Advanced Macroeconomics II

Lecture 8

Consumption: Asset Pricing

Isaac Baley

UPF & Barcelona GSE

February 10, 2016

Roadmap

1. Consumption with a risky asset
2. Portfolio choice (many assets)
3. Equity premium puzzle

Consumption with risky asset (1)

- Consider the consumption - savings problem:

$$V_0 = \max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_t \left[\sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \right]$$

- Until now, we had two assumptions:
  - Labor income y_t is risky
  - Wealth a_t is invested at the riskless interest rate r:

$$a_{t+1} = (1+r)a_t + y_t - c_t$$

- With this assumptions, we derive the Euler equation:

$$u'\left(c_t\right) = (1+r)\beta\mathbb{E}\left[u'\left(c_{t+1}\right)|y_t\right]$$

Consumption with risky asset (2)

- Suppose instead that you invest in an asset with risky return ξ_t:

$$a_{t+1} = (1 + \xi_t)\, a_t + y_t - c_t$$

- We assume ξ_t to follow a Markov process.
- Examples:
  (i) Risky discount bond (i.e. default):

$$\xi_{t+1} = \begin{cases} R_{t+1} & \text{if repay} \\ 0 & \text{if default} \end{cases} \quad \Longrightarrow \quad \xi_{t+1} = R_{t+1} Pr(repay)$$

  (ii) Shares of a company: One share cost p_t (in units of consumption good), and delivers a stochastic divident d_{t+1} next period:

$$1 + \xi_{t+1} = \frac{p_{t+1} + d_{t+1}}{p_t} = \underbrace{\frac{p_{t+1}}{p_t}}_{\text{capital gains}} + \underbrace{\frac{d_{t+1}}{p_t}}_{\text{dividend-price ratio}}$$

Consumption with risky asset (3): Euler

- Timing:
  - Enter period with a_t.
  - ξ_t and y_t are jointly determined at the beginning of period t.
  - Then household decides consumption c_t. (or new savings a_{t+1}).
- Budget implied by this timing:

$$a_{t+1} = (1 + \xi_t) a_t + y_t - c_t$$

- Euler is the same as before, but now that ξ_{t+1} is not known at time t:

$$u'(c_t) = \beta \mathbb{E} \left[u'(c_{t+1})(1 + \xi_{t+1}) | y_t, \xi_t \right]$$

- Divide both sides by $u'(c_t)$:

$$1 = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} (1 + \xi_{t+1}) \right]$$

Consumption with risky asset (4): SDF

- Define $M_{t+1} \equiv \beta u'(c_{t+1})/u'(c_t)$ as the stochastic discount factor (SDF).
- Also define $Z_{t+1} \equiv 1 + \xi_{t+1}$ the random return.
- Then Euler equation becomes:

$$1 = \mathbb{E}_t [M_{t+1} Z_{t+1}]$$

- We will use the Euler for different things:
  - Price assets (i.e. stocks)
  - Establish bounds on returns
- We will often rewrite the expectation of a product as product of expectations plus covariance:

$$1 = \mathbb{E}_t [M_{t+1} Z_{t+1}] = \mathbb{E}_t [M_{t+1}] \mathbb{E}_t [Z_{t+1}] + Cov_t [M_{t+1}, Z_{t+1}]$$

Consumption with risky asset (4): Price of an asset

- Euler equation can be used to derive the market price of assets.
- Let us compute the price of a stock:

$$1 = \mathbb{E}_t \left[M_{t+1} \frac{p_{t+1} + d_{t+1}}{p_t} \right]$$

Opening the expectation:

$$p_t = \mathbb{E}_t[M_{t+1}]\mathbb{E}_t \left[p_{t+1} + d_{t+1}\right] + Cov_t \left[M_{t+1}, p_{t+1} + d_{t+1}\right]$$

- The price of the stock (or bond) is determined by:
  1. Expected price plus future dividend (discounted by *expected* SDF)
  2. Risk, but *not* only the variance, also the **covariance with marginal utility of consumption (SDF).**
- This is the Consumption Capital Asset Pricing Model (CCAPM).

