Advanced Macroeconomics II

Lecture 5

Consumption: Permanent Income Hypothesis

Isaac Baley
UPF & Barcelona GSE

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Motivation

- Consumption is a large fraction (70%) of aggregate output.
- Households derive direct utility from consumption
  $\Rightarrow$ key determinant of welfare, both at short and long run.
- Savings vary a lot over the life cycle.
  $\Rightarrow$ key to understand investment in the long run (e.g. ageing populations)
- Relatively stable in the business cycle.
  $\Rightarrow$ still key for business cycle fluctuations
- Consumption depends on real interest rates.
  $\Rightarrow$ key to understand impact of *monetary policy*
- Consumption is affected by fiscal instruments (consumption tax, income tax through its effects on labor supply, etc)
  $\Rightarrow$ key to understand impact of *fiscal policy*

A little bit of history (1)

- **Keynesian consumption function (1936)**
  - Consumption is a constant fraction of disposable income.

$$C_t = \alpha Y_t, \quad \alpha \in (0,1)$$

  - Assumption of Solow's growth model.
  - Implication: Big role for government stimulus.
- **Friedman's Permanent Income Hypothesis [PIH] (1957)**
  - Individual consumption tracks permanent income, which is the "normal" level of income.

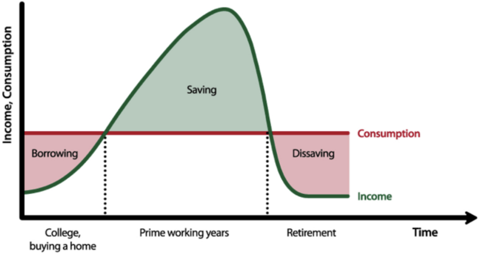
$$\begin{aligned} c^P_{i,t} &= \alpha_t Y^P_{i,t} \\ Y_{i,t} &= Y^P_{i,t} + \epsilon^Y_{i,t} \\ c_{i,t} &= c^P_{i,t} + \epsilon^C_{i,t} \end{aligned}$$

  - Implication: Smaller role for government stimulus.

A little bit of history (2)

- **Ando and Modigliani's Life Cycle Theory (1963)**
  - Savings smooth consumption over the life cycle while income varies.

$$c_{i,t} = \alpha_{t,a} Y_{i,t}, \quad a = age \quad i = individual$$



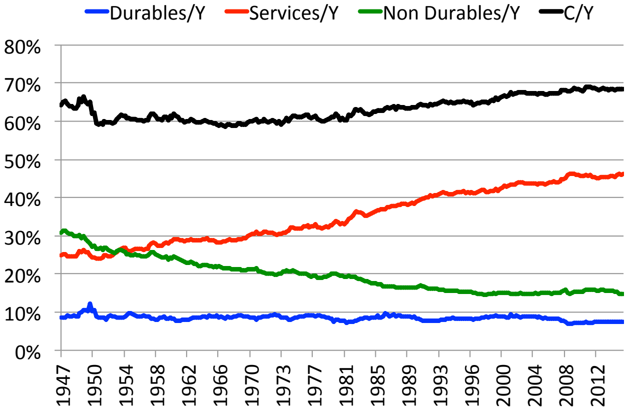
- **Afterwards, microfoundation of consumption optimization problem.**
  - PIH is a special case of a general framework.
  - Adding uncertainty, heterogenous agents, borrowing constraints, GE ...

Roadmap

1. **Facts about aggregate and household level consumption**
2. Consumption-savings problem with certainty
   - Euler Equation and Consumption Smoothing
   - Full Solution
   - Application: Ricardian Equivalence
3. Consumption-savings problem with uncertainty
   - Special case: Quadratic utility
   - Propensity to consume and income persistence
4. Empirical evidence

Facts about consumption

- **Fact 1: Consumption is a large fraction of aggregate output.**
  - Focus on consumption of non-durables + services ($\sim$ 60% of output).
  - Later we will study durables.



Facts about consumption (cont...)

- **Fact 2: Consumption is less volatile than output.**
- **Fact 3: Relatively low correlation of consumption with output.**
  - HP-detrended, quarterly time series from US during 1954 –1991.

Variable	**St Dev (%)**	**Correlation with GNP**
GNP	1.72	1
Consumption (non-durables)	0.86	0.77
Investment (gross private domestic)	8.24	0.91
Hours Worked	1.59	0.86

Cooley and Prescott (1996)

- This facts point towards **aggregate consumption smoothing.**

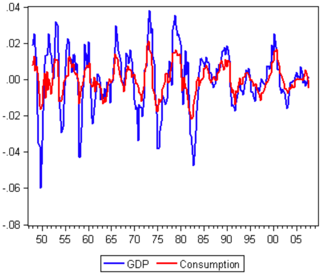
Aside: Hodrick-Prescott Filter (1997)

- Two sided filter that allows for isolating cyclical component of a time series.
- Let $y_t = \log Y_t$, which has a trend τ_t and a cyclical component y_t^c:

$$y_t = \tau_t + y_t^c + \varepsilon_t$$

- Trend solves following problem for a given λ (1600 for quarterly data):

