Advanced Macroeconomics II

Lecture 1

Stochastic Dynamic Programming

Isaac Baley

UPF & Barcelona GSE

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Introduction

- Stochastic dynamic programming is a very useful tool to think and solve a lot of economic problems.
- We review the theory of dynamic programming and extend to include uncertainty.
- Straightforward extension of what you did in Advanced Macro I.
- References:
  - Acemoglu (2009), Ch. 16.
  - Adda and Cooper (2003), Ch 2-3.
  - Ljungquvist and Sargent (2014), Ch 3-4.
  - Stokey, Lucas, and Prescott (1989), Ch 3-4 (deterministic), 7-9 (stochastic with measure theory).

Roadmap

1. **Sequence Problem**
2. Recursive Formulation
3. Role of Uncertainty
4. Contraction Mapping Theorem
5. Characterization: Euler + Transversality
6. Solution Methods

Sequence Problem (1): Setup

- Time is infinite and discrete.
- An agent solves the following problem:

$$\max_{\{y_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \quad U(x_t, y_t, z_t) \right]$$

Subject to:

$$\begin{aligned} (\textit{control}) \qquad & y_t \in \widetilde{G}(x_t, z_t) \\ (\textit{state}) \qquad & x_{t+1} \in \widetilde{f}(x_t, y_t, z_t) \\ (\textit{initial conditions}) \qquad & x_0, z_0 \text{ given} \end{aligned}$$

  - $y_t \in Y \subset \mathbb{R}^{K_y}$: control variables (choice variables, e.g. investment)
  - $x_t \in X \subset \mathbb{R}^{K_x}$: endogenous state variables (predetermined, e.g. capital)
  - $z_t \in Z$: exogenous state variables (stochastic shocks, e.g. productivity)
  - $U : X \times Y \times Z \to \mathbb{R}$ instantaneous payoff
  - $\beta \in (0, 1]$: discount factor

Sequence Problem (2): Exogenous Shocks

- z_t is a stationary shock.
- For simplicity, we assume z_t to be a first order Markov Chain.
  - N possible realizations:

$$Z \equiv \{z_1, z_2, ..., z_N\}$$

  - Transition probability from i to j, denoted q_{ji}

$$pr\left[z_t = z_j \mid z_{t-1} = z_i\right] = q_{ji}$$

  such that:

$$q_{ji} \geq 0 \quad and \quad \sum_{j=1}^{N} q_{ji} = 1 \text{ for any } i = 1, ..., N$$

  - $N \times N$ matrix with i in rows and j in columns, where rows sum 1.

Sequence Problem (3): Constraint Sets

$$\max_{\{y_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \quad U(x_t, y_t, z_t) \right]$$

$$\begin{aligned} (\textit{control}) \qquad & y_t \in \widetilde{G}(x_t, z_t) \\ (\textit{state}) \qquad & x_{t+1} \in \widetilde{f}(x_t, y_t, z_t) \\ (\textit{initial conditions}) \qquad & x_0, z_0 \text{ given} \end{aligned}$$

- Constraint sets:
  - $\widetilde{G}(x_t, z_t)$ constraint on admissible controls, for given states.
  - $\widetilde{f}(x_t, y_t, z_t)$ law of motion of the state.
- Using $\widetilde{f}$ we can substitute y_t as a function of x_{t+1}, x_t and z_t.
- Example: $k_{t+1} = (1-\delta)k_t + i_t$. Control variable can be either i_t or k_{t+1}.

Sequence Problem (4): Solution

- Substitute y_t using $\widetilde{f}$ and obtain the Sequence Problem (P1):

$$V(x_0, z_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t U(x_t, z_t, x_{t+1}) \right]$$

$$st \quad : \quad x_{t+1} \in G(x_t, z_t), \qquad x_0, z_0 \text{ given}$$

- $V : X \times Z \to \mathbb{R}$ is the value function.
- Problem is *stationary* in that U and G do not depend on time.
- Solution is an infinite sequence $\left\{x_{t+1}^*\right\}_{t=0}^{\infty}$
- **Idea of dynamic programming:** transform the problem into one of finding a time-invariant *function* $\pi(x_t, z_t)$ rather than an infinite *sequence*.
  - Example, instead of the infinite sequence of optimal capital $\left\{k_{t+1}^*\right\}_{t=0}^{\infty}$, you find $k_{t+1}^* = \pi(k_t, z_t)$, the time invariant capital policy function.