Stock pricing (1)

- Starting from stock Euler:

$$p_t = \mathbb{E}_t \left[M_{t+1} d_{t+1}\right] + \mathbb{E}_t \left[M_{t+1} p_{t+1}\right]$$

- Let's substitute recursively the sequence of $\{p_{t+j}\}_{j=1}^{\infty}$ and using the Law of Iterated Expectations, to obtain:

$$\begin{aligned} p_t &= \mathbb{E}_t \left[M_{t+1} d_{t+1}\right] + \mathbb{E}_t \left[M_{t+1} \left(\mathbb{E}_{t+1} \left[M_{t+2} d_{t+2}\right] + \mathbb{E}_{t+1} \left[M_{t+2} p_{t+2}\right]\right)\right] \\ &= \mathbb{E}_t \left[M_{t+1} d_{t+1}\right] + \mathbb{E}_t \left[M_{t+1} M_{t+2} d_{t+2}\right] + \mathbb{E}_t \left[M_{t+1} M_{t+2} p_{t+2}\right] \\ &= \dots \\ &= \mathbb{E}_t \left[\sum_{j=1}^{\infty} \left(\prod_{s=1}^{j} M_{t+s}\right) d_{t+j}\right] + \lim_{j\to\infty} \mathbb{E}_t \left[\left(\prod_{s=1}^{j} M_{t+s}\right) p_{t+j}\right] \end{aligned}$$

Stock pricing (2)

- Notice that $M_{t+1}M_{t+2} = \beta\frac{u'(c_{t+1})}{u'(c_t)}\beta\frac{u'(c_{t+2})}{u'(c_{t+1})} = \beta^2\frac{u'(c_{t+2})}{u'(c_t)}$.
- In general:

$$\left(\prod_{s=1}^{j} M_{t+s}\right) = \beta^j \frac{u'\left(c_{t+j}\right)}{u'\left(c_t\right)}$$

- Hence we obtain the price of the stock:

$$p_t = \underbrace{\mathbb{E}_t\left[\sum_{j=1}^{\infty}\beta^j \frac{u'\left(c_{t+j}\right)}{u'\left(c_t\right)} d_{t+j}\right]}_{\text{discounted stream of dividends}} + \underbrace{\lim_{j\to\infty}\mathbb{E}_t\left[\beta^j \frac{u'\left(c_{t+j}\right)}{u'\left(c_t\right)} p_{t+j}\right]}_{\text{bubble term}}$$

- Thus the stock price = fundamental value + bubble.

Stock pricing (3): Bubbles

- Bubble term:

$$\lim_{j\to\infty} \mathbb{E}_t \left[\beta^j \frac{u'(c_{t+j})}{u'(c_t)} p_{t+j}\right]$$

- We will assume no bubble condition: Bubble term $= 0$
- Usually bubbles can be ruled out in general equilibrium models.
- Bubble may arise in OLG models (i.e. money is a bubble) or in models with borrowing constraints (papers by Martin and Ventura).
- Rational bubbles (wait for Alberto's class)
  - Suppose that $p_t = p_t^* + B_t$
  - p_t^* is the fundamental value of the asset
  - B_t is a "rational" bubble, which grows at the constant rate $B_{t+1} = RB_t$

Stock pricing (4): Risk neutral agent

- Assume that the household holding stocks (the "investor") is risk neutral ($u'(c_t)$ is constant) and $R\beta = 1$.
- In this case $M_{t+1} = \beta u'(c_{t+1})/u'(c_t) = \beta = \frac{1}{R}$ at any time t.
- Therefore, the pricing equation simplifies to:

$$p_t = \mathbb{E}_t\left[\frac{d_{t+1}}{R}\right] + \mathbb{E}_t\left[\frac{d_{t+2}}{R^2}\right] + \ldots + \frac{\mathbb{E}_t\left[p_{t+j}\right]}{R^j}$$

- Assuming no bubble condition $\lim_{j\to\infty} \frac{\mathbb{E}_t(p_{t+j})}{R^j} = 0$, we obtain:

$$p_t = \sum_{j=1}^{\infty} \frac{1}{R^j} \mathbb{E}_t[d_{t+j}]$$

- The price is the net present value of future dividends.
- Clearly, risk does not affect the price.