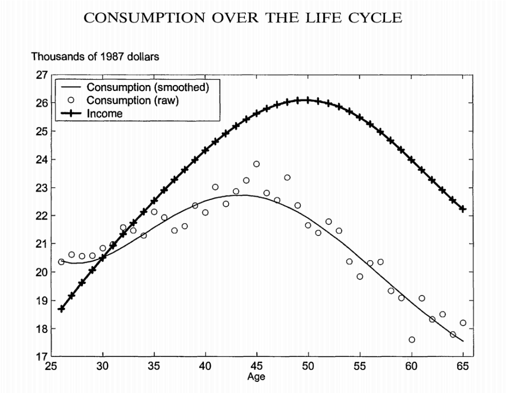
$$\min_{\{\tau_t\}_{t=1}^{T}} \left(\sum_{t=1}^{T} (y_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1} [(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2 \right)$$



- Other filters: Baxter and King, BandPass filters.

Facts about consumption (cont...)

- **Fact 4: Within household's life cycle, the correlation of consumption and income is large and positive during early and late years, and negative in the middle.**



Souce: Gourinchas & Parker (2002), "Consumption over the Life Cycle". CES data for US.

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The building blocks

- **Time:** Discrete and infinite $t = 0, 1, 2, ..., \infty$.
- **Many identical optimizing households who live forever**
  - Choosing consumption sequence to maximize utility $U\left(\{c_t\}_{t=0}^{\infty}\right)$
  - Dynasty where current generations care about future ones (bequests).
  - Discrete time version of the consumption problem in Ramsey model.
- **Endowments:**
  - Labor income $\{y_t\}_{t=0}^{\infty}$ is a non-stochastic sequence.
  - Initial wealth a_0.
- **Financial markets:** borrowing and saving through a one period risk-free bond with return $R = 1 + r$ (with certainty, one asset is complete markets).
- **Partial equilibrium approach:**
  - No production sector and no population growth.
  - Interest rate R fixed, and labor income y is exogenous.

The building blocks

- Preferences:

$$U = \max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

1. Stationarity: $u_s(\cdot) = u(\cdot)$ and $\beta_s = \beta$ for all periods s.
2. Homogeneity: $u^i = u$ and $\beta^i = \beta$ for all households i.
3. No habit formation: u at t only depends on consumption at t.
4. Monotonicity and Risk Aversion: $u' > 0, u'' < 0$

- Later on, we change assumptions:
  - $u''' > 0 \iff$ Prudence (precautionary saving)
  - Habits
  - Hyperbolic discounting

Net assets and budget constraint

- Define net assets a_{t+1} at the beginning of period $t+1$:

$$a_{t+1} = Ra_t - c_t + y_t \quad \text{with initial condition } a_0 \text{ given}$$

- Implicit timing assumption:
  1. Begin period with net assets a_t
  2. Receive interest payments ra_t and income y_t
  3. Decide consumption c_t (or equivalently net assets for next period a_{t+1}).
- We can explore different timings, but no big changes in results.
- The maximization problem is then:

$$V_0 = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to: $\quad a_{t+1} = Ra_t - c_t + y_t, \quad a_0$ given

What about transversality conditions?

- Substituting the budget constraint recursively forward, starting from $t = 0$:

$$a_0 = \frac{1}{R}(a_1 + c_0 - y_0) = \frac{1}{R}\left(\frac{1}{R}(a_2 + c_1 - y_1) + c_0 - y_0\right) = \dots$$

- Moving incomes to LHS and consumptions to RHS:

$$a_0 + \frac{1}{R}\left(y_0 + \frac{y_1}{R} + \dots + \frac{y_t}{R^t}\right) - \frac{a_{t+1}}{R^{t+1}} = \frac{1}{R}\left(c_0 + \frac{c_1}{R} + \dots + \frac{c_t}{R^t}\right)$$

- Keep substituting and take limits:

$$Ra_0 + \sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j y_j - \lim_{t\to\infty}\frac{a_{t+1}}{R^{t+1}} = \sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j c_j$$

- If allowed to increase its borrowing over time, the household will choose a_{t+1} equal to negative infinity, and consumption equal to infinity.

No Ponzi schemes

- To eliminate this possibility a "No Ponzi game" condition is imposed:

$$\lim_{t\to\infty} \frac{a_{t+1}}{R^{t+1}} \geq 0$$

  - This condition is an exogenous constraint, not an optimality condition necessary for the maximization.

- Since it is then not optimal for a_t to grow too large, the condition above must hold with exact equality for the household to maximise utility.

$$\lim_{t\to\infty} \frac{a_{t+1}}{R^{t+1}} = 0$$

- For now we abstract from borrowing constraints and only use "No Ponzi".

Different borrowing constraint

- Ad-hoc borrowing constraints (usually binding):
  - Borrowing only up to some value $B \geq 0$:

$$a_{t+1} \geq -B$$

  - Extreme case (no borrowing, only saving)

$$a_{t+1} \geq 0$$

- Natural borrowing constraint (NBC):

$$a_{t+1} \geq \widetilde{a}_t \quad \text{where} \quad \widetilde{a}_t \equiv -\sum_{j=1}^{\infty} \frac{y_{t+j}}{R^j}$$

Why is it "natural" not to borrow more than the stream future income? More on this later.

