_0 = \hat{k}^{*\pm}$

$$\overline{\mathbb{E}}^{\pm}\left[\int_0^{\tau'}\mathrm{d}\hat{k}_s^{n+1}\right] \;=\; -\nu(n+1)\overline{\mathbb{E}}^{\pm}\left[\int_0^{\tau'}\hat{k}_s^n\,\mathrm{d}s\right] \;+\; \sigma(n+1)\overline{\mathbb{E}}^{\pm}\left[\hat{k}_s^n\,\mathrm{d}Ws\right] \;+\; \frac{\sigma^2 n(n+1)}{2}\overline{\mathbb{E}}^{\pm}\left[\hat{k}_s^{n-1}\,\mathrm{d}s\right]. \tag{C.288}$$

The term on the LHS is the definite integral of a derivative. On the RHS, we use the OST in (A.1) to set the martingale in the second term to zero, $\overline{\mathbb{E}}^{\pm}\left[\hat{k}_s^n\,\mathrm{d}Ws\right] = 0$.

$$\overline{\mathbb{E}}^{\pm}\left[\hat{k}_{\tau'}^{n+1}\right] - (\hat{k}^{*\pm})^{n+1} \;=\; -\nu(n+1)\overline{\mathbb{E}}^{\pm}\left[\int_0^{\tau'}\hat{k}_s^n\,\mathrm{d}s\right] \;+\; \frac{\sigma^2 n(n+1)}{2}\overline{\mathbb{E}}^{\pm}\left[\int_0^{\tau'}\hat{k}_s^{n-1}\,\mathrm{d}s\right]. \tag{C.289}$$

Divide both sides by $\overline{\mathbb{E}}^{\pm}[\tau']$

$$\frac{\overline{\mathbb{E}}^{\pm}\left[\hat{k}_{\tau'}^{n+1}\right] - (\hat{k}^{*\pm})^{n+1}}{\overline{\mathbb{E}}^{\pm}[\tau']} \;=\; -\nu(n+1)\frac{\overline{\mathbb{E}}^{\pm}\left[\int_0^{\tau'}\hat{k}_s^n\,\mathrm{d}s\right]}{\overline{\mathbb{E}}^{\pm}[\tau']} \;+\; \frac{\sigma^2 n(n+1)}{2}\frac{\overline{\mathbb{E}}^{\pm}\left[\int_0^{\tau'}\hat{k}_s^{n-1}\,\mathrm{d}s\right]}{\overline{\mathbb{E}}^{\pm}[\tau']} \tag{C.290}$$

Use OMT in (A.2) to recover steady-state moments in the RHS using the occupancy measure (e.g. $\overline{\mathbb{E}}^{\pm}[\hat{k}^n] = \mathbb{E}^{\pm}\left[\int_0^{\tau'}\hat{k}_s^n\,\mathrm{d}s\right]/\overline{\mathbb{E}}^{\pm}[\tau']$)

$$\frac{\overline{\mathbb{E}}^{\pm}\left[\hat{k}_{\tau'}^{n+1}\right] - (\hat{k}^{*\pm})^{n+1}}{\overline{\mathbb{E}}^{\pm}[\tau']} \;=\; -\nu(n+1)\mathbb{E}^{\pm}[\hat{k}^n] \;+\; \frac{\sigma^2 n(n+1)}{2}\mathbb{E}^{\pm}[\hat{k}^{n-1}] \tag{C.291}$$

Solving for $\mathbb{E}^{\pm}[\hat{k}^n]$ and rearranging, we obtain a mapping for the conditional n-th moment:

$$\mathbb{E}^{\pm}[\hat{k}^n] \;=\; \frac{(\hat{k}^{*\pm})^{n+1} - \overline{\mathbb{E}}^{\pm}\left[\hat{k}_{\tau'}^{n+1}\right]}{\nu(n+1)\overline{\mathbb{E}}^{\pm}[\tau']} \;+\; \frac{\sigma^2 n}{2\nu}\mathbb{E}^{\pm}[\hat{k}^{n-1}]. \tag{C.292}$$

Applying similar steps as before, we compute the conditional moments centered at the economy-wide mean $\mathbb{E}[\hat{k}]$:

$$\mathbb{E}^{\pm}[(\hat{k}-\mathbb{E}[\hat{k}])^n] \;=\; \frac{(\hat{k}^{*\pm}-\mathbb{E}[\hat{k}])^{n+1} - \overline{\mathbb{E}}^{\pm}\left[(\hat{k}_{\tau'}-\mathbb{E}[\hat{k}])^{n+1}\right]}{\nu(n+1)\overline{\mathbb{E}}^{\pm}[\tau']} \;+\; \frac{\sigma^2 n}{2\nu}\mathbb{E}^{\pm}[(\hat{k}-\mathbb{E}[\hat{k}])^{n-1}]. \tag{C.293}$$

Conditional means To obtain the conditional means, we first evaluate (C.292) at $n = 1$,

$$\mathbb{E}^{\pm}[\hat{k}] = \frac{(\hat{k}^{*\pm})^2 - \overline{\mathbb{E}}^{\pm}\left[\hat{k}^2_{\tau'}\right]}{2\nu\overline{\mathbb{E}}^{\pm}[\tau']} + \frac{\sigma^2}{2\nu}. \tag{C.294}$$

Unconditional mean To compute the unconditional mean, we average the conditional means using the adjusted shares $r^{\pm} = \frac{\mathcal{N}^{\pm}}{\mathcal{N}}\frac{\overline{\mathbb{E}}^{\pm}[\tau']}{\overline{\mathbb{E}}[\tau']}$, where $r^- + r^+ = 1$.

$$\begin{aligned}
\mathbb{E}[\hat{k}] &= r^-\mathbb{E}^-[\hat{k}] + r^+\mathbb{E}^+[\hat{k}] && \text{(C.295)}\\
&= r^-\frac{\overline{\mathbb{E}}^-\left[(\hat{k}^{*-})^2 - \hat{k}^2_{\tau'}\right]}{2\nu\overline{\mathbb{E}}^-[\tau']} + r^+\frac{\overline{\mathbb{E}}^+\left[(\hat{k}^{*+})^2 - \hat{k}^2_{\tau'}\right]}{2\nu\overline{\mathbb{E}}^+[\tau']} + (r^- + r^+)\frac{\sigma^2}{2\nu} && \text{(C.296)}\\
&= \frac{\mathcal{N}^-}{\mathcal{N}}\frac{\overline{\mathbb{E}}^-[\tau']}{\overline{\mathbb{E}}[\tau']}\frac{\overline{\mathbb{E}}^-\left[(\hat{k}^{*-})^2 - \hat{k}^2_{\tau'}\right]}{2\nu\overline{\mathbb{E}}^-[\tau']} + \frac{\mathcal{N}^+}{\mathcal{N}}\frac{\overline{\mathbb{E}}^+[\tau']}{\overline{\mathbb{E}}[\tau']}\frac{\overline{\mathbb{E}}^+\left[(\hat{k}^{*+})^2 - \hat{k}^2_{\tau'}\right]}{2\nu\overline{\mathbb{E}}^+[\tau']} + \frac{\sigma^2}{2\nu} && \text{(C.297)}\\
&= \frac{\frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-\left[(\hat{k}^{*-})^2 - \hat{k}^2_{\tau'}\right] + \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+\left[(\hat{k}^{*+})^2 - \hat{k}^2_{\tau'}\right]}{2\nu\overline{\mathbb{E}}[\tau']} + \frac{\sigma^2}{2\nu} && \text{(C.298)}\\
&= \frac{\overline{\mathbb{E}}[(\hat{k}^*)^2] - \overline{\mathbb{E}}[\hat{k}^2_{\tau'}]}{2\overline{\mathbb{E}}[\Delta\hat{k}]} + \frac{\sigma^2}{2\nu}. && \text{(C.299)}
\end{aligned}$$

In the last step, we substitute $\nu = \overline{\mathbb{E}}[\Delta\hat{k}']/\overline{\mathbb{E}}[\tau']$ and use the relationship $\overline{\mathbb{E}}[\cdot] = \frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\cdot] + \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\cdot]$.

□

C.8 Proof of Proposition 8

Proposition 8. ***(Recovering the variance and covariance)*** *The variance $\mathbb{V}ar[\hat{k}]$ and the covariance $\mathbb{C}ov[\hat{k}, a]$ are recovered from the microdata as:*

$$\mathbb{V}ar[\hat{k}] = \frac{1}{3}\frac{\mathbb{E}\left[(\hat{k}^* - \mathbb{E}[\hat{k}])^3\right] - \overline{\mathbb{E}}\left[(\hat{k}_{\tau'} - \mathbb{E}[\hat{k}])^3\right]}{\hat{k}^* - \overline{\mathbb{E}}[\hat{k}_{\tau'}]}. \tag{60}$$

$$\mathbb{C}ov[\hat{k}, a] = \frac{1}{2\nu}\left(\mathbb{V}ar[\hat{k}] + \sigma^2\mathbb{E}[a] - \frac{\overline{\mathbb{E}}\left[(\hat{k}_{\tau'} - \mathbb{E}[\hat{k}])^2\tau'\right]}{\overline{\mathbb{E}}[\tau]}\right). \tag{61}$$

Proof. **Variance** To obtain the variance, we evaluate (C.293) at $n = 2$:

$$\mathbb{E}^{\pm}[(\hat{k} - \mathbb{E}^{\pm}[\hat{k}])^2] = \frac{\overline{\mathbb{E}}^{\pm}\left[(\hat{k}^{*\pm} - \mathbb{E}^{\pm}[\hat{k}])^3 - (\hat{k}_{\tau'} - \mathbb{E}^{\pm}[\hat{k}])^3\right]}{3\nu\overline{\mathbb{E}}^{\pm}[\tau']} + \frac{\sigma^2}{\nu}\mathbb{E}^{\pm}[(\hat{k} - \mathbb{E}^{\pm}[\hat{k}])]. \tag{C.300}$$

Substituting the definition of variance on the LHS and setting the second term on the RHS to zero, we obtain

$$\mathbb{V}ar^{\pm}[\hat{k}] = \frac{1}{3}\frac{(\hat{k}^{*\pm} - \mathbb{E}^{\pm}[\hat{k}])^3 - \overline{\mathbb{E}}^{\pm}\left[(\hat{k}_{\tau'} - \mathbb{E}^{\pm}[\hat{k}])^3\right]}{\nu\overline{\mathbb{E}}^{\pm}[\tau']}. \tag{C.301}$$

To obtain the unconditional average, we average the conditional variances using relative adjusting shares to get the unconditional values:

$$\mathbb{V}ar[\hat{k}] \;=\; \frac{1}{3}\frac{(\hat{k}^* - \mathbb{E}[\hat{k}])^3 - \overline{\mathbb{E}}\left[(\hat{k}_{\tau'} - \mathbb{E}[\hat{k}])^3\right]}{\nu\overline{\mathbb{E}}[\tau']}. \tag{C.302}$$

Finally, substitute the denominator for (C.275).