Roadmap

1. Consumption with a risky asset
2. Portfolio choice (many assets)
3. Equity premium puzzle

Portfolio choice (1)

- For simplicity, we assume no labor income.
- Assume there are two assets:
  - Bonds: risk-free and pay return R.
  - Stocks: risky and pay return Z_{t+1} (unknown at t).
- Consumer may choose how much to invest in each asset.
- Let ω_t be the fraction in stocks and $1-\omega_t$ in bonds.
- The budget constraint becomes:

$$a_{t+1} = \left(\omega_t Z_{t+1} + (1-\omega_t) R\right)(a_t - c_t)$$

Portfolio choice (2)

- Value function:

$$V(a_t, Z_t) = \max_{c_t,\omega_t} \quad u(c_t) + \beta \mathbb{E}_t \left[V(a_{t+1}, Z_{t+1}) \right]$$

$$a_{t+1} = (\omega_t Z_{t+1} + (1-\omega_t) R)(a_t - c_t)$$

- FOCs:

$$\begin{aligned} u'(c_t) - \beta \mathbb{E}_t \left[(\omega_t Z_{t+1} + (1-\omega_t) R) \frac{\partial V(a_{t+1}, Z_{t+1})}{\partial a_{t+1}} \right] &= 0 \qquad (c_t) \\ (a_t - c_t) \beta \mathbb{E}_t \left[(Z_{t+1} - R) \frac{\partial V(a_{t+1}, Z_{t+1})}{\partial a_{t+1}} \right] &= 0 \qquad (\omega_t) \end{aligned}$$

- Envelope condition:

$$\frac{V(a_t, Z_t)}{\partial a_t} = \beta \mathbb{E}_t \left[(\omega_t Z_{t+1} + (1-\omega_t) R) \frac{V(a_{t+1}, Z_{t+1})}{\partial a_{t+1}} \right]$$

Portfolio choice (3)

- Combining the FOC w.r.t. c_t and the envelope condition:

$$u'(c_t) = \frac{V(a_t, Z_t)}{\partial a_t}$$

Substituting back into the envelope condition:

$$u'(c_t) = \beta \mathbb{E}_t \left[\left(\omega_t Z_{t+1} + (1 - \omega_t) R \right) u'(c_{t+1}) \right]$$

and opening the expectation (note that ω_t is chosen at t so comes out)

$$u'(c_t) = \omega_t \beta \mathbb{E}_t \left[Z_{t+1} u'(c_{t+1}) \right] + (1 - \omega_t) R \beta \mathbb{E}_t \left[u'(c_{t+1}) \right]$$

- From the FOC w.r.t. ω_t (assuming $a_t \neq c_t$)

$$R \mathbb{E}_t \left[\frac{V(a_{t+1}, Z_{t+1})}{\partial a_{t+1}} \right] = \mathbb{E}_t \left[Z_{t+1} \frac{V(a_{t+1}, Z_{t+1})}{\partial a_{t+1}} \right]$$

substitute the fact that $u'(c_t) = \frac{V(a_{t+1}, Z_{t+1})}{\partial a_{t+1}}$ and it becomes:

$$R \mathbb{E}_t \left[u'(c_{t+1}) \right] = \mathbb{E}_t \left[Z_{t+1} u'(c_{t+1}) \right]$$

Portfolio choice (4)

- So we have two equations:

$$
\begin{aligned}
R\mathbb{E}_t\left[u'\left(c_{t+1}\right)\right] &= \mathbb{E}_t\left[Z_{t+1}u'\left(c_{t+1}\right)\right] \\
u'\left(c_t\right) &= \beta\left\{\omega_t\mathbb{E}_t\left[Z_{t+1}u'\left(c_{t+1}\right)\right] + \left(1-\omega_t\right)R\mathbb{E}_t\left[u'\left(c_{t+1}\right)\right]\right\}
\end{aligned}
$$