Back to the problem

- Reformulate the sequential problem in terms of a_{t+1} by substituting the budget constraint:

$$V_0(a_0) = \max_{\{a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(Ra_t + y_t - a_{t+1}\right)$$

  - FOC: $-u'(c_t) + \beta R u'(c_{t+1}) = 0$.

- Alternatively, use the recursive problem

$$V(a, y) \;\; = \;\; \max_{a'} \;\; u\left(Ra + y - a'\right) + \beta V(a', y')$$

  - FOC: $-u'(c) + \beta \frac{\partial V(a', y')}{\partial a'} = 0$
  - Envelope: $\frac{\partial V(a,y)}{\partial a} = Ru'(c)$
  - FOC + Forward Envelope: $-u'(c) + \beta R u'(c') = 0$.

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Euler Equation and Consumption Smoothing (1)

- The Euler equation for this problem is given by:

$$u'(c_t) = \beta R u'(c_{t+1})$$

- Interpretation:
  - Let $\rho =$ subjective intertemporal discount rate, such that $\beta = \frac{1}{1+\rho}$.
  - $\rho = 0.02$ means consuming tomorrow rather than today reduces u by 2%.

- Euler Eq can be written comparing subjective and objective discounts:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{1+r}{1+\rho}$$

Since $u''(c) < 0$, then

  - If $r = \rho$, then perfect smoothing $c_0 = c_1 = ... = c_t = c_{t+1}$.
  - If $r > \rho$, intertemporal saving motive: $c_t < c_{t+1}$.
  - If $r < \rho$, intertemporal borrowing motive: $c_t > c_{t+1}$.

Euler Equation and Consumption Smoothing (2)

- Example: Constant Relative Risk Aversion (CRRA) Utility

$$u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$
$$u'(c_t) = c_t^{-\gamma}$$

- Euler equation:

$$\frac{c_t^{-\gamma}}{c_{t+1}^{-\gamma}} = \frac{1+r}{1+\rho} \implies \frac{c_{t+1}}{c_t} = \left(\frac{1+r}{1+\rho}\right)^{\frac{1}{\gamma}}$$

- Denote consumption growth with $g^c = \log\left(\frac{c_{t+1}}{c_t}\right)$.
- Then Euler implies a constant trend:

$$g^c = \frac{1}{\gamma}\log\left(\frac{1+r}{1+\rho}\right) \approx \frac{r-\rho}{\gamma}$$

Euler Equation and Consumption Smoothing (3)

- More in general, we can linearize the FOC.

$$u'(c_t) = \frac{1+r}{1+\rho} u'(c_{t+1})$$

- A first order linear approximation of the RHS around c_t:

$$u'(c_t) \approx \frac{1+r}{1+\rho} \left[u'(c_t) + u''(c_t)(c_{t+1} - c_t) \right]$$

- Dividing both sides by $u'(c_t)$ and multiply and divide by c_t rearranging:

$$1 \approx \frac{1+r}{1+\rho} \left[1 + \frac{u''(c_t)}{u'(c_t)} (c_{t+1} - c_t) \right] \approx \frac{1+r}{1+\rho} \left[1 + \underbrace{\frac{u''(c_t)\, c_t}{u'(c_t)}}_{-\gamma(c_t)} \frac{c_{t+1} - c_t}{c_t} \right]$$

where the coefficient of relative risk aversion is defined as:

$$\gamma(c_t) \equiv -\frac{u''(c_t)\, c_t}{u'(c_t)}$$

Euler Equation and Consumption Smoothing (4)

- Under the linear approximation, consumption growth is given by:

$$
\begin{aligned}
-\gamma\left(c_t\right)\frac{c_{t+1}-c_t}{c_t} &\approx \frac{1+\rho}{1+r}-1 \\
\frac{c_{t+1}-c_t}{c_t} &\approx \frac{r-\rho}{R\gamma\left(c_t\right)}
\end{aligned}
$$

  - Consumption trend depends on ρ, r and $\gamma\left(c_t\right)$.

- Under the example of CRRA utility: $\gamma\left(c_t\right)=\gamma$. Proof.

$$
u\left(c_t\right)=\frac{c_t^{1-\gamma}}{1-\gamma};\ u'\left(c_t\right)=c_t^{-\gamma};\ u''\left(c_t\right)=-\gamma c_t^{-(1+\gamma)}
$$

$$
-\frac{u''\left(c_t\right)c_t}{u'\left(c_t\right)}=\frac{\gamma c_t^{-(1+\gamma)}c_t}{c_t^{-\gamma}}=\gamma
$$

  - Trend becomes constant: $\frac{r-\rho}{R\gamma}$
  - Consumption is smoothed over time, even without uncertainty.

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Full solution

- Solution of the problem: consumption as a function of a_0, $\{y_t\}_{t=0}^{\infty}$. Close form solution in the deterministic case.