Joint moments of $\hat{k}$ and a We prove this proposition of any joint moment of $(\hat{k}, a)$. Consider the function $Y_s = (\hat{k}_s - \mathbb{E}[\hat{k}])^{n+1}s$ and apply Itō's lemma to obtain:

$$\begin{aligned} \mathrm{d}Y_s \;=\;& (\hat{k}_s - \mathbb{E}[\hat{k}])^{n+1}\,\mathrm{d}s \;-\; \nu(n+1)(\hat{k}_s - \mathbb{E}[\hat{k}])^n s\,\mathrm{d}s \;+\; \sigma(n+1)(\hat{k}_s - \mathbb{E}[\hat{k}])^n s\,\mathrm{d}W_s \\ &+\; \frac{\sigma^2}{2}n(n+1)(\hat{k}_s - \mathbb{E}[\hat{k}])^{n-1}s\,\mathrm{d}s. \end{aligned} \tag{C.303}$$

We integrate this expression from 0 to τ', take expectations conditional on the previous reset point, use OST in (A.1) to set martingales to zero, and divide both sides by $\overline{\mathbb{E}}^{\pm}[\tau']$:

$$\frac{\overline{\mathbb{E}}^{\pm}\left[(\hat{k}_{\tau'} - \mathbb{E}[\hat{k}])^{n+1}\tau'\right]}{\overline{\mathbb{E}}^{\pm}[\tau']} \;=\; \mathbb{E}^{\pm}[(\hat{k}-\mathbb{E}[\hat{k}])^{n+1}] - \nu(n+1)\mathbb{E}^{\pm}[(\hat{k}-\mathbb{E}[\hat{k}])^n a] + \frac{\sigma^2}{2}n(n+1)\mathbb{E}^{\pm}[(\hat{k}-\mathbb{E}[\hat{k}])^{n-1}a]. \tag{C.304}$$

Rearranging:

$$\mathbb{E}^{\pm}[(\hat{k} - \mathbb{E}[\hat{k}])^n a] \;=\; \frac{1}{\nu(n+1)}\left[\mathbb{E}^{\pm}[(\hat{k} - \mathbb{E}[\hat{k}])^{n+1}] \;+\; \frac{\sigma^2}{2}n(n+1)\mathbb{E}^{\pm}[(\hat{k} - \mathbb{E}[\hat{k}])^{n-1}a] \;-\; \frac{\overline{\mathbb{E}}^{\pm}[(\hat{k}_{\tau'} - \mathbb{E}[\hat{k}])^{n+1}\tau']}{\overline{\mathbb{E}}^{\pm}[\tau']}\right]. \tag{C.305}$$

To obtain the unconditional average, we average the conditional joint moments of $\hat{k}$ and a using relative adjusting shares to get the unconditional joint moments:

$$\mathbb{E}[(\hat{k} - \mathbb{E}[\hat{k}])^n a] \;=\; \frac{1}{\nu(n+1)}\left[\mathbb{E}[(\hat{k} - \mathbb{E}[\hat{k}])^{n+1}] \;+\; \frac{\sigma^2}{2}n(n+1)\mathbb{E}[(\hat{k} - \mathbb{E}[\hat{k}])^{n-1}a] \;-\; \frac{\overline{\mathbb{E}}[(\hat{k}_{\tau'} - \mathbb{E}[\hat{k}])^{n+1}\tau']}{\overline{\mathbb{E}}[\tau']}\right]. \tag{C.306}$$

Covariance Finally, to compute the covariance between $(\hat{k}, a)$, we evaluate expression (C.306) at $n = 1$ to obtain

$$\mathbb{C}ov[\hat{k}, a] = \frac{1}{2\nu}\left(\mathbb{V}ar[\hat{k}] + \sigma^2\mathbb{E}[a] - \frac{\overline{\mathbb{E}}[(\hat{k}_{\tau'} - \mathbb{E}[\hat{k}])^2\tau']}{\overline{\mathbb{E}}[\tau']}\right) \tag{C.307}$$

□

C.9 Proof of Proposition 9

Proposition 9. ***(Recovering the irreversibility term)*** *The CIR's irreversibility term is recovered from the microdata as*

$$\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s\left[\mathrm{d}(\hat{k}_s\mathcal{M}(\hat{k}_s))\right]\right] \;=\; \frac{\overline{\mathbb{E}}[\hat{k}_{\tau'}\mathcal{M}(\hat{k}_{\tau'})] - \overline{\mathbb{E}}[\hat{k}^*\mathcal{M}(\hat{k}^*)]}{\overline{\mathbb{E}}[\tau]}, \tag{62}$$

where departing deviations $\mathcal{M}(\hat{k}^{*\pm})$ and ending deviations $\mathcal{M}(\hat{k}_{\tau'})$ are recovered in Proposition 3.

Proof. **C.9.1 Local drift**

Apply Ito's lemma to the product $\hat{k}_s\mathcal{M}(\hat{k}_s)$

$$\mathbb{E}_s[\mathrm{d}(\hat{k}_s\mathcal{M}(\hat{k}_s))] = \left[-\nu\left[\mathcal{M}(\hat{k}_s) + \hat{k}_s\mathcal{M}'(\hat{k}_s)\right] + \frac{\sigma^2}{2}\mathbb{E}\left[2\mathcal{M}'(\hat{k}_s) + \hat{k}_s\mathcal{M}''(\hat{k}_s)\right]\right]\mathrm{d}s \tag{C.308}$$

Taking the integral between 0 and τ', using the OST in (A.1) to set martingales to zero, and the OMT in (A.2) to convert occupancy measures into cross-sectional moments:

$$\frac{\overline{\mathbb{E}}\left[\mathbb{E}\left[\hat{k}_{\tau'}\mathcal{M}(\hat{k}_{\tau'})|\Delta\hat{k}\right]\right] - \overline{\mathbb{E}}\left[\mathbb{E}\left[\hat{k}^*\mathcal{M}(\hat{k}^*)|\Delta\hat{k}\right]\right]}{\overline{\mathbb{E}}[\tau]} = \mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s\left[\mathrm{d}(\mathcal{M}(\hat{k}_s)\hat{k}_s)\right]\right] \tag{C.309}$$

□

C.10 Proof of proposition 10

Proposition 10. ***(Recovering reset points)*** *Let* $\Phi \equiv \log\big(\alpha/(\mathcal{U} - (1-\alpha)\nu - (1-\alpha)^2\sigma^2/2)\big)$. *The two reset points* $\{\hat{k}^{*-}, \hat{k}^{*+}\}$ *are recovered from the microdata as:*

$$\hat{k}^{*-} = \frac{1}{1-\alpha}\left(\Phi - \log p + \log\frac{1-\overline{\mathbb{E}}^-\left[e^{-\mathcal{U}\tau^* + (1-\alpha)(\hat{k}^{*-}-\hat{k}_{\tau'})}\right]}{1-\overline{\mathbb{E}}^-\left[\frac{p(\Delta\hat{k}')}{p}e^{-\mathcal{U}\tau^*}\right]}\right), \tag{65}$$

$$\hat{k}^{*+} = \frac{1}{1-\alpha}\left(\Phi - \log p(1-\omega) + \log\frac{1-\overline{\mathbb{E}}^+\left[e^{-\mathcal{U}\tau^* + (1-\alpha)(\hat{k}^{*+}-\hat{k}_{\tau'})}\right]}{1-\overline{\mathbb{E}}^+\left[\frac{p(\Delta\hat{k}')}{p(1-\omega)}e^{-\mathcal{U}\tau^*}\right]}\right). \tag{66}$$

C.10.1 Without irreversibility

We begin showing how to recover the unique reset point $\hat{k}^*$ without irreversibility. Recall the HJB for Tobin's q in sequential form in equation (11), evaluated at $q^* = 1$:

$$q(\hat{k}) = \mathbb{E}\left[\int_0^\tau \frac{\alpha e^{-\mathcal{U}s+(\alpha-1)\hat{k}_s}}{p}\mathrm{d}s + e^{-\mathcal{U}\tau}1\right] \tag{C.310}$$

Evaluating at the optimum $\hat{k}^*$

$$1 = q(\hat{k}^*) = \overline{\mathbb{E}}\left[\int_0^\tau \frac{\alpha e^{-\mathcal{U}s-(1-\alpha)\hat{k}_s}}{p}\mathrm{d}s + e^{-\mathcal{U}\tau}1\right] \tag{C.311}$$

Next, we characterize in terms of observables. Define $Y_s = e^{-\mathcal{U}s-(1-\alpha)\hat{k}_s}$ and apply Ito's lemma:

$$\mathrm{d}Y_s = Y_s[-\mathcal{U}\,\mathrm{d}s - (1-\alpha)\,\mathrm{d}\hat{k}_s + \frac{(1-\alpha)^2}{2}\mathrm{d}\hat{k}_s^2] \tag{C.312}$$

$$= Y_s\left[-\mathcal{U}\,\mathrm{d}s - (1-\alpha)(-\nu\,\mathrm{d}s + \sigma\,\mathrm{d}W_s) + \frac{(1-\alpha)^2\sigma^2}{2}\mathrm{d}s\right] \tag{C.313}$$

$$= -\underbrace{\left[\mathcal{U} - (1-\alpha)\nu - \frac{(1-\alpha)^2\sigma^2}{2}\right]}_{\phi} Y_s\,\mathrm{d}s - (1-\alpha)\sigma Y_s\,\mathrm{d}W_s \tag{C.314}$$

where we define $\phi \equiv \mathcal{U} - (1-\alpha)\nu - \frac{(1-\alpha)^2\sigma^2}{2}$. Integrating both sides from 0 to τ^*, taking expectations conditional on the initial condition $k_0 = k^*$, and using the OST in (A.1) to set the expectation of martingales to zero, we obtain:

$$\underbrace{\overline{\mathbb{E}}\left[\int_0^{\tau^*} \mathrm{d}Y_s\,\mathrm{d}s\right]}_{\overline{\mathbb{E}}[Y_\tau - Y_0\ ]} = -\phi\overline{\mathbb{E}}\left[\int_0^{\tau^*} Y_s\,\mathrm{d}s\right] - (1-\alpha)\sigma\underbrace{\overline{\mathbb{E}}\left[\int_0^{\tau^*} Y_s\,\mathrm{d}W_s\right]}_{=0}. \tag{C.315}$$

or simply

$$\frac{\overline{\mathbb{E}}\left[Y_0 - Y_{\tau^*}\right]}{\phi} = \overline{\mathbb{E}}\left[\int_0^{\tau^*} Y_s\,\mathrm{d}s\right] \tag{C.316}$$

Since $Y_{\tau^*} = e^{-\mathcal{U}\tau^* - (1-\alpha)(\hat{k}^* - \Delta\hat{k})}$ and $Y_0 = e^{-(1-\alpha)\hat{k}^*}$, then $Y_0 - Y_{\tau^*} = e^{-(1-\alpha)\hat{k}^*}[1 - e^{-\mathcal{U}\tau^* + (1-\alpha)\Delta\hat{k}}]$. Substituting back and rearranging, we find an expression for the first term of (C.311)

$$\overline{\mathbb{E}}\left[\int_0^{\tau^*} Y_s\,\mathrm{d}s\right] = \frac{e^{-(1-\alpha)\hat{k}^*}\left(1 - \overline{\mathbb{E}}\left[e^{-\mathcal{U}\tau^* + (1-\alpha)\Delta\hat{k}}\right]\right)}{\phi} \tag{C.317}$$

Using this term, we solve for $\hat{k}^*$ from (C.311) to get:

$$p = \alpha\frac{e^{-(1-\alpha)\hat{k}^*}\overline{\mathbb{E}}\left[1 - e^{-\mathcal{U}\tau^* + (1-\alpha)\Delta\hat{k}}\right]}{\phi} + p\overline{\mathbb{E}}\left[e^{-\mathcal{U}\tau^*}\right] \tag{C.318}$$

$$e^{(1-\alpha)\hat{k}^*} = \alpha\frac{1 - \overline{\mathbb{E}}\left[e^{-\mathcal{U}\tau^* + (1-\alpha)\Delta\hat{k}}\right]}{\phi p\left(1 - \overline{\mathbb{E}}\left[e^{-\mathcal{U}\tau^*}\right]\right)} \tag{C.319}$$

$$\hat{k}^* = \frac{1}{1-\alpha}\log\left(\frac{\alpha}{\phi p}\frac{1 - \overline{\mathbb{E}}\left[e^{-\mathcal{U}\tau^* + (1-\alpha)\Delta\hat{k}}\right]}{1 - \overline{\mathbb{E}}\left[e^{-\mathcal{U}\tau^*}\right]}\right) \tag{C.320}$$

$$\hat{k}^* = \frac{1}{1-\alpha}\left(\Phi - \log p + \log\frac{1 - \overline{\mathbb{E}}\left[e^{-\mathcal{U}\tau^* + (1-\alpha)\Delta\hat{k}}\right]}{1 - \overline{\mathbb{E}}\left[e^{-\mathcal{U}\tau^*}\right]}\right) \tag{C.321}$$

where $\Phi \equiv \log(\alpha/\phi) = \log(\alpha) - \log\left(\mathcal{U} - (1-\alpha)\nu - (1-\alpha)^2\sigma^2/2\right)$.

C.10.2 With irreversibility

With irreversibility we characterize the two reset states as a function of the data. The investment price is now a function of the investment sign. Moreover, since future reset points are unknown and to avoid confusion, we distinguish between past and future investments by denoting with primes future variables, such as $\hat{k}'_\tau$ and $\Delta\hat{k}'$.

Using the sequential formulation of the Tobin's q

$$q(\hat{k}) = \mathbb{E}\left[\int_0^{\tau}\frac{\alpha e^{-\mathcal{U}s + (\alpha-1)\hat{k}_s}}{p}\,\mathrm{d}s + e^{-\mathcal{U}\tau}Q^*\left(\Delta\hat{k}\right)\right] \tag{C.322}$$

Because there are two reset points, at this step, we must condition on the appropriate initial condition to evaluate the previous condition. If the last reset point was $\hat{k}_0 = \hat{k}^{*-}$ (there was a capital purchase), then the optimality condition is $q(\hat{k}^{*-}) = 1$:

$$\text{(C.323)} \qquad 1 \;=\; \frac{\alpha}{p}\overline{\mathbb{E}}^{-}\left[\int_0^{\tau^*} e^{-\mathcal{U}s-(1-\alpha)\hat{k}_s}\,\mathrm{d}s \;+\; Q(\Delta\hat{k}')e^{-\mathcal{U}\tau^*}\right].$$

If the last reset point is $\hat{k}_0 = \hat{k}^{*+}$ (there was a capital sale), then the optimality condition is $q(\hat{k}^{*+}) = 1-\omega$:

$$\text{(C.324)} \qquad 1-\omega \;=\; \frac{\alpha}{p}\overline{\mathbb{E}}^{+}\left[\int_0^{\tau^*} e^{-\mathcal{U}s-(1-\alpha)\hat{k}_s}\,\mathrm{d}s \;+\; Q(\Delta\hat{k}')\;e^{-\mathcal{U}\tau^*}\right].$$

where $\overline{\mathbb{E}}^{+} = \overline{\mathbb{E}}[\cdot|k^{*+}, u_0]$ denotes expectations conditional on a negative investment.