- Together, they imply two Euler Equations that must be satisfied:

$$
\begin{aligned}
1 &= R\mathbb{E}_t\left[\beta\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}\right] \\
1 &= \mathbb{E}_t\left[Z_{t+1}\beta\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}\right]
\end{aligned}
$$

- And using other definition of SDF:

$$
\begin{aligned}
1 &= R\mathbb{E}_t\left[M_{t+1}\right] \\
1 &= \mathbb{E}_t\left[M_{t+1}Z_{t+1}\right]
\end{aligned}
$$

- Keep this in mind: $\frac{1}{R} = \mathbb{E}_t\left[M_{t+1}\right]$

Portfolio choice (5)

- For each Euler, we open the expectations of the product as the product of the expectation plus the covariance.

- Rearranging, we obtain an expression for excess returns:

$$\mathbb{E}_t[Z_{t+1}] - R = -R\beta cov\left[Z_{t+1}, \frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}\right]$$

- Excess returns are positive if covariance is negative.

Portfolio choice (6)

- Finally, recall the linearisation of Euler eq.: $\frac{u'(c_{t+1})}{u'(c_t)} = 1 - \gamma \frac{c_{t+1} - c_t}{c_t}$:

$$E_t Z_{t+1} - R = \gamma R \beta cov \left(Z_{t+1}, \frac{c_{t+1} - c_t}{c_t} \right) \tag{1}$$

  where γ is the coefficient of relative risk aversion.

- But data tells LHS>RHS (Mehra-Prescott, JME 1985, updated data 2003)

- **Equity premium puzzle**

Portfolio choice (7)

- Start again from:

$$
\begin{aligned}
1 &= R\mathbb{E}_t\left[M_{t+1}\right] \\
1 &= \mathbb{E}_t\left[M_{t+1}Z_{t+1}\right]
\end{aligned}
$$

- Now subtract the first equation from the second:

$$\mathbb{E}_t\left[M_{t+1}(Z_{t+1} - R)\right] = 0$$

  and define excess returns $\hat{Z}_{t+1} \equiv Z_{t+1} - R$ to get:

$$\mathbb{E}_t\left[M_{t+1}\hat{Z}_{t+1}\right] = 0$$

- This is a key moment condition used in empirical asset pricing (Hansen and Singleton, 1982)

Market price of risk and HJ bounds (1)

- Consider again:

$$\mathbb{E}_t\left[M_{t+1}\hat{Z}_{t+1}\right] = 0$$

- Open the expectation and write as:

$$\mathbb{E}_t[M_{t+1}]\mathbb{E}_t\left[\hat{Z}_{t+1}\right] = -Cov_t\left[M_{t+1}, \hat{Z}_{t+1}\right]$$

- Recall the Cauchy-Schwarz inequality applied to the covariance (this comes from the definition of correlation coefficient between 0 and 1):

$$|Cov_t[M_{t+1}, \xi_{t+1}]| \leq \sigma_t[M_{t+1}]\sigma_t[\xi_{t+1}]$$

where σ_t denotes the conditional standard deviation. which also says

$$-\sigma_t[M_{t+1}]\sigma_t[\xi_{t+1}] \leq Cov_t[M_{t+1}, \xi_{t+1}] \leq \sigma_t[M_{t+1}]\sigma_t[\xi_{t+1}]$$

and in particular:

$$-Cov_t[M_{t+1}, \xi_{t+1}] \leq \sigma_t[M_{t+1}]\sigma_t[\xi_{t+1}]$$

Market price of risk and HJ bounds (2)