- Solution: Euler equation + budget constraint

- So we have two equations:

$$\sum_{t=0}^{\infty} \left(\frac{1}{R}\right)^t c_t = Ra_0 + \sum_{t=0}^{\infty} \left(\frac{1}{R}\right)^t y_t$$

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta R$$

Solution with constant consumption (perfect smoothing)

- Let us impose $r = \rho$ or equivalently $\beta R = 1$.
  - This is a sensible assumption: in GE, with homogeneous agents, the steady state implies constant consumption.
- From EE it yields immediately $c_t = \overline{c} \;\; \forall t$.
- It implies that the LHS of the budget reads:

$$\sum_{t=0}^{\infty} \left(\frac{1}{R}\right)^t c_t = \overline{c} \sum_{t=0}^{\infty} \left(\frac{1}{R}\right)^t = \frac{\overline{c}}{1 - \frac{1}{R}} = \overline{c}\frac{R}{r}$$

- Therefore:

$$\overline{c} = \underbrace{\frac{r}{1+r}}_{\text{annuity value}} \underbrace{\left[R a_0 + \sum_{t=0}^{\infty} \left(\frac{1}{R}\right)^t y_t \right]}_{\equiv w_0^p \text{ permanent wealth}} \equiv y^p$$

- Consumption is constant since the value of w_0^p is perfectly known at $t = 0$.

Permanent Income Hypothesis

- This is the *permanent income hypothesis (PIH).*

$$c_t = \overline{c} = y^p = \frac{r}{1+r} w_0^p \text{ for any } t$$

- Consumption equals permanent income, or the annuity value of wealth.
- An increase in current income has a very minor effect in current consumption, because the increase is spent evenly during all lifetime.

$$\frac{\partial c_t}{\partial y_t} = \frac{\partial c_t}{\partial y_t^p} \frac{\partial y_t^p}{\partial y_t} = 1 \frac{\partial y_t^p}{\partial y_t} = \frac{r}{1+r}$$

where the last equality can be obtained from:

$$y_t^p = \frac{r}{1+r} w_t^p = \frac{r}{1+r} \left[Ra_t + y_t + \sum_{j=1}^{\infty} \left(\frac{1}{1+r} \right)^j y_{t+j} \right]$$

- Hence: $\frac{\partial c_t}{\partial y_t} = \frac{r}{1+r} < 1$

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The Ricardian Equivalence (1)

- Imagine a government has to finance a given amount of spending.
- Does the timing of taxation matter?
- Does it matter whether
  a) it sets taxes today or
  b) it issues debt and collects taxes in the future to pay the debt back?
- Ricardian Equivalence: No, it does not matter.

The Ricardian Equivalence (2)

- In the Keynesian model, temporary tax cuts can have a large stimulating effect on demand.

- PIH suggests that a consumer will spread out the gains from a temporary tax cut over a long horizon, and so the stimulus effect will be much smaller.

- Furthermore, PIH implies that a increase in income today that is accompanied by a decrease of income in the future with the same present value ($\Rightarrow$ the permanent income does not change) will not change consumption decisions.

- Ricardian Equivalence is consistent with PIH.

- Let's add a **government** to the previous model.

The Ricardian Equivalence (3)

- The government collects lump sum taxes τ_t and issues debt B_t to finance a given stream of public spending $\{g_t\}_{t=0}^{\infty}$.
- The government has its own budget constraint:

$$RB_t + g_t = B_{t+1} + \tau_t$$

- Substituting forward recursively:

$$B_t = \frac{1}{R}\left(\frac{1}{R}\left(B_{t+2} + \tau_{t-1} - g_{t-1}\right) + \tau_t - g_t\right)$$

- After imposing a transversality condition, obtain the intertemporal budget:

$$\sum_{j=0}^{\infty} \frac{g_{t+j}}{R^j} = \sum_{j=0}^{\infty} \frac{\tau_{t+j}}{R^j} - RB_t$$

The present value of government spending has to be equal to its *paying capacity*, which is equal to the present value of taxes minus debts.

The Ricardian Equivalence (3)

- The household budget constraint becomes:

$$c_t + a_{t+1} = (y_t - \tau_t) + Ra_t$$

The corresponding intertemporal budget constraint will be:

$$\sum_{j=0}^{\infty} \frac{c_{t+j}}{R^j} = Ra_t + \sum_{j=0}^{\infty} \frac{y_{t+j} - \tau_{t+j}}{R^j}$$

- If we impose constant consumption and we solve, we get:

$$c_t = r \left\{ a_t + \frac{1}{R} \left[\sum_{j=0}^{\infty} \frac{y_{t+j}}{R^j} - \sum_{j=0}^{\infty} \frac{\tau_{t+j}}{R^j} \right] \right\}$$

Timing of the taxes does not matter, only the NPV of taxes matters.

Ricardian Equivalence: Failure

1. **Capital Market imperfections**
   - If there are borrowing constraints.
2. **Finite lifetimes**
   - What if a cut of taxes today comes with higher taxes in 30 years time, once a big fraction of today's population is already dead?
3. **Distortionary taxation**
   - If taxes are distortionary, then there exist an optimal intertemporal profile of taxes (the solution to the Ramsey problem).
4. **Income uncertainty**
   - If changes in government taxation alter the degree of uncertainty that households face in their income.

Ricardian Equivalence: Why bother?

- The Ricardian equivalence has value as a reference point.
  - It defines the conditions we need for the equivalence to hold and we can easily work out cases when it does not.
  - Having in mind the Ricardian equivalence (or PIH) when we look at the real world is a good framework to understand reality.
- From the policy point of view, it is a quantitative issue.
- Are the departures quantitatively important?
- Two papers to read for next week in Box.