To express these moments in terms of microdata, we follow similar steps as in the proof without irreversibility but taking into account that investments happen at two different reset points. Consider the optimality condition of a firm that has bought capital at a price p^{buy} to reset its capital-productivity ratio to $\hat{k}^{*-}$ in (C.323) (the proof for a capital purchase is analogous). As before, we define $Y_s = e^{-\mathcal{U}s-(1-\alpha)\hat{k}_s}$ and apply Ito's lemma: $\mathrm{d}Y_s = -\phi Y_s\,\mathrm{d}s - (1-\alpha)\sigma Y_s\,\mathrm{d}W_s$. Integrating both sides from 0 to τ^*, taking expectations conditional on a positive investment, i.e., with respect to the initial condition $k_0 = k^{*-}$, and using the OST to set the expectation of martingales to zero, we obtain:

$$\text{(C.325)} \qquad \frac{\overline{\mathbb{E}}^{-}\left[Y_0 - Y_{\tau^*}\right]}{\phi} \;=\; \overline{\mathbb{E}}^{-}\left[\int_0^{\tau^*} Y_s\,\mathrm{d}s\right].$$

Since $Y_{\tau^*} = e^{-\mathcal{U}\tau^*-(1-\alpha)\hat{k}'_\tau}$ and $Y_0 = e^{-(1-\alpha)\hat{k}^{*-}}$, then $Y_0 - Y_{\tau^*} = e^{-(1-\alpha)\hat{k}^{*-}} - e^{-\mathcal{U}\tau^*-(1-\alpha)\hat{k}'_\tau}$. We find a common factor but remain alert about the difference between current and future investments:

$$\text{(C.326)} \qquad Y_0 - Y_{\tau^*} \;=\; e^{-(1-\alpha)\hat{k}^{*-}}\left[1 - e^{-\mathcal{U}\tau^*-(1-\alpha)(\hat{k}'_\tau-\hat{k}^{*-})}\right].$$

Substituting back these difference into the LHS of (C.325),

$$\text{(C.327)} \qquad \frac{e^{-(1-\alpha)\hat{k}^{*-}}\overline{\mathbb{E}}^{-}\left[1 - e^{-\mathcal{U}\tau^*-\mathcal{U}(\hat{k}'_\tau-\hat{k}^{*-})}\right]}{\phi} \;=\; \overline{\mathbb{E}}^{-}\left[\int_0^{\tau^*} Y_s\,\mathrm{d}s\right].$$

Now, we substitute this expression into the first term of (C.323) and multiply and divide the second term by p^{buy} to get:

$$\text{(C.328)} \qquad 1 \;=\; \frac{\alpha}{p^{\text{buy}}}\frac{e^{-(1-\alpha)\hat{k}^{*-}}\overline{\mathbb{E}}^{-}\left[1 - e^{-\mathcal{U}\tau^*-(1-\alpha)(\hat{k}'_\tau-\hat{k}^{*-})}\right]}{\phi} \;+\; \overline{\mathbb{E}}^{-}\left[Q(\Delta\hat{k}')e^{-\mathcal{U}\tau^*}\right]$$

$$\text{(C.329)} \qquad p^{\text{buy}} \;=\; \alpha\frac{e^{-(1-\alpha)\hat{k}^{*-}}\overline{\mathbb{E}}^{-}\left[1 - e^{-\mathcal{U}\tau^*-(1-\alpha)(\hat{k}'_\tau-\hat{k}^{*-})}\right]}{\phi} \;+\; p^{\text{buy}}\overline{\mathbb{E}}^{-}\left[\frac{p(\Delta\hat{k}')}{p^{\text{buy}}}e^{-\mathcal{U}\tau^*}\right]$$

$$\text{(C.330)} \qquad e^{(1-\alpha)\hat{k}^{*-}} \;=\; \frac{\alpha}{\phi p^{\text{buy}}}\frac{1-\overline{\mathbb{E}}^{-}\left[e^{-\mathcal{U}\tau^*-(1-\alpha)(\hat{k}'_\tau-\hat{k}^{*-})}\right]}{1-\overline{\mathbb{E}}^{-}\left[\frac{p(\Delta\hat{k}')}{p^{\text{buy}}}e^{-\mathcal{U}\tau^*}\right]}$$

$$\text{(C.331)} \qquad \hat{k}^{*-} \;=\; \frac{1}{1-\alpha}\left(\Phi \;-\; \log p^{\text{buy}} \;+\; \log\frac{1-\overline{\mathbb{E}}^{-}\left[e^{-\mathcal{U}\tau^*-(1-\alpha)(\hat{k}'_\tau-\hat{k}^{*-})}\right]}{1-\overline{\mathbb{E}}^{-}\left[\frac{p(\Delta\hat{k}')}{p^{\text{buy}}}e^{-\mathcal{U}\tau^*}\right]}\right)$$

The previous expression characterizes the reset point after a positive investment. As a final step, noting that

$\hat{k}'_\tau = \hat{k}^*(\Delta\hat{k}') - \Delta\hat{k}'$, we can rewrite (C.331) as:

$$\hat{k}^{*-} = \frac{1}{1-\alpha}\left(\Phi - \log p^{\text{buy}} + \log \frac{1-\overline{\mathbb{E}}^{-}\left[e^{-(r+\xi)\tau^* - (1-\alpha)(\hat{k}^*(\Delta\hat{k}') - \hat{k}^{*-} - \Delta\hat{k}')}\right]}{1-\overline{\mathbb{E}}^{-}\left[\frac{p(\Delta\hat{k}')}{p^{\text{buy}}} e^{-(r+\xi)\tau^*}\right]}\right) \tag{C.332}$$

With one reset point, $\hat{k}^*(\Delta\hat{k}') = \hat{k}^*$ and the expression collapses to that in (C.321). Here, because reset points might be different, $\hat{k}^{*-}$ appears on both sides of (C.331); thus, it is only characterizes it implicitly. We propose an iterative method to compute this value from the microdata.

With analogous steps, we obtain the reset point for negative investments:

$$\hat{k}^{*+} = \frac{1}{1-\alpha}\left(\Phi - \log p^{\text{sell}} + \log \frac{1-\overline{\mathbb{E}}^{+}\left[e^{-(r+\xi)\tau^* - (1-\alpha)(\hat{k}^*(\Delta\hat{k}') - \hat{k}^{*+} - \Delta\hat{k}')}\right]}{1-\overline{\mathbb{E}}^{+}\left[\frac{p(\Delta\hat{k}')}{p^{\text{sell}}} e^{-(r+\xi)\tau^*}\right]}\right) \tag{C.333}$$

Given the two reset points, the distance between them (the length of the inner inaction region) equals:

$$\hat{k}^{*+} - \hat{k}^{*-} = \frac{1}{1-\alpha}\left(\log\frac{1}{1-\omega} + \log \frac{1-\overline{\mathbb{E}}^{+}\left[e^{-(r+\xi)\tau^* - (1-\alpha)(\hat{k}^*(\Delta\hat{k}') - \hat{k}^{*+} - \Delta\hat{k}')}\right]}{1-\overline{\mathbb{E}}^{-}\left[e^{-(r+\xi)\tau^* - (1-\alpha)(\hat{k}^*(\Delta\hat{k}') - \hat{k}^{*-} - \Delta\hat{k}')}\right]} - \log \frac{1-\overline{\mathbb{E}}^{+}\left[\frac{p(\Delta\hat{k})}{p(1-\omega)} e^{-(r+\xi)\tau^*}\right]}{1-\overline{\mathbb{E}}^{-}\left[\frac{p(\Delta\hat{k})}{p} e^{-(r+\xi)\tau^*}\right]}\right)$$

D A General Equilibrium Framework

This section provides a general equilibrium model that microfounds the parsimonious investment model presented, allowing for examining macroeconomic fluctuations. Its core components are a small open economy and "capital quality shocks."

D.1 Economic environment

Time is continuous, and it extends forever. Four types of agents live in the economy: (i) A representative household, (ii) a capital–goods producer, (iii) a final–good producer, and (iv) a unit mass of intermediate–good firms indexed by $f \in [0,1]$ who are subject to capital adjustment frictions.

(i) Representative household. The household chooses the stochastic processes for consumption C_s, risk-free bonds B_s, and equity for each firm E_{fs}, subject to the law of motion for nominal wealth:

$$W_s = \int_0^1 P_{ft} E_{ft} \, \mathrm{d}f + B_t, \tag{D.1}$$

$$\mathrm{d}B_s + \int_0^1 P_{fs} \, \mathrm{d}E_{fs} \, \mathrm{d}f = (\mathcal{Y}_s - C_s) \, \mathrm{d}s, \tag{D.2}$$

where P_{fs} is the price of equity for firm f and $\mathcal{Y}_s$ is the after-tax available income, given by:

$$\mathcal{Y}_s = \left(\int_0^1 D_{fs} E_{fs} \, \mathrm{d}f + \tilde{\rho}_s B_s \right). \tag{D.3}$$

Here, D_{fs} represents firm f's dividend payments, and $\tilde{\rho}_s$ is the world interest rate. We omit the profits of the final-good producer and the capital-good producer, as they exhibit constant returns to scale and do not generate profits for the household. Thus, we exclude those sectors' profits from the household budget constraint. Taking the prices of equity $\{P_{fs}\}_{fs}$ and the real interest rate $\tilde{\rho}_s$ as given, the household's problem is to maximize its expected utility (discounted at rate χ):

$$\max_{\{C_s, B_s, \{E_{ft}\}_f\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\int_0^{\infty} e^{-\chi t} \log C_s \, \mathrm{d}s \right], \tag{D.4}$$

subject to the budget constraints in (D.2) and (D.3), and the initial conditions B_0 and $\{E_{f0}\}_f$.

(ii) Capital–good producer. The capital–good producer manufactures firm-specific investment goods $\{i_{ft}\}_{f \in [0,1]}$ in a competitive market according to a linear technology:

$$\int_0^1 \left(\frac{\varphi(i_{ft}) i_{fs}}{u_{fs}} \right) \mathrm{d}f = i_s, \tag{D.5}$$

where

$$\varphi(i_{ft}) = \begin{cases} \varphi^- & \text{if } i_{ft} > 0 \\ \varphi^+ & \text{if } i_{ft} \leq 0 \end{cases}. \tag{D.6}$$

We refer to u_{fs} as capital quality shocks. The parameters φ^- and φ^+ measure the level of partial irreversibility, with $\varphi^- > \varphi^+$. Taking the prices of firm-specific investment goods p^k_{js} as given, the capital–good firm maximizes its profits:

$$\max_{\{i_{ft}, i_t\}_{t=0}^{\infty}} \left(\int_0^1 p^k_{fs} i_{fs} \, \mathrm{d}f - i_s \right), \tag{D.7}$$

subject to the technology described in (D.5). Here, i_s is the aggregate investment to produce capital. Note that i_{ft} may be positive or negative, as its sign has no technological constraint.

(iii) Final–good producer. The final–good producer assembles output Y_s using intermediate inputs $\{\hat{y}_{fs}\}_{f\in[0,1]}$ according to a linear aggregator:

$$Y_s = \int_0^1 \left(\frac{\hat{y}_{fs}}{u_{fs}} \right) \mathrm{d}f, \tag{D.8}$$

where capital quality u_{fs} decreases the marginal product of the intermediate good f. Taking the prices of intermediate inputs p_{ft} as given, the producer's problem entails choosing final–good supply Y_s and input demands $\hat{y}_{fs}$ to maximize profits:

$$\max_{Y_s, \hat{y}_{fs}} \left(Y_t - \int_0^1 p_{ft} \hat{y}_{ft} \, \mathrm{d}f \right), \tag{D.9}$$

subject to the aggregator in (D.8).