- Substituting back:

$$\mathbb{E}_t[M_{t+1}]\mathbb{E}_t\left[\hat{Z}_{t+1}\right] = -Cov_t\left[M_{t+1}, \hat{Z}_{t+1}\right] \leq \sigma_t[M_{t+1}]\sigma_t\left[\xi_{t+1}\right]$$

- Rearrange and obtain a bound on the risk-adjusted return of an asset:

$$\underbrace{\frac{\mathbb{E}_t\left[\hat{Z}_{t+1}\right]}{\sigma_t\left[\hat{Z}_{t+1}\right]}}_{\text{risk-adjusted return}} \leq \underbrace{\frac{\sigma_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}}_{\text{market price of risk}}$$

  For a final touch, recall that $\mathbb{E}_t[M_{t+1}] = \frac{1}{R}$.

- The market price of risk comes from preferences.
- This conditions is called Hansen-Jaganathan bounds and can be checked empirically.

Market price of risk and HJ bounds (3)

- This expression applied to a risky asset with price p_t and return Z_{t+1} says:

$$p_t \geq \frac{1}{R} \left\{ \mathbb{E}_t[Z_{t+1}] - \underbrace{\frac{\sigma_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}}_{\text{market price of risk}} \sigma_t \left[Z_{t+1}\right] \right\}$$

- The market price of risk gives us the rate at which the price of the asset falls (relative to the price of the riskless bond $\frac{1}{R}$) as the conditional volatility of its returns increase.

Failure of CRRA to attain HJ bounds (4)

$$\underbrace{\frac{\mathbb{E}_t\left[\hat{Z}_{t+1}\right]}{\sigma_t\left[\hat{Z}_{t+1}\right]}}_{\text{risk-adjusted return}} \leq \beta \underbrace{\frac{\sigma_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right]}{\mathbb{E}_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right]}}_{\text{market price of risk}}$$

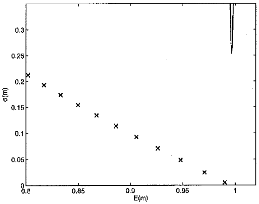

Figure 15.6.1: Solid line: Hansen-Jagannathan volatility bounds for quarterly returns on the value-weighted NYSE and Treasury Bill, 1948-2005. Crosses: Mean and standard deviation for intertemporal marginal rate of substitution for CRRA time separable preferences. The coefficient of relative risk aversion, γ takes on the values 1, 5, 10, 15, 20, 25, 30, 35, 40, 45, 50 and the discount factor β=0.995.

Roadmap

1. Consumption with a risky asset
2. Portfolio choice (many assets)
3. Equity premium puzzle
    - Empirical challenges
    - Solutions

Equity Premium Puzzle (1)

- Mehra and Prescott (1985) consider a simple "pure exchange economy", with one representative household that maximises intertemporal consumption $\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\gamma}}{1-\gamma}\right]$.
- One representative firm produces y_t, which is an exogenous stochastic process:

$$y_{t+1} = x_{t+1}y_t$$

- $x_t \in \{(1+\mu-\delta),(1+\mu+\delta)\}$ is a two state symmetric Markov process with persistence ϕ.
- These parameters match the average, the standard deviation and the first order autocorrelation of the growth rate of per capital consumption.
- In equilibrium the representative household owns the representative firm, and consumes a dividend equal to output: $c_t = y_t$ (no capital, no storage technology, no savings!).

Equity Premium Puzzle (2)

- So the price of this security is:

$$p_t = \beta \mathbb{E}_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} (p_{t+1} + d_{t+1}) \right\}$$

- Which becomes (recall that $u'(c_t) = c_t^{-\gamma}$):

$$p_t = \beta \mathbb{E}_t \left\{ \left(\frac{y_t}{y_{t+1}} \right)^{\gamma} (p_{t+1} + y_{t+1}) \right\}$$

- Since we know the law of motion of y_t, we can compute the equilibrium price and return $\mathbb{E}_t[Z_{t+1}]$.