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Adding uncertainty

- Basic problem same as before.
- Two potential sources of exogenous uncertainty:
  - Labor income: $\{y_t\}_{t=0}^{\infty}$ is stochastic (we focus on this now).
  - Capital income $\{R_t\}_{t=0}^{\infty}$ is stochastic (later with asset pricing).
- First we focus on stochastic labor income and assume a **stationary process.**
- Only asset that can be traded is a one period bond with return R.
  - Incomplete financial markets: With a continuum of states, a one-period bond is not enough to move wealth across future states.
- Interest rate $R_t = R$ is still exogenous and constant over time.
  - GE with aggregate uncertainty: R_t is stochastic and correlated to y_t.

Stationary Income Process

- A process $\{y_t\}_{t=0}^{\infty}$ is strictly stationary if the joint probability distribution does not change when shifted in time:

$$F(y_{t_1}, y_{t_2}, \ldots, y_{t_k}) = F(y_{t+\tau}, y_{t_1+\tau}, \ldots, y_{t_k+\tau}), \quad \forall k, \tau, t_1, t_2, \ldots$$

- It implies 2^{nd} order stationarity: unconditional mean $\mathbb{E}[y_t] = \mu_y$, variance $\mathbb{V}[y_t] = \sigma_y^2$ and autocovariance $Cov[y_t, y_{t+\tau}] = \gamma(\tau)$ are finite and invariant.
- For example, a stationary process for income could be an AR(1) structure:

$$y_t = (1-\rho)\overline{y} + \rho y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim iid(0, \sigma_\varepsilon^2)$$

Homework: $\mu_y = \overline{y} \quad \gamma(\tau) = \rho\gamma(\tau - 1), \quad \gamma(0) = \sigma_y^2 = \dfrac{\sigma_\varepsilon^2}{1-\rho^2}$

- Cases:
  - If $\rho = 1$, y_t is a random walk
  - if $\rho = 0$, y_t is *iid*
  - if $\rho \in (0, 1)$, y_t is persistent

Consumption-saving problem with uncertainty

- The problem:

$$\max_{\{c_t\}_{t=0}^{\infty}} \quad \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \right]$$

  subject to

$$a_{t+1} = Ra_t + y_t - c_t, \qquad a_0 \text{ given}$$

$$y_t = (1-\rho)\overline{y} + \rho y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim iid(0, \sigma_\varepsilon^2)$$

$$\lim_{t\to\infty} \mathbb{E}_0 \left[\frac{a_{t+1}}{R^{t+1}} \right] \geq 0$$

- Timing:
  1. Start period t with net assets a_t.
  2. Receive interest payments ra_t.
  3. ε_t is realized, and household learns realization of y_t
  4. Consumption $c_t(a_t, y_t)$ is decided.
  5. Residual wealth a_{t+1} is invested (if positive) or borrowed (if negative).
- For now we assume there are no borrowing constraints or they don't bind.

Recursive problem with uncertainty

- Problem in recursive form (substituting budget constraint):

$$V(a_t, y_t) = \max_{a_{t+1}} \quad u(Ra_t + y_t - a_{t+1}) + \beta \mathbb{E}_t [V(a_{t+1}, y_{t+1}) | y_t]$$

- Note: y_t is a state variable for two reasons:
  - it determines disposable income $Ra_t + y_t$
  - it provides information on future income when $\rho \neq 0$

- FOC & Envelope & Forward Envelope:

$$\begin{aligned} u'(c_t) &= \beta \mathbb{E}_t \left[\frac{\partial V(a_{t+1}, y_{t+1})}{\partial a_{t+1}} \right] \\ \frac{\partial V(a_t, y_t)}{\partial a_t} &= Ru'(c_t) \implies \frac{\partial V(a_{t+1}, y_{t+1})}{\partial a_{t+1}} = Ru'(c_{t+1}) \end{aligned}$$

- Euler equation:

$$u'(c_t) = \beta R \mathbb{E}_t [u'(c_{t+1})]$$

Policy

- The solution is a policy function $c_t = c(a_t, y_t)$ that satisfies:
  - Bellman equation:

$$c(a_t, y_t) = \arg\max \quad u[c(a_t, y_t)] + \beta \mathbb{E}_t [V(a_{t+1}, y_{t+1})]$$

  - Feasibility:

$$a_{t+1} = Ra_t + y_t - c(a_t, y_t)$$

  - Euler equation: $\forall (a_t, y_t)$

$$u'[c(a_t, y_t)] = \beta R \mathbb{E}_t \left[u' \left(c \left(\overbrace{Ra_t + y_t - c(a_t, y_t)}^{a_{t+1}}, \; y_{t+1} \right) \right) \right]$$

- We discuss the Euler equation first, then full solution.