(iv) Intermediate-good firms. These are the most important economic agents for our question as they make investment choices subject to adjustment costs. Intermediate–good firm $f \in [0,1]$ produces output y_{fs} using capital k_{fs} according to a production function with decreasing returns to scale:

$$y_{fs} = u_s^{1-\alpha} k_{fs}^{\alpha}, \quad \alpha < 1. \tag{D.10}$$

An idiosyncratic component drives the firm's total productivity:

$$\mathrm{d}\log(u_{fs}) = \mu \, \mathrm{d}s + \sigma \, \mathrm{d}W_{fs}, \qquad W_{fs} \sim \text{Wiener}, \tag{D.11}$$

where the processes W_{ft} are independent across intermediate–good firms. The profit rate is given by:

$$\pi_{fs} = p_{fs} y_{fs}. \tag{D.12}$$

Taking the prices of the intermediate goods p_{ft}, the marginal investor discount factor Q_t, and firm-specific capital goods $p^k_{ft}(\hat{i})$ as given, together with the adjustment friction θ_{ft}, each firm f chooses a sequence of capital adjustment dates $\{T_{fh}\}_{h=1}^{\infty}$ and investments $\{i_{f,T_{fh}}\}_{h=1}^{\infty}$ to maximize its expected discounted stream of profits:

$$\max_{\{T_{fh}, \hat{i}_{f,T_{fh}}\}_{h=1}^{\infty}} \mathbb{E} \left[\int_0^{\infty} Q_t \pi_{ft} \, \mathrm{d}s - \sum_{h=1}^{\infty} Q_{T_{fh}} p^k_{f,T_{fh}} \left(\theta_{fT_{fh}} + p^k_{fT_{fh}}(\hat{i}_{fT_{fh}}) \hat{i}_{fT_{fh}} \right) \right], \tag{D.13}$$

subject to the profit function in (D.12) and the law of motion for its capital stock:

$$\log(k_{ft}) = \log(k_{f0}) - \zeta t + \sum_{h: T_{fh} \le t} \log \left(1 + \frac{\hat{i}_{f,T_{fh}}}{k_{T_{fh}^-}} \right). \tag{D.14}$$

Market structure. There are three types of goods (respectively, markets) in the economy: (i) final goods, (ii) intermediate goods, and (iii) firm-specific investment goods. There are two assets: (i) risk-free bonds and (ii) equity. All good and asset markets are competitive. We assume equity can only be held by the representative household. Thus, we have segmented the equity market, and the bond market freely trades across countries. The market clearing conditions, respectively, are as follows:

(D.15) $$E_{fs} = 1 \quad \text{for all } t \text{ and } f,$$

(D.16) $$\hat{y}_{fs} = y_{fs} \quad \text{for all } s \text{ and } f,$$

(D.17) $$\hat{i}_{fs} = i_{fs} \quad \text{for all } s \text{ and } f.$$

Equilibrium. *Given a stochastic processes for capital quality* $\{u_{fs}\}_{fs}$*, and adjustment costs* θ_{ft}*, an equilibrium is a set of stochastic processes for prices* $\{\tilde{\rho}_s, \{p_{fs}, p^k_{fs}(i), P_{fs}\}_{f\in[0,1]}\}_{s=0}^{\infty}$*, the household's policy* $\{C_s, B_s, \{E_{fs}\}_{f\in[0,1]}\}_{t=0}^{\infty}$*, the final–good producer's policy* $\left\{Y_s, \{\hat{y}_{fs}\}_{f\in[0,1]}\right\}_{t=0}^{\infty}$*, the capital–good producer's policy* $\left\{\{i_{fs}\}_{f\in[0,1]}, i_s\right\}_{t=0}^{\infty}$*, and the intermediate–good firms' policy* $\left\{\{T_{fh}, i_{f,T_{hf}}\}_{h=1}^{\infty}\right\}$ *such that:*

(i) Given prices $\{\tilde{\rho}_s, P_{fs}\}$*, the household solves* (D.4).

(ii) Given prices $\{p^k_{fs}\}$*, the capital–good producer solves* (D.7).

(iii) Given prices $\{p_{fs}\}$*, the final–good producer solves* (D.9).

(iv) Given prices $\{Q_s, p_{fs}, p^k_{fs}\}$*, intermediate–good firms solve* (D.13).

(v) Market clears in (D.15) *to* (D.17).

D.2 Equilibrium characterization

We now describe the equilibrium determination of prices and quantities in that order. From now on, we assume that the world interest rate is constant: $\tilde{\rho}_s = \tilde{\rho}$. We derive the aggregate macroeconomic outcomes from their individual counterparts.

Equilibrium determination of prices. The household's optimality conditions over bonds and equity are:

(D.18) $$\begin{aligned} \tilde{\rho}\,\mathrm{d}s &= \chi\,\mathrm{d}s - \frac{\mathrm{d}(1/C_s)}{1/C_s} \quad \forall s \\ \frac{\mathbb{E}[dP^i_{fs}] + D^i_{fs}\,\mathrm{d}s}{P^i_{fs}} &= \chi\,\mathrm{d}s - \frac{\mathrm{d}(1/C_s)}{1/C_s} \quad \forall s, f \end{aligned}$$

The differential equations in (D.18) jointly imply a unique equilibrium for the price of equity. Under the equilibrium condition of unit supply of equity in (D.15), we find:

$$V_0 = P_0 = \mathbb{E}_0\left[\int_0^{\infty} e^{-\tilde{\rho}s} D_s\,\mathrm{d}s\right].$$

Finally, the zero-profit conditions for the final- and capital-good producers imply the following relationships for the input and output prices of the respective goods:

$$p_{ft} = \frac{1}{u_{ft}} \quad ; \quad p^k_{ft}(i) = p_{ft}\varphi(i),$$

where $\varphi(i) = \varphi^+ \mathbf{1}_{i<0} + \varphi^- \mathbf{1}_{i>0}$, and $p^k_{ft}(i)$ represents the relative price of capital.

Equilibrium policy of intermediate good firms. With these facts about equilibrium prices established, we turn to the problem facing an individual intermediate–good firm. Let $V(k, u)$ be the value of a firm with capital k and productivity u. The sufficient optimality conditions satisfied by a firm's policy are (i) the HJB equation valid during periods of inactivity, (ii) the value matching conditions, and (iii) the smooth pasting conditions. The firm policy consists of an inaction region $\mathcal{R} \equiv \{(k, u) : k^-(u) \le k \le k^+(u)\}$, where $k^-(u)$ and $k^+(u)$ are the lower and upper inaction thresholds, together with reset capitals $k^{*-}(u)$ and $k^{*+}(u)$ for positive and negative investments upon adjustment.

Let $r \equiv \tilde{\rho} - \mu$ (without subtracting $\sigma^2/2$, in contrast to the main text) be the adjusted discount factor and let $v(\hat{k}) : \mathbb{R} \to \mathbb{R}$ be a function of the log capital-productivity ratio equal to

$$v(\hat{k}) = \max_{\tau, \Delta\hat{k}} \mathbb{E}\left[\int_0^\tau e^{-rs+\alpha\hat{k}_s}\, \mathrm{d}s + e^{-r\tau}\left(-\theta - p(\Delta\hat{k})(e^{\hat{k}_\tau + \Delta\hat{k}} - e^{\hat{k}_\tau}) + v(\hat{k}_\tau + \Delta\hat{k})\right)\Big|\hat{k}_0 = \hat{k}\right] \tag{D.19}$$

where the price function with taxes is given by:

$$p(i_s) = \left(\varphi^- \mathbf{1}_{i_s>0} + \varphi^+ \mathbf{1}_{i_s<0}\right). \tag{D.20}$$

Then the firm value equals $V_0 = v(\hat{k}_0)$.

A few remarks about the firms' investment policy are in order. The formulations of the capital quality shocks and the adjustment costs allow us to collapse the state-space of the firms (k, u) into the capital-to-productivity ratio $\hat{k} = k/u$. Note that the value of the firm $v(\hat{k}_0)$ is not scaled by the level of productivity to recover the time-0 value V_0. The prices of intermediate goods p_{ft} and capital goods p^k_{ft}, as well as the adjustment costs θ_{ft}, are proportional to capital quality u_{ft}, making profits and investment scaled by total productivity the relevant variables for the firm.

Equilibrium determination of macroeconomic outcomes. With equilibrium prices and firms' policies, we can determine equilibrium aggregate quantities. Lemma D.1 characterizes the equilibrium detrended aggregate quantities: It shows that all aggregates are functions of the distribution of capital-to-productivity ratios.

Lemma D.1. *Let $g(\hat{k})$ be the density of capital-to-productivity ratios and define the following expectations:* $\mathbb{E}\left[\exp(\hat{k})\right] \equiv \int_{\hat{k}^-}^{\hat{k}^+} \exp(\hat{k}) g(\hat{k})\, \mathrm{d}\hat{k}$. *Then, the equilibrium aggregate output is*

$$\hat{Y}_t \quad \equiv \quad \int_0^1 p_{ft} y_{ft}\, \mathrm{d}f \quad = \quad \mathbb{E}\left[\exp(\alpha\hat{k}_t)\right], \tag{D.21}$$

We cannot sum firms' capital since they are different goods. Still, as in the main text, the capital-productivity ratio is the only input to determine output in this economy. Thus, we define aggregate capital as

$$\hat{K}_t = \mathbb{E}\left[\exp(\hat{k}_t)\right]. \tag{D.22}$$

Equations (D.21) and (D.22) show an important property in this economy: Without fixed costs of adjustment and partial irreversibility, the supply side of this model collapses to a neoclassical firm with technology $\hat{Y} = \hat{K}^\alpha$. Doing a first-order approximation on equation (D.22), it follows the CIR(δ) definition in 32.

With these facts over aggregate quantities, we now describe misallocation. Misallocation is defined as the dispersion of the log of productivity-weighted marginal revenue given by

$$\mathbb{V}\left[\log\left(u_f \frac{\mathrm{d}p_f y_f}{\mathrm{d}k_f}\right)\right] = \mathbb{V}\left[\log\left(\frac{u_f}{k_f}\left(\frac{k_f}{u_f}\right)^\alpha\right)\right] = (1-\alpha)\mathbb{V}\left[\hat{k}\right]. \tag{D.23}$$

The argument for why we need to weigh according to capital quality comes from the technology to produce investment. If the transformation rate from consumption goods to firm-specific investment goods is one, then there is no need to weigh it with capital. This is not the case in our economy since the transformation rate from consumption goods to firm-specific investment goods is given by u_f. This is why we need to weigh the idiosyncratic capital quality shocks to obtain misallocation or productivity in this economy.

D.3 Remarks on the economic framework

General equilibrium structure Capital quality u_{ft} was first used by Baley and Blanco (2021) in the investment context. In the pricing literature, an analogous formulation was first employed by Woodford (2009) to maintain the tractability of their model. It is also used by Midrigan (2011), Alvarez and Lippi (2014), Baley and Blanco (2019), and Blanco (2020), among others. This formulation implies that aggregate feasibility depends only on firms' capital-to-productivity ratios rather than capital and productivity separately. As a result, capital quality shocks reduce the dimensionality of the aggregate state space from the joint distribution of capital and productivity to the distribution of their ratio.

Partial irreversibility The price wedge is a technological constraint for the capital–good producer and, therefore, is exogenous. This formulation follows Veracierto (2002) and Khan and Thomas (2008). Alternatively, partial irreversibility could be the outcome of distortionary taxation. For example, Chen *et al.* (2023) uses China's 2009 VAT reform to study changes in the level of partial irreversibility. It would be straightforward to extend our framework to micro-found partial irreversibility as an outcome of a tax system, as in Chen *et al.* (2023). See Lanteri (2018) for a model that endogenizes partial irreversibility.

Financial markets We assume that the representative household can trade in the bond market, but the economy is closed to the equity market. Only the households in the small open economy own firms, providing the firms' discount factor. While these are extreme assumptions, they are a reasonable approximation for small, open economies. Empirically, it is well known that central banks, firms, and households in emerging economies tend to save in dollar-denominated risk-free assets (e.g., T-bills). Moreover, despite the globalization of finance and financial institutions, market participants commonly allocate most of their wealth to domestic assets. This *home bias* may be due to regulatory constraints, information, and transaction costs, though some attribute it to preferences. While the current version of the model represents an extreme form of the *home bias* phenomenon, it provides a useful starting point for analyzing business cycle fluctuations in an investment model.

Tractability Given the novelty of the general equilibrium framework, further discussion of the assumptions and economic adjustment mechanisms is warranted. The tractability of our framework arises from three main features. First, all aggregate variables are expressed regarding the distribution of capital-to-productivity ratios. This result follows from the introduction of capital quality shocks and the structure of capital adjustment costs. Second, the model produces a constant real interest rate due to the small open economy assumption. Third, the closed equity market assumption allows us to determine the firms' discount factor as a function of the world interest rate.

The theory developed in the main text assumes that the cross-sectional distribution of capital-to-productivity ratios is the relevant aggregate state and that the interest rate is exogenous. The general equilibrium framework presented here provides a microfoundation for the model in the main text.