Equity Premium Puzzle (3)

- Suppose now that households can trade a security which guarantees a safe return next period equal to 1.
- Then the price of this security p_t^{safe} must satisfy:

$$p_t^{safe} = \beta \mathbb{E}_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} 1 \right\} = \beta \mathbb{E}_t \left\{ \left(\frac{y_t}{y_{t+1}} \right)^{\gamma} \right\}$$

  and net return is $R = 1 + r^{safe} = \frac{1}{p^{safe}}$

- If there is no risk, then $y_t = y_{t+1}$ and $p_t^{safe} = \beta$, which implies $R\beta = 1$.
- If y_t is stochastic, more volatility implies higher p_t^{safe} and lower R. For a given volatility more risk aversion implies the same.
- Note: nobody buys and sells this security in equilibrium, because all households are homogeneous (Lucas trick).

Equity Premium Puzzle (4)

- Results obtained are consistent with the risk premium as derived before:

$$E_t Z_{t+1} - R = \gamma R \beta cov \left(Z_{t+1}, \frac{c_{t+1} - c_t}{c_t} \right)$$

- Model tells us what risk is: covariance with consumption growth.

- Data to test:
  - $E_t Z_{t+1} - 1$: Return NYSE market index 1889 - 1978: 6.98%
  - $R - 1$: Return 3 months T-bill = 0.8% –> equity premium 6.18
  - $cov \left(Z_{t+1}, \frac{c_{t+1} - c_t}{c_t} \right)$: Covariance between stock returns and consumption growth = 0.0027
  - If $\beta R = 1$ the implied risk aversion is $\gamma = 0.0618/0.0027 = 23$.
  - But realistic values of γ are between 1 and 4.

Equity Premium Puzzle (5)

	Mean	Variance-Covariance		
		$1+r^s_{t+1}$	$1+r^b_{t+1}$	c_{t+1}/c_t
$1+r^s_{t+1}$	1.070	0.0274	0.00104	0.00219
$1+r^b_{t+1}$	1.010		0.00308	−0.000193
c_{t+1}/c_t	1.018			0.00127

Table 15.3.1: Summary statistics for U.S. annual data, 1889–1978. The quantity $1+r^s_{t+1}$ is the real return to stocks, $1+r^b_{t+1}$ is the real return to relatively riskless bonds, and c_{t+1}/c_t is the growth rate of per capita real consumption of nondurables and services. Source: Kocherlakota (1996a, Table 1), who uses the same data as Mehra and Prescott (1985).

Hansen and Singleton (1982)

- Recall the first order condition for any asset with payoff Z_{t+1}^{i}:

$$u'\left(c_t\right) = \mathbb{E}_t\left[\beta u'\left(c_{t+1}\right) Z_{t+1}^{i}\right]$$

- Divide both sides by $u'\left(c_t\right)$ to obtain the moment condition:

$$\mathbb{E}_t\left[Z_{t+1}^{i}\frac{\beta u'\left(c_{t+1};\theta\right)}{u'\left(c_t;\theta\right)}\right] - 1 = 0$$

  where i indicates asset i.

- θ are the structural parameters of the utility function.

- Since this is a moment condition, they use (invent) GMM to estimate the parameters θ and β such that the empirical counterpart of this condition is as close as possible to 0.

Hansen and Singleton (1982)

- Moreover, since this FOC is conditional to the current information set, it must be that:
$$\mathbb{E}\left[\left(Z^i_{t+1}\frac{\beta u'\left(c_{t+1};\theta\right)}{u'\left(c_t;\theta\right)}-1\right)y_t\right]=0$$
for any time t variable y_t.

- So if the model is correct, any lagged variable y_t is a valid instrument to estimate θ.

- HS use lagged asset returns as instruments (recall Hall) and find:

  1. Overidentifying restrictions strongly reject the model
  2. θ way too high, just like Mehra-Prescott.

- Conclusion: something's really wrong with the model.

- Interested in reading more: Mehra and Prescott (2003) pretty accessible.

Solutions to Equity Puzzle (homework)

- Habits
- Disasters
- Distorted beliefs
- Asymmetric shocks