Stochastic Euler Equation

- Euler Equation:

$$u'(c_t) = \beta R \mathbb{E}_t [u'(c_{t+1})]$$

- Implies that $u'(c_t)$ is a sufficient statistic to predict $u'(c_{t+1})$:

$$\mathbb{E}_t [u'(c_{t+1})] = \frac{u'(c_t)}{R\beta}$$

- Let the expectation error be given by $\varepsilon^u_{t+1} \equiv u'(c_{t+1}) - \mathbb{E}_t [u'(c_{t+1})]$, then

$$u'(c_{t+1}) = \frac{u'(c_t)}{R\beta} + \varepsilon^u_{t+1}, \qquad \mathbb{E}_t \left[\varepsilon^u_{t+1}\right] = 0$$

- From rational expectations, ε^u_{t+1} is independent of all the information available at time t, and of c_t in particular.
    - Orthogonality conditions for GMM estimation:

$$\mathbb{E}_t \left[\varepsilon^u_{t+1} x_t\right] = 0, \quad \text{where} \quad x_t \in \{c_t, y_t, ...\}$$

Special case $R\beta = 1$

- Assume $R\beta = 1$
$$u'(c_{t+1}) = u'(c_t) + \varepsilon^u_{t+1}$$
- Marginal utility is a random walk (a thus a martingale).
  - Martingale: $\mathbb{E}_t[x_{t+1}] = x_t$
  - Submartingale: $\mathbb{E}_t[x_{t+1}] \geq x_t$
  - Supermartingale: $\mathbb{E}_t[x_{t+1}] \leq x_t$
- Recall that with certainty, $R\beta = 1$ meant perfect smoothing:
$$u'(c_{t+1}) = u'(c_t) \iff c_{t+1} = c_t = \overline{c}$$
- Which preferences would imply that consumption itself is a martingale?
$$\mathbb{E}_t[c_{t+1}] = c_t$$

Roadmap

1. Facts about aggregate and household level consumption
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Special case: Quadratic utility

- Consider a linear quadratic utility function

$$u(c_t) = \alpha_0 c_t - \frac{\alpha_1}{2} c_t^2, \qquad \alpha_i > 0$$

- Provided that $c_t < \frac{\alpha_0}{\alpha_1}$, the utility satisfies required conditions:

$$u'(c_t) = \alpha_0 - \alpha_1 c_t > 0 \qquad u''(c_t) = -\alpha_1 < 0$$

- Substituting in the Euler equation:

$$\mathbb{E}_t [u'(c_{t+1})] = \frac{u'(c_t)}{\beta R} \implies \mathbb{E}_t [\alpha_0 - \alpha_1 c_{t+1}] = \frac{\alpha_0 - \alpha_1 c_t}{R\beta}$$

- Solving for expected consumption at $t+1$:

$$\mathbb{E}_t [c_{t+1}] = \frac{1}{\beta R} c_t + \frac{\alpha_0}{\alpha_1} \left(1 - \frac{1}{\beta R}\right)$$

Special case: Quadratic utility and $\beta R = 1$

- By further assuming that $\beta R = 1$:

$$\mathbb{E}_t\left(c_{t+1}\right) = c_t$$

or alternatively

$$c_{t+1} = c_t + \varepsilon_{t+1}, \qquad \mathbb{E}_t\left[\varepsilon_{t+1}\right] = 0$$

- **Consumption is a random walk (martingale).**
- By the Law of Iterated Expectations and the martingale property:

$$\mathbb{E}_t[c_{t+2}] = \mathbb{E}_t[\mathbb{E}_{t+1}[c_{t+2}]] = \mathbb{E}_t[c_{t+1}] = c_t$$

therefore in general for any future period:

$$\mathbb{E}_t[c_{t+j}] = c_t, \qquad \forall j \geq 0$$

i) The best predictor of future consumption is current consumption.
ii) No other information available at the current period can help predict next periods' consumption.

Special case: Quadratic utility and $\beta R = 1$

- To solve the model, we use both the information in the Euler equation ($\mathbb{E}_t[c_{t+j}] = c_t, \forall j \geq 0$) and the budget constraint.
- Let us iterate forward the budget constraint from time t, use the No Ponzi condition, and obtain:

$$\sum_{j=0}^{\infty} \frac{1}{R^j} \mathbb{E}_t[c_{t+j}] = \underbrace{Ra_t}_{\text{Financial Wealth}} + \underbrace{\sum_{j=0}^{\infty} \frac{1}{R^j} \mathbb{E}_t[y_{t+j}]}_{H_t \quad \text{Human Wealth}}$$

- Note: We used conditional expectations to deal with uncertain future realizations of income and consumption as seen from time t.
- Note: Wealth is now stochastic (vs. certainty case where was a constant).

Special case: Quadratic utility and $\beta R = 1$

- Use the martingale property on the LHS:

$$\sum_{j=0}^{\infty} \frac{1}{R^j} \mathbb{E}_t[c_{t+j}] = c_t \sum_{j=0}^{\infty} \frac{1}{R^j} = \left(\frac{R}{R-1} \right) c_t = \left(\frac{1+r}{r} \right) c_t$$

and substitute back to get the optimal consumption:

$$c_t = \underbrace{\frac{r}{1+r}}_{annuity \;\; value} \underbrace{\left(Ra_t + \sum_{j=0}^{\infty} \frac{1}{R^j} \mathbb{E}_t[y_{t+j}] \right)}_{W_t \; = \; \text{Wealth}} \equiv y_t^p$$

where we define permanent income y_t^P as the annuity value of total wealth.