E Establishment-level investment data

This section describes the sources, the construction of variables, and the filters we apply to clean the data to construct the investment series at the firm level in order

E.1 Source, description and data cleaning

Data come from the *Encuesta Nacional Industrial Anual* (ENIA). The sample period covers 31 years, from 1980 to 2011, with an average of 543 manufacturing plants per year. We have a total number of plant-year observations of 154,591.

1. First, we drop the 3,984 permanently small firms (i.e. with less than 10 workers throughout the sample period, 4% of the sample). This filter is motivated by the lack of good quality data with respect to these firms since ENIA is directed to plants with more than 10 workers.
2. Second, we drop 5,343 observations with a non-positive total value of book capital, wage bill or sales.
3. Third, we drop 12,161 observations that had a frequency of non-zero investment lower than 10% of the sample period.
4. Finally, we drop 5,556 plants with less than 3 years of coverage (6% of the sample). Note that we consider new plants (and give a new ID) that disappear from the sample more than three years and reappear in the sample after that.
5. In total, we drop about 18% of the year-plant observations and keep 127,631 observations. Within this remaining sample, a balanced panel would maintain 101,160 plant-years.

Table E.1 summarizes the cleaning process and shows the number of observations dropped at each step.

Table E.1: Data cleaning

Description	**Chile**
Start year	1980
End year	2011
Avg. number of plants per year	543
Plant-year observations	154,591
Cleaning	**Removed observations**
Less than 10 employees	3,984
Non-positive wage bill, capital, or sales	5,343
Frequency of non-zero investment less than 10	12,161
Less than 3 years of coverage	5,556
Remaining observations	127,631
% of total	82.6
With more than 10 years of data	101,160
% of remaining observations	79.3

Sources: Authors' calculations using establishment-level survey data from Chile. Less than 10 employees refers to plants with less than 10 employees for all the years in the sample.

E.2 Perpetual Inventory Method

To deal with reporting and measurement errors in the surveys, we construct capital series using the standard perpetual inventory method (PIM) with the addition of an investment price wedge.

Capital stocks. Let firm i's stock of capital on year t be given by:

$$k_{i,t} = (1-\xi^k)k_{i,t-1} + \frac{I_{i,t}}{p(I_t)D_t} \quad \text{for } k_{i,t_0} \text{ given.} \tag{E.1}$$

We consider the following elements to construct the capital series:

- Capital types considered are $j \in \{\text{structures}, \text{machinery and equipment}, \text{vehicles}\}$.
- Gross investment: $I_{i,j,t}$ is the gross nominal investment into the capital of type j at time t, and it is based on the information on purchases, reforms and improvements, and sales of fixed assets reported by each plant in the ENIA and EAM data sets.

$$I_{i,j,t} = puchases_{i,j,t} + reforms_{i,j,t} + improvements_{i,j,t} - sales_{i,j,t} \tag{E.2}$$

- Depreciation rate: $\xi^k = 0.09$ is a the depreciation rate.
- Price deflators: $D_{j,t}$ are gross fixed capital formation deflators by capital type from Penn World Tables (PWT).
- Investment prices and wedge.
- Initial capital: K_{i,j,t_0} is given by:

$$K_{i,j,t_0} = \frac{\tilde{K}_{i,j,t_0}}{D_{t_0}} \quad \text{if } \tilde{K}_{i,j,t_0} \geq 0, \tag{E.3}$$

where $\tilde{K}_{i,j,t_0}$ is firm i's self-reported nominal stock of capital of type j at current prices on the starting year $t_0 = t_{0,i,j}$, which is the first year in which firm i reports a non-negative capital stock of type j.

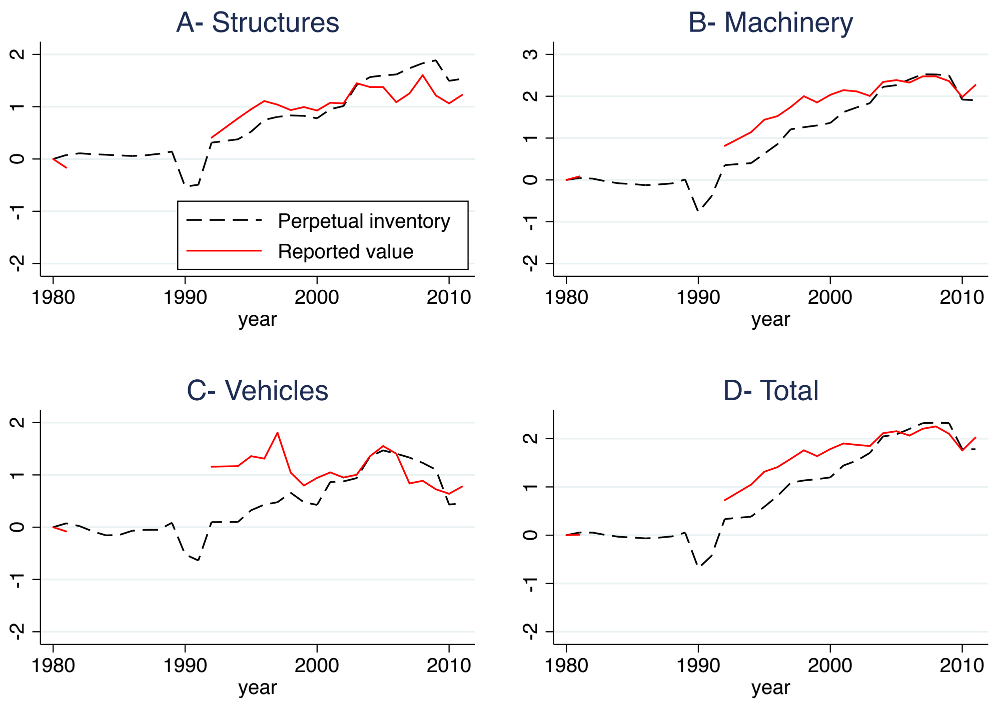
Investment rates. Once we construct the investment and capital stock series, we generate the investment rate $i_{i,j,t}$ by dividing investment by initial capital:

$$i_{i,j,t} = \frac{I_{i,j,t}}{K_{i,j,t-1}}, \tag{E.4}$$

Outliers. Once we generate the series of investment rates, we eliminate investment rates below the 2nd percentile and above the 98th percentile of the investment rate distribution.

Figure E.1 plots the aggregate capital stock computed with the perpetual inventory method and compares it to the reported book value. In the aggregate, we observe that the reported book value is consistent with the PIM series for each capital type and the total stock. This shows the sound quality of the micro-data. Moreover, given the similarity in the series, we validate our choice of using the initial book value reported by the plant as the initial condition for the PIM construction.

Figure E.1: **Chile:** Reported Book Value vs Perpetual Inventory



Notes: Aggregate capital stock in Chile's manufacturing sector reported by plants and computed through the PIM. All the variables are in logs and real terms, normalized to zero in 1980.

E.3 Comparison with National Accounts

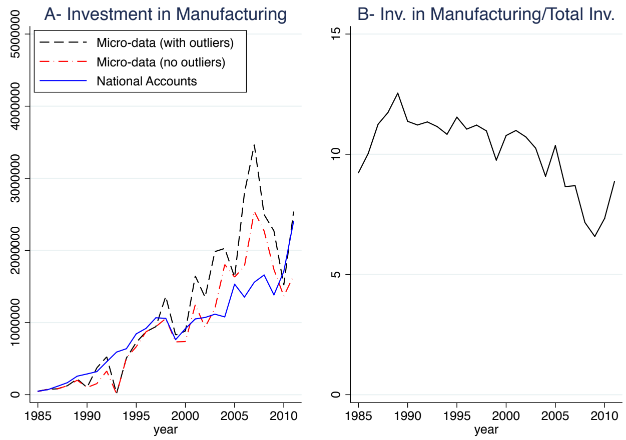
This section verifies that the information in the survey data is consistent with aggregated information from National Accounts.

The national account office in Chile uses the ENIA survey to compute several indices, such as variations in inventories or value added by type of industry. Nevertheless, National Accounts does not use ENIA to compute total investment or investment in the manufacturing sector; for that purpose, it uses sources related to the supply of capital goods (i.e., balance of payments, National Statistical Institute, Corporacion de Desarrollo Tecnologico de Bienes de Capital). Therefore, National Accounts serve as an orthogonal source to verify that the micro-data from the survey is consistent with the total investment in the manufacturing sector.

Panel A of Figure E.2 describes the total nominal investment constructed from the ENIA (dashed black line) and the total nominal investment in the manufacturing sector constructed using National Accounts (solid blue line), in current millions of pesos. As we can see, the two series are very close, with a correlation of 0.62. Total investment from the micro-data for the period 2005-2009 seems to grow much faster than National Accounts, but we found that a few outliers mainly explain this. For example, suppose we drop observations with investment rates larger than 5% of aggregate investment (dashed-dotted red line). In that case, the fit between the aggregate investment from the micro-data and the national account is much better, both in levels and cyclicality. Finally, Panel B of E.2 describes the proportion of total investment done in the manufacturing sector, which represents on average 7% in the sample period but has been declining.

For 2003-2009, the National Accounts calculates the investment distribution by capital types at the sector level. Table E.2 describes the composition of capital across different types from the ENIA and National Accounts.

Figure E.2: **Chile**: Micro-data vs. National Accounts



Notes: Panels A describes investments in the manufacturing sector. Black dashed line plots aggregate nominal investment constructed from ENIA, the red dashed-dotted line plots the same variable but dropping outliers (i.e., investment larger than 5% of aggregate investment), and the blue solid line plots the total investment from National Accounts. Panel B describes investment in the manufacturing sector over total investment. Nominal investment from the national account uses the concatenated investments from the base year 2015.

The proportions invested in structures are similar between National Accounts and ENIA, but the decomposition between machinery and vehicles differs across datasets.

Table E.2: Chile: Distribution of Investment Across types of Capital

	Structures	Machinery	Vehicles
National Accounts	35.4	51.4	13.1
ENIA	29.1	68.6	2.1

Notes: Proportion of investment across different types of capital in the ENIA and national accounts.

E.4 Mappings from microdata to macro outcomes

This section describes the application of the theory with producer-level investment data. Let I_{ft} be the nominal investment in period t for firm f. Below, we show the steps to compute all the macroeconomic outcomes described in Table II.

(I) **Construction of** $\left\{\Delta\hat{k}_{fh}, \tau_{fh}\right\}_{fh}$

- We follow subsection E.1 to drop observations that do not satisfy a set of criteria (e.g., positive sales, wage bill, minimum number of year, etc.).
- We apply the perpetual inventory method to construct the capital stock k_{ft} of firm f in period t

$$k_{ft} \;=\; (1-\xi)k_{ft-1} \;+\; I_{ft}/(p(I_{ft})D_t), \tag{E.1}$$

  where ξ is the physical depreciation rate; I_{ft} is the nominal value of the investment; $p(I_{ft})$ is the investment pricing function, which considers different prices for capital purchases and sales; D_t is the gross fixed capital formation deflator, and k_{f0} is a plant's self-reported nominal capital stock at current prices for the first year enter in the data or its firms investment level.

- We construct the change in the capital-productivity ratio upon action $\Delta\hat{k}_h$ as

$$\Delta\hat{k}_{fh} \;=\; \begin{cases} \log\left(1+\frac{I_{ft}/(p(I_{ft})D_t)}{k_{ft-1}}\right) & \text{if} \quad |\iota_h| > \underline{\iota}, \\ 0 & \text{if} \quad |\iota_h| < \underline{\iota}. \end{cases} \tag{E.2}$$

- With $\Delta\hat{k}_{fh}$, we construct τ_{fh} as the number of periods between non-zero investments.

(II) **Construction of weights** ω_f Unobserved heterogeneity in the frequency of non-zero investment can generate a bias in estimating moments relevant to the theory. Let N_f be the adjustment frequency of firm f. We construct weights ω_f as the inverse frequency of non-zero investments:

$$\omega_f \propto \frac{1}{N_f}. \tag{E.3}$$

We normalized ω_f s.t.

$$\sum_f \sum_{h=1}^{N_f} \omega_f = 1 \tag{E.4}$$

(III) **Estimation of** $(\hat{k}^{*-}, \hat{k}^{*+}, \nu, \sigma)$**:** First, we estimate the drift as:

$$\nu = \frac{\sum_{fh}\Delta\hat{k}_{fh}\omega_f}{\sum_{fh}\tau_{fh}\omega_f}, \tag{E.5}$$

With the estimate of the drift, we design an iterative method to estimate $(\hat{k}^{*-}, \hat{k}^{*+}, \sigma^2)$. The method constructs a sequence $(\hat{k}^{*-}_j, \hat{k}^{*+}_j, \sigma^2_j)_{j=0}^{\infty}$ that converges to the solution of the implicit equations (57), (65), and (66) from Section 4.