- Note that permanent income is a random variable (vs. deterministic case).
- **Result: If preferences are quadratic and $\beta R = 1$, then consumption follows a martingale process and equals permanent income.**

Certainty equivalence

- Recall that without uncertainty and $\beta R = 1$, the FOC was given by:

$$u'\left(c_{t+1}\right) = u'\left(c_t\right) \iff c_{t+1} = c_t$$

- And the solution was given by:

$$\overline{c} = c_t = \frac{r}{1+r}\left[Ra_t + \sum_{j=0}^{\infty} \frac{1}{R^j} y_{t+j}\right]$$

- **Certainty equivalence:** to solve the stochastic problem one can
  1. solve the deterministic problem
  2. substitute conditional expectations of the forcing variables $\left(y_{t+j}\right)$ in place of the variables themselves.
- **Implication**: Variance and higher moments of the income process DO NOT matter for the determination of consumption, only its expectation.
- However, ex-post consumption is not constant because y_t^p is not constant.

Consumption and news

- Let us compute how consumption changes between periods:

$$\Delta c_{t+1} = c_{t+1} - c_t = c_{t+1} - E_t[c_{t+1}] = \frac{r}{1+r}\left(W_{t+1} - \mathbb{E}_t[W_{t+1}]\right)$$

- Now lets compute the innovation (unexpected change) in wealth:

$$\begin{aligned} W_{t+1} - \mathbb{E}_t[W_{t+1}] &= R(a_{t+1} - \underbrace{\mathbb{E}_t[a_{t+1}]}_{a_{t+1} \text{ known at } t}) + \sum_{j=0}^{\infty} \frac{1}{R^j}\left(\mathbb{E}_{t+1}[y_{t+1+j}] - \underbrace{\mathbb{E}_t[\mathbb{E}_{t+1}[y_{t+1+j}]]}_{\mathbb{E}_t[y_{t+1+j}]}\right) \\ &= \sum_{j=0}^{\infty} \frac{1}{R^j}\left(\mathbb{E}_{t+1} - \mathbb{E}_t\right) y_{t+1+j} \end{aligned}$$

- Therefore, change in consumption is given by:

$$\Delta c_{t+1} = \frac{r}{1+r} \sum_{j=0}^{\infty} \frac{1}{R^j}\left(\mathbb{E}_{t+1} - \mathbb{E}_t\right) y_{t+1+j}$$

- **Result: Changes in consumption are proportional to the revision in expected earnings due to the new information (news) arriving in that same period.**

Random Walks

- Change in consumption is given by:

$$\Delta c_{t+1} = \frac{r}{1+r} \sum_{j=0}^{\infty} \frac{1}{R^j} \left(\mathbb{E}_{t+1} - \mathbb{E}_t \right) y_{t+1+j}$$

- From budget constraint, we find the change in assets:

$$\begin{aligned} \Delta a_{t+1} &= r a_t + y_t - c_t \\ &= r a_t + y_t - \frac{r}{R} \left[R a_t + \sum_{j=0}^{\infty} \frac{1}{R^j} \mathbb{E}_t [y_{t+j}] \right] \\ &= y_t - \frac{r}{R} \sum_{j=0}^{\infty} \frac{1}{R^j} \mathbb{E}_t [y_{t+j}] \end{aligned}$$

- Since Δc_{t+1} and Δa_{t+1} are stationary, it means that the original series have unit roots (integrated of order 1).
- **Implication: c_t and a_t do not converge.**

Cointegration

- **Cointegration:** Two non-stationary series of order 1 are said to be cointegrated if there exists a linear combination that is stationary.
- Claim: a_t and c_t are cointegrated series.
  - From the solution of consumption, we obtain a linear combination of a_t and c_t that is stationary:

$$\begin{bmatrix} 1 & -r \end{bmatrix} \begin{bmatrix} c_t \\ a_t \end{bmatrix} = c_t - ra_t = \frac{r}{R} \sum_{j=0}^{\infty} \frac{1}{R^j} \mathbb{E}_t[y_{t+j}]$$

  - According to Granger and Engle, $[1 \quad -r]'$ is a **cointegrating vector** that, when applied to the non-stationary vector process $[c_t \quad a_t]'$, yields a process that is asymptotically stationary.
  - Note: The cointegration vector is not unique. $[\frac{1}{r} \quad -1]$ (and many others) are also cointegrating vectors.

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Propensity to consume and income persistence (1)

- Since consumption depends only on permanent income:

$$c_t = y_t^p$$

- The marginal propensity to consume out of current income is given by:

$$\frac{\partial c_t}{\partial y_t} = \underbrace{\frac{\partial c_t}{\partial y_t^p}}_{=1} \frac{\partial y_t^p}{\partial y_t} = \frac{\partial y_t^p}{\partial y_t}$$

- In the stochastic case, $\frac{\partial y_t^p}{\partial y_t}$ depends on the process governing income.

  1. Permanent shocks: getting a better job
  2. Transitory shocks: win lottery, become temporarily unemployed
  3. Persistent shocks

Propensity to consume and income persistence (2)

- Let us assume that income follows an AR(1) process:

$$y_{t+1} = (1-\rho)\overline{y} + \rho y_t + \varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim iid(0, \sigma_\varepsilon^2), \qquad \rho \in [0,1]$$

- In the problem set, you will compute the policy using two methods.

$$c\left(a_t, y_t\right) = ra_t + \frac{R-1}{R-\rho} y_t$$

- Propensity to consume out of current income: $\frac{\partial c_t}{\partial y_t} = \frac{R-1}{R-\rho}$
  (i) **If shocks are permanent ($\rho = 1$), then $\frac{\partial c_t}{\partial y_t} = 1$.**
    - Consumption responds one to one to changes in income.

  (ii) **If shocks are purely transitory ($\rho = 0$), then $\frac{\partial c_t}{\partial y_t} = \frac{r}{1+r}$.**
    - Extra income is spread evenly among all periods.

  (iii) **Propensity to consume increases with persistence (ρ)**
    - Current income carries information about future realizations and the update of permanent income is bigger.

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Prediction I

Prediction I: c_t summarizes all time t information to forecast c_{t+1}.

- If $\beta R = 1$ and certainty equivalence, then consumption is a random walk:

$$\mathbb{E}_t \left(c_{t+1} \right) = c_t \quad \Longrightarrow c_{t+1} - c_t = \varepsilon_{t+1}$$

- Time t variables, besides c_t, don't have additional predictive power on Δc_{t+1}.
- Hall (1978) tests Prediction I.
  - Confirms random walk hypothesis when adding lagged consumption...
  - ... but rejects it when lagged stock market prices appear significant.
- This evidence stimulated research in the direction of borrowing constraints and precautionary saving.

Prediction II

Prediction II: Predictable changes in y_t have no effect on c_t.

- The optimal consumption rule is $c_t = y_t^p$ and $c_{t+1} = y_{t+1}^p$, which implies:

$$y_{t+1}^p - y_t^p = y_{t+1}^p - \mathbb{E}_t\left(y_{t+1}^p\right) = \varepsilon_{t+1}$$

- ε_{t+1} is the shock (unpredictable component) to permanent income, it is the only thing that should alter consumption.
- Campbell-Mankiw (NBER Macro Annuals, 1989) test Prediction II.
  - Full rationality is too strong: many agents are not rational or constrained.
  - Fraction λ of agents consume all income and $1-\lambda$ follow PIH:

$$c_t - c_{t-1} = \lambda\left(y_t - y_{t-1}\right) + (1-\lambda)\,\varepsilon_t$$

  - Recall that: $\varepsilon_t = y_t^p - E_{t-1}\left(y_t^p\right)$
  - Clearly if the PIH is correct λ should be zero.

Prediction II

- Implementation: treat ε_t as an error, and estimate:

$$\Delta c_t = \lambda \Delta y_t + \upsilon_t$$

$$\upsilon_t = (1 - \lambda)\, \varepsilon_t$$

  - What is the problem you face if you estimate $\widehat{\lambda}$ with OLS?
  - Instrument using Δc_{t-j} with $j \geq 1$.

- They find a consistent estimate of λ around 0.4 - 0.5:

  - Rejects PIH: consumption responds to lagged (or predictable) information.
  - A finding of $\lambda > 0$ is called **excess sensitivity of consumption.**
  - Around 40-50% are rule of thumb consumers?

Prediction III

Prediction III: $\mathbb{V}[\Delta c_t]$ should be equal to $\mathbb{V}[\Delta y_t^p]$.

- Consumption changes 1 to 1 with respect to unexpected changes in permanent income, thus variances should be the same.
- If $\Delta c_t = \Delta y_t^p$, then it must be that $\mathbb{V}(\Delta c_t) = \mathbb{V}(\Delta y_t^p)$.
- Empirical evidence shows that $\widehat{\mathbb{V}[\Delta c_t]} < \widehat{\mathbb{V}[\Delta y_t^p]}$.
  - Consumption seems to be excessively smooth with respect to unexpected changes in future income.
  - This is called **excess smoothness of consumption.**

Prediction III

- Important: y_t^p is not observed, so $\mathbb{V}[\Delta y_t^p]$ is computed by estimating a stochastic process for y_t.
- Idea behind? If $\rho = 1$ (income is a random walk), then

$$\Delta y_t = \eta_t \qquad \Longrightarrow \qquad \mathbb{V}[\Delta y^p] = \mathbb{V}[\Delta y]$$

- In fact, the process that fits the data best is

$$\Delta y_t = \rho \Delta y_{t-1} + \eta_t \qquad \textit{with } \rho = 0.44$$

- In this case, η_t has more than a one-for-one permanent effect on income: one shock η_t and one lagged effect $\rho\eta_t$.
- Hence the Deaton paradox: $\mathbb{V}[\Delta c] = \mathbb{V}[\Delta y^p] > \mathbb{V}[\Delta y]$
  - But data: $\mathbb{V}[\Delta c] < \mathbb{V}[\Delta y]$
  - Consumption is excessively smooth compared to what we would expect from the model!!

Problems with PIH

- Empirical tests of PIH basically fail.
- Because of certainty equivalence, risk does not matter!
- We will introduce two instances where risk (volatility of income shocks) will matter for consumption and saving decisions:
  1. Prudence
  2. Borrowing constraints (similar to investment irreversibility)
- Both elements will generate precautionary savings that increase with the volatility of income process.