0. Fix a convergence parameter $\psi > 0$ and a dampening parameter $\Gamma \in (0, 1)$.

1. **Construct $(\hat{k}_0^{*-}, \hat{k}_0^{*+}, \sigma_0)$ assuming no irreversibility to compute σ_0 and the right-hand side of (65) and (66):** Without irreversibility, we can estimate σ_0^2 as

$$\sigma_0^2 = \frac{\sum_{fh} \Delta\hat{k}_{fh}^2 \omega_f}{\sum_{fh} \tau_{fh}\omega_f} - 2\nu \left(\frac{\sum_{fh} \Delta\hat{k}_{fh}\omega_f}{2}(1 - \mathbb{CV}^2[\tau]) + \frac{\mathbb{C}ov[\Delta hk, \tau]}{\sum_{fh} \tau_{fh}\omega_f} \right) \tag{E.6}$$

$$\mathbb{CV}^2[\tau] := \frac{\sum_{fh}(\tau_{fh} - \overline{\mathbb{E}}[\tau])^2\omega_f}{\left(\sum_{fh} \tau_{fh}\omega_f\right)^2}, \quad \text{where} \quad \overline{\mathbb{E}}[\tau] = \sum_{fh} \tau_{fh}\omega_f, \tag{E.7}$$

$$\mathbb{C}ov[\Delta\hat{k}, \tau] := \sum_{fh}(\tau_{fh} - \overline{\mathbb{E}}[\tau])(\Delta\hat{k}_{fh} - \overline{\mathbb{E}}[\Delta\hat{k}])\omega_f, \quad \text{where} \quad \overline{\mathbb{E}}[\Delta\hat{k}] = \sum_{fh} \Delta\hat{k}_{fh}\omega_f. \tag{E.8}$$

With σ_0^2, we compute

$$\Phi(\nu, \sigma_0^2) = \log\left(\frac{\alpha}{\mathcal{U} - (1-\alpha)\nu - \frac{\sigma_0^2(1-\alpha)^2}{2}} \right) \tag{E.9}$$

$$\hat{k}_0^{*-} = \frac{1}{1-\alpha}\left(\Phi(\nu, \sigma_0^2) - \log(p^{\text{buy}}) + \log\left(\frac{1 - \mathcal{N}um_0^-}{1 - \mathcal{D}en_0^-} \right) \right) \tag{E.10}$$

$$\hat{k}_0^{*+} = \frac{1}{1-\alpha}\left(\Phi(\nu, \sigma_0^2) - \log(p^{\text{sell}}) + \log\left(\frac{1 - \mathcal{N}um_0^+}{1 - \mathcal{D}en_0^+} \right) \right) \tag{E.11}$$

where the numerators and denominators in the last terms are computed as:

$$\mathcal{N}um_0^+ = \frac{\sum_{fh} \exp(-\mathcal{U}\tau'_{fh} + (1-\alpha)\Delta\hat{k}'_{fh}) I(\Delta\hat{k}_{fh} < 0)\omega_f}{\sum_{fh} I(\Delta\hat{k}_{fh} < 0)\omega_f} \tag{E.12}$$

$$\mathcal{D}en_0^+ = \frac{\sum_{fh} \exp(-\mathcal{U}\tau'_{fh}) I(\Delta\hat{k}_{fh} < 0)\omega_f}{\sum_{fh} I(\Delta\hat{k}_{fh} < 0)\omega_f} \tag{E.13}$$

$$\mathcal{N}um_0^- = \frac{\sum_{fh} \exp(-\mathcal{U}\tau'_{fh} + (1-\alpha)\Delta\hat{k}'_{fh}) I(\Delta\hat{k}_{fh} > 0)\omega_f}{\sum_{fh} I(\Delta\hat{k}_{fh} > 0)\omega_f} \tag{E.14}$$

$$\mathcal{D}en_0^- = \frac{\sum_{fh} \exp(-\mathcal{U}\tau'_{fh}) I(\Delta\hat{k}_{fh} > 0)\omega_f}{\sum_{fh} I(\Delta\hat{k}_{fh} > 0)\omega_f} \tag{E.15}$$

Define $\hat{k}_0^*(\Delta\hat{k})$ and $\hat{k}_{\tau,0}(\Delta\hat{k})$ as:

$$\hat{k}_0^*(\Delta\hat{k}) = \hat{k}_j^{*-}\mathbb{1}_{\{\Delta\hat{k}>0\}} + \hat{k}_j^{*+}\mathbb{1}_{\{\Delta\hat{k}<0\}}, \tag{E.16}$$

$$\hat{k}_{h,0}(\Delta\hat{k}) = \hat{k}_0^*(\Delta\hat{k}) - \Delta\hat{k}, \tag{E.17}$$

(E.18)

2. For $j = 1, 2, 3, \ldots$, compute $(\hat{k}_j^{*-}, \hat{k}_j^{*+}, \tilde{\sigma}_j^2)$ as

$$\tilde{\sigma}_j^2 = \frac{\sum_{fh}(\hat{k}^*_{\tau,j-1}(\Delta\hat{k}'_{fh}) + \nu\tau'_{fh})\omega_f - \sum_{fh}\hat{k}^*_{j-1}(\Delta\hat{k}_{fh})\omega_f}{\sum_{fh}\tau_{fh}\omega_f} \tag{E.19}$$

$$\Phi(\nu, \sigma_{j-1}^2) = \log\left(\frac{\alpha}{\mathcal{U} - (1-\alpha)\nu - \frac{\sigma_{j-1}^2(1-\alpha)^2}{2}}\right) \tag{E.20}$$

$$\hat{k}_j^{*-} = \frac{1}{1-\alpha}\left(\Phi - \log(p^{\text{buy}}) + \log\left(\frac{1-\mathcal{N}um_j^-}{1-\mathcal{D}en_j^-}\right)\right) \tag{E.21}$$

$$\hat{k}_j^{*+} = \frac{1}{1-\alpha}\left(\Phi - \log(p^{\text{sell}}) + \log\left(\frac{1-\mathcal{N}um_j^+}{1-\mathcal{D}en_j^+}\right)\right) \tag{E.22}$$

where the numerators and denominators in the last terms are computed as:

$$\mathcal{N}um_j^+ = \frac{\sum_{fh}\exp\left(-\mathcal{U}\tau'_{fh} + (1-\alpha)\left(\hat{k}^{*+} - \hat{k}^*_{\tau,j-1}(\Delta\hat{k}'_{fh})\right)\right)I(\Delta\hat{k}_{fh} < 0)\omega_f}{\sum_{fh}I(\Delta\hat{k}_{fh} < 0)\omega_f} \tag{E.23}$$

$$\mathcal{D}en_j^+ = \frac{\sum_{fh}\frac{p(\Delta\hat{k}'_{fh})}{p(\Delta\hat{k}_{fh})}\exp\left(-\mathcal{U}\tau'_{fh}\right)I(\Delta\hat{k}_{fh} < 0)\omega_f}{\sum_{fh}I(\Delta\hat{k}_{fh} < 0)\omega_f} \tag{E.24}$$

$$\mathcal{N}um_j^- = \frac{\sum_{fh}\exp\left(-\mathcal{U}\tau'_{fh} + (1-\alpha)\left(\hat{k}^{*-}_{j-1} - \hat{k}_{\tau,j-1}(\Delta\hat{k}'_{fh})\right)\right)I(\Delta\hat{k}_{fh} > 0)\omega_f}{\sum_{fh}I(\Delta\hat{k}_{fh} > 0)\omega_f} \tag{E.25}$$

$$\mathcal{D}en_j^- = \frac{\sum_{fh}\frac{p(\Delta\hat{k}'_{fh})}{p(\Delta\hat{k}_{fh})}\exp\left(-\mathcal{U}\tau'_{fh}\right)I(\Delta\hat{k}_{fh} > 0)\omega_f}{\sum_{fh}I(\Delta\hat{k}_{fh} > 0)\omega_f} \tag{E.26}$$

Update $(\hat{k}_j^{*-}, \hat{k}_j^{*+}, \sigma_j^2)$ with the dampening parameter Γ

$$\sigma_j^2 = (1-\Gamma)\tilde{\sigma}_j^2 + \Gamma\sigma_{j-1}^2 \tag{E.27}$$

$$\hat{k}_j^{*-} = (1-\Gamma)\tilde{\hat{k}}_j^{*-} + \Gamma\hat{k}_{j-1}^{*+} \tag{E.28}$$

$$\hat{k}_j^{*+} = (1-\Gamma)\tilde{\hat{k}}_j^{*+} + \Gamma\hat{k}_{j-1}^{*+} \tag{E.29}$$

3. Repeat step 2 until there is a j, such that, $||\left(\sigma_j^2 - \sigma_{j-1}^2, \hat{k}_j^{*-} - \hat{k}_{j-1}^{*-}, \hat{k}_j^{*+} - \hat{k}_{j-1}^{*+}\right)|| < \psi$.

(IV) We construct the reset points $\hat{k}^*(\Delta\hat{k})$ and stopped capitals $\hat{k}_\tau(\Delta\hat{k})$ as:

$$\hat{k}^*(\Delta\hat{k}) = \hat{k}^{*-}\mathbb{1}_{\{\Delta\hat{k}>0\}} + \hat{k}^{*+}\mathbb{1}_{\{\Delta\hat{k}<0\}} \tag{E.30}$$

$$\hat{k}_\tau(\Delta\hat{k}) = \hat{k}^*(\Delta\hat{k}) - \Delta\hat{k} \tag{E.31}$$

(V) Cross-sectional mean and variances: We estimate the unconditional mean and variances of capital–productivity ratios as

$$\mathbb{E}[\hat{k}] = \frac{\sum_{fh}\hat{k}^*(\Delta\hat{k}_{fh})^2\omega_f - \sum_{fh}\hat{k}_\tau(\Delta\hat{k}'_{fh})^2\omega_f}{2\sum_{fh}\Delta\hat{k}_{fh}\omega_f} + \frac{\sigma^2}{2\nu} \tag{E.32}$$

$$\mathbb{V}[\hat{k}] = \frac{\sum_{fh}(\hat{k}^*(\Delta\hat{k}_{fh}) - \mathbb{E}[\hat{k}])^3\omega_f - \sum_{fh}(\hat{k}_\tau(\Delta\hat{k}'_{fh}) - \mathbb{E}[\hat{k}])^3\omega_f}{3\sum_{fh}\Delta\hat{k}_{fh}\omega_f} \tag{E.33}$$

We can also compute conditional means

$$\mathbb{E}^{\pm}[\hat{k}] = \frac{(\hat{k}^{*\pm})^2 - \sum_{fh} \hat{k}_\tau(\Delta\hat{k}'_{fh})^2 I(\pm\Delta\hat{k}_{fh} < 0)\omega_f}{2(\hat{k}^{*\pm} - \sum_{fh} \hat{k}_\tau(\Delta\hat{k}'_{fh}) I(\pm\Delta\hat{k}_{fh} < 0)\omega_f)} + \frac{\sigma^2}{2\nu} \tag{E.34}$$

(E.35)

(VI) **Estimation of covariance:** We estimate the covariance using the sample moment of (61) given by

$$\mathbb{C}ov[\hat{k}, a] = -\frac{\sum_{fh}(\hat{k}_{\tau'}(\Delta\hat{k}'_{fh}) - \mathbb{E}[\hat{k}])^2 \tau'_{fh}\omega_f}{2\nu\mathbb{E}[\tau]} + \frac{\mathbb{V}[\hat{k}]}{2\nu} + \frac{\sigma^2}{2\nu}\frac{\mathbb{E}[\tau]}{2}(1 + \mathbb{CV}(\tau)), \tag{E.36}$$

where $\mathbb{CV}(\tau)$ and $\bar{\tau}$ are estimated using (E.7), $\mathbb{V}[\hat{k}]$ is estimated using (E.33), and $\mathbb{E}[\hat{k}]$ is estimated using (E.34).

(VII) **Estimation of irreversibility term:** We estimate the CIR's irreversibility term following its sample counterpart:

$$\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s\left[\mathrm{d}(\mathcal{M}(\hat{k}_s)\hat{k}_s)\right]\right] = \frac{\sum_{fh}\left(\hat{k}_\tau(\Delta\hat{k}')\mathcal{M}\left(\Delta\hat{k}'_{fh}\right) - \hat{k}^*(\Delta\hat{k})\mathcal{M}\left(\Delta\hat{k}_{fh}\right)\right)\omega_f}{\overline{\mathbb{E}}[\tau]}. \tag{E.37}$$

The objects are given by:

- $\mathcal{M}(\Delta\hat{k}) := \mathcal{M}(\hat{k}^{*-})\mathbb{1}_{\{\Delta\hat{k}_{fh}>0\}} + \mathcal{M}(\hat{k}^{*+})\mathbb{1}_{\{\Delta\hat{k}_{fh}<0\}}$ with

$$\mathcal{M}(\hat{k}^{*-}) = (\mathbb{E}^-[\hat{k}] - \mathbb{E}[\hat{k}])\overline{\mathbb{E}}^-[\tau]\,\frac{\mathbb{E}[\mathbb{P}^+]}{\mathbb{P}^{-+}} \tag{E.38}$$

$$\mathcal{M}(\hat{k}^{*+}) = (\mathbb{E}^+[\hat{k}] - \mathbb{E}[\hat{k}])\overline{\mathbb{E}}^+[\tau]\,\frac{\mathbb{E}[\mathbb{P}^-]}{\mathbb{P}^{+-}}. \tag{E.39}$$

- Conditional durations of inaction as:

$$\mathbb{E}^-[\tau] = \frac{\sum_{fh}\tau'_{fh}\mathbb{1}_{\{\Delta\hat{k}_{fh}>0\}}\omega_f}{\sum_{fh}\mathbb{1}_{\{\Delta\hat{k}_{fh}>0\}}\omega_f}, \tag{E.40}$$

$$\mathbb{E}^+[\tau] = \frac{\sum_{fh}\tau'_{fh}\mathbb{1}_{\{\Delta\hat{k}_{fh}<0\}}\omega_f}{\sum_{fh}\mathbb{1}_{\{\Delta\hat{k}_{fh}<0\}}\omega_f}, \tag{E.41}$$

- Transition probabilities as:

$$\mathbb{P}^{-+} = \frac{\sum_{fh}\mathbb{1}_{\{\Delta\hat{k}'_{fh}<0\}}\mathbb{1}_{\{\Delta\hat{k}_{fh}>0\}}\omega_f}{\sum_{fh}\mathbb{1}_{\{\Delta\hat{k}_{fh}>0\}}\omega_f}, \tag{E.42}$$

$$\mathbb{P}^{+-} = \frac{\sum_{fh}\mathbb{1}_{\{\Delta\hat{k}'_{fh}>0\}}\mathbb{1}_{\{\Delta\hat{k}_{fh}<0\}}\omega_f}{\sum_{fh}\mathbb{1}_{\{\Delta\hat{k}_{fh}<0\}}\omega_f} \tag{E.43}$$

- Expected probabilities

$$\mathbb{E}[\mathbb{P}^+] = \frac{\sum_{fh}\tau'_{fh}\mathbb{1}_{\{\Delta\hat{k}'_{fh}<0\}}\omega_f}{\overline{\mathbb{E}}[\tau]}, \tag{E.44}$$

$$\mathbb{E}[\mathbb{P}^-] = \frac{\sum_{fh}\tau'_{fh}\mathbb{1}_{\{\Delta\hat{k}'_{fh}>0\}}\omega_f}{\overline{\mathbb{E}}[\tau]}. \tag{E.45}$$

E.5 Calculating $(\lambda^-, \lambda^+, J^-, J^+)$

Let $\left\{\Delta\hat{k}_{fh}, \tau_{fh}\right\}_{fh}$ and weights ω_f be the sample of investment rates and durations with firms' weights ω_f computed in step (i) in Section E.4. We follow step (ii) in Section E.4 to compute $\left(\nu, \sigma^2, \hat{k}^{*+}, \hat{k}^{*-}\right)$. We now the steps to compute the $(\lambda^-, \lambda^+, J^-, J^+)$. From now on, we assume that the sample is i.i.d. Observe that this assumption is incorrect due to partial irreversibility (positive investment begets future positive investment). We find similar results whenever we estimate

(I) **Estimate $h(\Delta k)$ using a parametric specification:** From now on, we assume that the sample is i.i.d.[35] We parameterize $h(\Delta\hat{k})$ with two Gamma distributions where their parameters can differ between positive and negative investment rates.

$$h(\Delta\hat{k}) = \begin{cases} \frac{\Upsilon}{\Gamma(\varrho_-)\varsigma_-^{\varrho_-}}\left(\Delta\hat{k}\right)^{\varrho_- -1} e^{-\frac{\Delta\hat{k}}{\varsigma_-}} & \text{if } \Delta\hat{k} > 0 \\ \frac{1-\Upsilon}{\Gamma(\varrho_+)\varsigma_+^{\varrho_+}}\left((-\Delta\hat{k})\right)^{\varrho_+ -1} e^{-\frac{(-\Delta\hat{k})}{\varsigma_+}} & \text{if } \Delta\hat{k} < 0 \end{cases}. \tag{E.46}$$

Here, $\Upsilon \in (0,1)$ is the fraction of positive investment rates, $\varrho_\pm > 0$ are the shape parameters of Gamma distributions and $\varsigma_\pm > 0$ are the scale parameters. We have two methods to estimate the parameters $(\Upsilon, \varrho_-, \varrho_+, \varsigma_-, \varsigma_+)$: Maximum likelihood estimator and method of moments.

(I.a) **Method of Moments:** Using the method of moments, we have that

$$\Upsilon = \frac{\mathcal{N}^-}{\mathcal{N}^- + \mathcal{N}^+}, \tag{E.47}$$

$$\varrho_- = \frac{\bar{\mathbb{E}}\left[\Delta\hat{k}|\Delta\hat{k} > 0\right]^2}{\bar{\mathbb{V}}\left[\Delta\hat{k}|\Delta\hat{k} > 0\right]}, \tag{E.48}$$

$$\varsigma_- = \frac{\bar{\mathbb{V}}[\Delta\hat{k}|\Delta\hat{k} > 0]}{\bar{\mathbb{E}}[\Delta\hat{k}|\Delta\hat{k} > 0]} \tag{E.49}$$

$$\varrho_+ = \frac{\bar{\mathbb{E}}\left[\Delta\hat{k}|\Delta\hat{k} < 0\right]^2}{\bar{\mathbb{V}}\left[\Delta\hat{k}|\Delta\hat{k} < 0\right]}, \tag{E.50}$$

$$\varsigma_+ = \frac{\bar{\mathbb{V}}[\Delta\hat{k}|\Delta\hat{k} < 0]}{\bar{\mathbb{E}}[\Delta\hat{k}|\Delta\hat{k} < 0]}. \tag{E.51}$$

By replacing equations (E.47)-(E.51) with their sample counter-part with weights w_f, we have the estimates of $(\Upsilon, \varrho_-, \varrho_+, \varsigma_-, \varsigma_+)$.

(I.b) **Maximum Likelihood Estimation:** To simplify notation, let $i = 1, 2, \ldots, N$ denotes the index of the sample with weights w_i. To write the likelihood, let us define $\mathcal{A}_-$ the set of positive investment rates with cardinality $\#\mathcal{A}_- = N_-$ (i.e., the total number of positive investment rates in the sample) and $\mathcal{A}_+$ be the set of negative investment rate with cardinality $\#\mathcal{A}_+ = N_+$ (i.e., the total number of negative investment rates in the sample). By construction, $N_- + N_+ = \sum_f N_f$, i.e., the sample size. Under these assumptions, we can

[35] Observe that this assumption is not correct due to partial irreversibility (positive investment begets future positive investment). We find similar results whenever we estimate $h(\Delta k_h)$ conditional on a positive or negative investment rate and then Baye's rule.

Table E.1: Estimation of $h(\Delta\hat{k})$ with Method of Moments and Maximum Likelihood

	Method of Moments		Maximum Likelihood	
Parameters				
Share positive investment (Υ)	0.947		0.947	
	$\Delta\hat{k} > 0$	$\Delta\hat{k} < 0$	$\Delta\hat{k} > 0$	$\Delta\hat{k} < 0$
Shape parameter-Gamma distribution (ϱ)	0.758	1.413	0.744	0.915
Scale parameter-Gamma distribution (ς)	0.273	0.031	0.279	0.048

write the likelihood as

(E.52)

$$L(\Upsilon, \varrho_-, \varsigma_-, \varrho_+, \varsigma_+, \alpha | \{\Delta\hat{k}\}) = \sum_{i \in \mathcal{A}_-} w_i \log\left(\frac{\Upsilon}{\Gamma(\varrho_-)\varsigma_-^{\varrho_-}}\left(\Delta\hat{k}_i\right)^{k_- - 1} e^{-\frac{\Delta\hat{k}_i}{\varsigma_-}}\right) + \sum_{i \in \mathcal{A}^+} w_i \log\left(\frac{(1-\Upsilon)}{\Gamma(\varrho_+)\varsigma_+^{\varrho_+}}\left(\Delta\hat{k}_i\right)^{k_+ - 1} e^{-\frac{\Delta\hat{k}_i}{\varsigma_+}}\right)$$

(E.53)
$$= (k_- - 1)\sum_{i \in \mathcal{A}_-} w_i \log\left(\Delta\hat{k}_i\right) - \sum_{i \in \mathcal{A}_-} \frac{w_i \Delta\hat{k}_i}{\varsigma_-} - N_- \sum_{i \in \mathcal{A}_-} w_i \sim (\varrho_- \log(\varsigma_-) - \log(\Gamma(\varrho_-)))\ldots$$

(E.54)
$$\cdots + (k_+ - 1)\sum_{i \in \mathcal{A}_+} w_i \log\left(\Delta\hat{k}_i\right) - \sum_{i \in \mathcal{A}_+} \frac{w_i \Delta\hat{k}_i}{\varsigma_+} - N_+ \sum_{i \in \mathcal{A}_+} w_i \left(\varrho_+ \log(\varsigma_+) - \log(\Gamma(\varrho_+))\right)$$

(E.55)
$$\cdots + N_- \sum_{i \in \mathcal{A}_-} w_i \log(\Upsilon) + N_+ \sum_{i \in \mathcal{A}_+} w_i \log(1-\Upsilon)$$

From the likelihood optimization, we have that

(E.56)
$$\log(\varrho_-) - \frac{\Gamma'(\varrho_-)}{\Gamma(\varrho_-)} = \log\left(\sum_{i \in \mathcal{A}_-} \frac{w_i \Delta\hat{k}_i}{N_- \sum_{i \in \mathcal{A}_-} w_i}\right) - \frac{\sum_{i \in \mathcal{A}_-} w_i \log\left(\Delta\hat{k}_i\right)}{N_- \sum_{i \in \mathcal{A}_-} w_i}$$

(E.57)
$$\log(\varrho_+) - \frac{\Gamma'(\varrho_+)}{\Gamma(\varrho_+)} = \log\left(\sum_{i \in \mathcal{A}_+} \frac{w_i \Delta\hat{k}_i}{N_+ \sum_{i \in \mathcal{A}_+} w_i}\right) - \frac{\sum_{i \in \mathcal{A}_+} w_i \log\left(\Delta\hat{k}_i\right)}{N_+ \sum_{i \in \mathcal{A}_+} w_i}$$

(E.58)
$$\varsigma_- = \frac{\sum_{i \in \mathcal{A}_-} w_i \Delta\hat{k}_i}{N_- \sum_{i \in \mathcal{A}_-} w_i \varrho_-},$$

(E.59)
$$\varsigma_+ = \frac{\sum_{i \in \mathcal{A}_+} w_i \Delta\hat{k}_i}{N_+ \sum_{i \in \mathcal{A}_+} w_i \varrho_+},$$

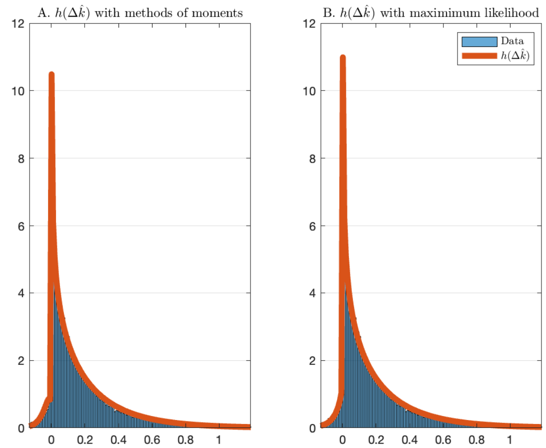
(E.60)
$$\Upsilon = \frac{N_- \sum_{i \in \mathcal{A}_-} w_i \sum}{N_- \sum_{i \in \mathcal{A}_-} w_i + N_+ \sum_{i \in \mathcal{A}_+} w_i}.$$

Figure E.1 shows the estimated distributions (E.46) under methods of moments and maximum likelihood. As the figure shows, we have an almost perfect histogram approximation under both methods. Table XX shows the estimates under the two methods.

(II) **Compute $g(\hat{k})$ and $\Lambda(\hat{k})$ using $h(\Delta\hat{k})$:** We use the finite difference to solve the KFE and back up the distribution $g(\hat{k})$. The hazard rate of adjustment is given by

(E.61)
$$\Lambda(\hat{k}) = \begin{cases} \frac{\mathcal{N} h(\hat{k}^{*+} - \hat{k})}{g(\hat{k})} & \text{if } \hat{k} > \hat{k}^{*+}, \\ \frac{\mathcal{N} h(\hat{k}^{*-} - \hat{k})}{g(\hat{k})} & \text{if } \hat{k} < \hat{k}^{*+} \end{cases}.$$

Figure E.1: Estimated $h(\Delta\hat{k})$ under Maximum Likelihood and Method of Moments



Notes: Panel A shows the histogram of investment rates (blue bars) and the estimated distribution (E.46) using the method of moments. Panel B uses the maximum likelihood to show the histogram of investment rates (blue bars) and the estimated distribution (E.46).

E.6 Results for Chile

Next, we present yearly averages of cross-sectional statistics. Inaction frequency is the fraction of observations with investment below 1% in absolute value; positive spikes are investments above 20% and negative spikes below −20%.

Table E.1 presents the yearly average of cross-sectional statistics by capital category for a balanced panel within Chile's ENIA establishment-level survey data. Note that the column total considers the statistics for the total capital stock, which is not the average of the statistics by capital type. For comparison, we include information for the US in Cooper and Haltiwanger (2006) and Zwick and Mahon (2017). Following these papers, investment rates reported in this table are computed as Investment divided by Initial Capital. We use perpetual inventories to compute capital stock.

Table E.1: Investment Rate Statistics (Chile: by capital type)

	Structures	**Machinery**	**Vehicles**	**Total**	**US I**	**US II**
Average Investment	7.3	17.0	17.1	15.8	12.2	10.4
Positive Fraction ($i > 1\%$)	22.2	54.9	25.5	56.8	81.5	—
Negative Fraction ($i < -1\%$)	0.5	1.5	5.3	3.1	10.4	—
Inaction rate ($\lvert i\rvert <= 1\%$)	77.3	43.7	69.2	40.1	8.1	23.7
Spike rate ($\lvert i\rvert > 20\%$)	8.9	23.2	21.2	22.8	20.4	14.4
Positive spikes ($i > 20\%$)	8.9	23.2	18.7	22.7	18.6	—
Negative spikes ($i < -20\%$)	0.0	0.0	2.5	0.1	1.8	—
Serial correlation	0.0	0.0	0.0	0.0	0.1	0.4

Notes: Authors calculations using establishment-level survey data for Chile (balanced panel). US I shows data from Cooper and Haltiwanger (2006) and US II shows data reported in Zwick and Mahon (2017) for the balanced panel. Following these papers, investment rates reported in this table are computed as investment divided by initial capital. We use the perpetual inventory method to compute capital stocks. We eliminate investment rates below the 1st percentile and above the 99th percentile of the investment rate distribution.

E.7 Comparative Statics

This section conducts a comparative statics exercise concerning the returns to scale α. Other parameters as in the main calibration.

Table E.2: Aggregate Capital Behavior: Comparative Statics

		$\omega = 0.15$		$\alpha = 0.5$	
	Benchmark	$\alpha = 0.4$	$\alpha = 0.6$	$\omega = 0.05$	$\omega = 0.25$
Productivity process					
ν	0.11	0.11	0.11	0.12	0.11
σ	0.23	0.23	0.24	0.23	0.24
Investment Policy					
Difference in reset capitals ($\hat{k}^{*+} - \hat{k}^{*-}$)	0.568	0.472	0.697	0.221	0.914
Exogenous price wedge	0.325	0.271	0.406	0.102	0.575
Endogenous response	0.243	0.201	0.291	0.118	0.339
Capital Allocation					
Variance	0.098	0.097	0.099	0.096	0.098
Capital Valuation					
Tobin's q	1.06	1.07	1.05	1.07	1.05
Productivity	1.09	1.10	1.08	1.08	1.10
Irreversibility	-0.03	-0.03	-0.03	-0.01	-0.05
Capital Fluctuations					
CIR	3.07	3.71	2.50	3.40	2.62
Responsiveness	2.29	2.37	2.17	2.51	1.93
Irreversibility	0.77	1.33	0.33	0.89	0.69

Notes: Objects recovered from theory mappings applied to establishment-level data from Chile. Comparative statics with respect to the wedge ω and the returns to scale α. Other parameters are described in the main text.

F Price wedges in the literature

Asplund (2000) and Ramey and Shapiro (2001) were the first to provide direct empirical evidence on the degree of partial irreversibility of capital investments using data from particular industries. Using data on equipment sales of three aerospace plants, Ramey and Shapiro find an average return on replacement costs of 28 cents per dollar, i.e., an average price wedge of 0.72. Nonetheless, they find that the wedge varies depending on the sale type, private liquidation, or auction, and they find an insider premium on the buyer. Meanwhile, Asplund examined prices for used metalworking machinery in Swedish manufacturing industries. He estimates wedges between 50 and 80 percent for “new” machines once installed. More recently, Kermani and Ma (2023) rely on estimated asset liquidation values of non-financial firms that filed for Chapter 11 bankruptcy. They find a liquidation recovery rate of 35 percent or a 0.65 irreversibility wedge for the average industry.

Contributions to the microeconomic implications of irreversibilities motivated the study of their macroeconomic consequences. Veracierto (2002) proposes a macro model with microeconomic (S,s) policy rules from optimal decision rules of profit-maximizing establishments. The author simulates the model economy with different values of the irreversibility wedge, ranging from 0 (fully flexible) to 1 (entirely irreversible investment), and finds aggregate fluctuations to behave similarly. Bloom (2009) predicts that an uncertainty shock generates a rapid drop and rebound in aggregate output and employment driven by a sudden halt in investment and hiring following the shock; adjustment costs and irreversibilities explain the investment halt. Bloom sets a 34 percent irreversibility wedge based on a simulated method of moments approach which includes joint (cross-sectional and dynamic) moments of the investment, employment, and sales growth series and second-and fourth-order correlations of the investment, employment growth, and sales series. Bloom’s estimates are used by Senga and Varotto (2024) in their study of cyclical capital misallocation with partial irreversibilities.

Similarly, Lanteri, Medina and Tan (2023) investigate capital reallocation following an import competition shock. The authors use the method of moments to calibrate their model and set an irreversibility wedge of 0.409 while allowing for an additional wedge for liquidating firms. The frequency of negative investments and the slope of exit thresholds (average slope of survival iso-probability lines) are informative moments for these wedges. Fang (2023) asks about monetary policy effectiveness in a setting with investment frictions. He targets the covariance of the capital gap with the time elapsed since the last adjustment to calibrate an irreversibility wedge of 0.3.

Smaller wedges are used by Khan and Thomas (2013) and by Lanteri (2018). Khan and Thomas base their 0.046 on several steady-state moments reported in Cooper and Haltiwanger (2006). Despite this small wedge, the authors find that negative shocks to borrowing conditions have strong and persistent effects through their capital distribution impact. Lanteri found a 0.067 average wedge in simulations of a GE model, with heterogeneous firms and imperfect substitutability between new and used capital. Despite this, his model features an endogenous resell to purchasing price, which he finds to be pro-cyclical; thus, it takes more work to reverse past investment decisions during recessions.

Table F.3: Price wedges in the literature

Direct evidence	Source	Wedge ω
1. Asplund (2000)	Metalwork machinery in Swedish manufacturing	0.5–0.8
2. Ramey and Shapiro (2001)	Equipment in US aerospace manufacturing	0.72
3. Kermani and Ma (2023)	US firms filling for Chapter 11	0.65

Quantitative models	Calibration	Wedge ω
4. Gilchrist, Sim and Zakrajšek (2014)	Book-value of leverage	0.5
5. Lanteri, Medina and Tan (2023)	Frequency of negative investments	0.41
6. Senga and Varotto (2024)	Investment, employment and sales moments	0.40
7. Bloom (2009); Bloom *et al.* (2018)	Investment, employment and sales moments	0.34
8. Fang (2023)	Covariance of the capital gap with age	0.30
9. Lanteri (2018)	Capital reallocation to expenditures	0.07
10. Khan and Thomas (2013)	Std dev, autocorrelation of investment rates, and spikes	0.05
11. Cooper and Haltiwanger (2006)	Investment spikes and serial correlation	0.025

Surveys	Notes	Wedge ω
12. Dibiasi, Mikosch and Sarferaz (2021)	Survey of Swiss firms	0.47
13. Dibiasi (2022)	Firm surveys and car resale prices	0.21–0.42

Notes: Price wedges are presented in descending order. See Appendix F for further details on these values.

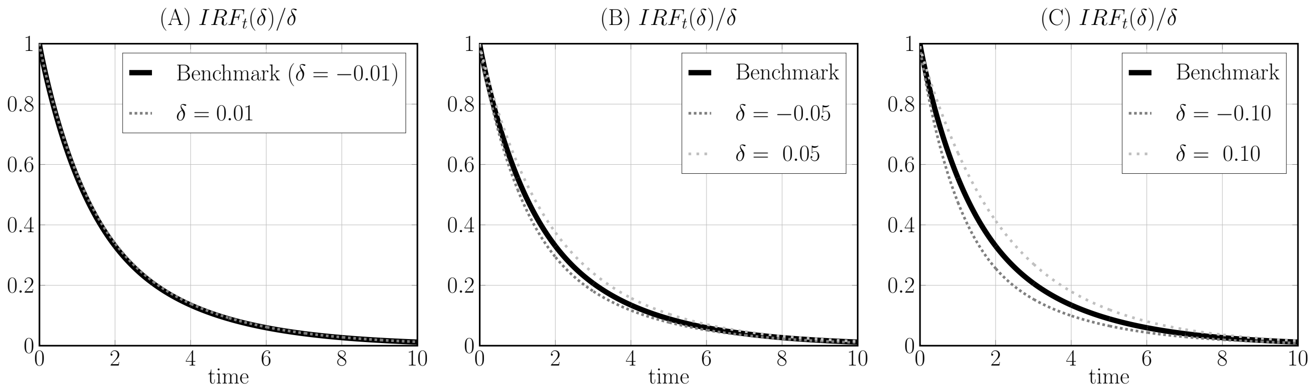
G Asymmetries and Non-Linearities

While the main text focuses on small productivity shocks, this section of the Online Appendix explores non-linearities and asymmetries by calculating impulse-response functions and the CIR(δ) for various aggregate shocks δ, both in terms of sign and magnitude. Our primary finding is that non-linearities and asymmetries are quantitatively insignificant in the generalized hazard model computed in Section 5 with a price wedge of 12% (i.e., $\omega = 0.12$). This result aligns with the distribution of investment rates and the CIR components (to first order).

Figure G.1 displays the impulse-response function normalized by the size of the shocks, i.e., $IRF_t(\delta)/\delta$. Panel A shows results for $\delta = -0.01$ (represented by the black solid line, as analyzed in the main text) and $\delta = 0.01$ (represented by the gray dotted line). A notable property that emerges is symmetry: positive and negative shocks produce symmetric effects on the dynamics of the average capital-productivity ratio. This outcome exemplifies the certainty equivalent principle, where first-order perturbations to aggregate shocks are independent of the volatility of those shocks. The same property can be observed in CIR(δ)/δ. As shown in Table G.1, the numerical computation of CIR(δ)/δ for a small negative shock of -1% is 1.94, while for a small positive shock of 1% it is 1.98. Both values are close to the first-order approximation of CIR$'(0)$, which is 1.93, since the CIR(δ) is differentiable at zero.

Panels B and C of Figure G.1 display impulse-response functions for positive and negative shocks of 5% and 10%. It is important to note that empirical estimates of the standard deviation of productivity shocks are generally less than 1%—see Galí (1999) and Justiano, Primiceri and Tambalotti (2010). Therefore, and under the assumption of a normal distribution, the probability of having a shock larger than 3% is around 0.0027. The figure illustrates that the responses show only minor differences compared to the reaction for $\delta = -0.01$. The main text shows the rationale for the minimal presence of non-linearities in Figure VII. To match the sufficient statistics for the CIR's implying substantial size and dispersion of investment rates, the model incorporates a relatively flat hazard function. This results in a low proportion of firms near the inaction region's boundary, leading to minimal asymmetries and non-linearities.

Figure G.1: Impuse-Response Function for Different Productivity Shocks



Notes: This figures shows the impulse-response function for the general hazard model computed in Section 5. Panel A shows the impulse-response function for $\delta = -0.01$ (black solid line) and for $\delta = 0.01$ (gray dotted line). Panel B and C show the impulse-response functions for $\delta = \pm 0.05$ and for $\delta = 0.10$, respectively.

Table G.1: CIR(δ) for Generalized Hazard Model and $\omega = 12$

	δ					
	-0.10	-0.05	-0.01	0.01	0.05	0.10
$CIR(\delta)/\delta$	1.78	1.87	1.94	1.98	2.05	2.14

Notes: The Table computes the normalized CIR(δ) for different aggregate shocks.