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Recursive Formulation (1): Principle of Optimality

- The Principle of Optimality allows to express the problem in **recursive form.**
  - In an optimal policy, whatever the initial state and decision are, the remaining decisions must be an optimal policy with regard to the state resulting from the first decision.

- Consider the original sequence problem (P1):

$$V(x_0, z_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \left[U(x_0, z_0, x_1) + \beta U(x_1, z_1, x_2) + \dots \right]$$

$$st \quad : \quad x_{t+1} \in G(x_t, z_t), \qquad x_0, z_0 \text{ given}$$

- Suppose $\left\{x_{t+1}^*\right\}_{t=0}^{\infty}$ is a solution to the problem and $V(x_0, z_0)$ is finite.

Recursive Formulation (2): Bellman Equation

- In period 1, the state variables are x_1^* and z_1, and the sequence $\left\{x_{t+1}^*\right\}_{t=0}^{\infty}$ is also optimal from period 1 onwards:

$$\begin{aligned} V^*(x_0, z_0) &= U(x_0, z_0, x_1^*) + \beta \mathbb{E}_0 \left[U(x_1^*, z_1, x_2^*) + \dots \right] \\ &= U(x_0, z_0, x_1^*) + \beta \mathbb{E}_0 \left[V(x_1^*, z_1) \right] \end{aligned}$$

- Therefore P1 must be equal to maximizing the two period problem:

$$V(x_0, z_0) = \max_{x_1 = \pi(x_0, z_0)} U(x_0, z_0, x_1) + \beta \mathbb{E}_0 \left[V(x_1, z_1) \right], \qquad x_0, z_0 \text{ given}$$

- For any t, we obtain the Recursive Problem (P2), a Bellman Equation:

$$V(x_t, z_t) = \max_{x_{t+1} = \pi(x_t, z_t)} \left\{ U(x_t, z_t, x_{t+1}) + \beta \mathbb{E}_t \left[V(x_{t+1}, z_{t+1}) \right] \right\} \qquad \forall x \in X$$

  - x_t and z_t are states and x_{t+1} is the vector of controls (tomorrow's state)
  - We usually assume that z is an exogenous first order stochastic process: $\mathbb{E}_t(z_{t+1})$ only depends on z_t.

Recursive Formulation (3): Important Notes

1. The infinite horizon plan is reduced to a two-period problem (today + continuation value).
   - Often gives better economic intuition
2. Once we have $V(\cdot)$, the policy function $x_{t+1} = \pi(x_t, z_t)$ can be found from:

$$V(x_t, z_t) = U(x_t, \pi(x_t, z_t), z_t) + \beta \mathbb{E}_t [V(\pi(x_t, z_t), z_{t+1})] \qquad \forall x_t \in X$$

3. P2 is a *functional equation*, i.e. a function of a functions.
   - Under certain assumptions, we can guarantee existence and properties of the solutions.
   - Contraction Mapping Theorem.
4. P2 is *recursive* in that $V(\cdot)$ appears both on the LHS and the RHS.
   - Powerful numerical tools to find the solution.
   - Contraction Mapping Theorem.

Recursive Formulation (4): Additional Assumptions

- We will make the following assumptions:
  (i) $\lim_{n\to\infty} \mathbb{E}_0 \left[\sum_{t=0}^{n} \beta^n U\left(x_t^*\left[z^{t-1}\right], x_{t+1}^*\left[z^t\right], z_t\right)\right]$ exists and is finite.
  (ii) X is a compact subset of $\mathbb{R}^K$.
  (iii) $G(x, z)$ is nonempty, compact valued, and continuous. Moreover, it is convex in x for any $z \in Z$.
  (iv) U is continuous, concave, differentiable and increasing in the state x for any $z \in Z$.

- With the previous assumptions, we can establish the following results:
  1. Equivalence of P1 (sequence problem) and P2 (recursive problem).
  2. $V : X \to \mathbb{R}$ exists, is unique, bounded, continuous, concave, increasing and differentiable.
  3. There exists a unique optimal plan with

$$x_{t+1}^* = \pi(x_t^*, z_t)$$

Roadmap

1. Sequence Problem
2. Recursive Formulation
3. **Role of Uncertainty**
4. Contraction Mapping Theorem
5. Characterization: Euler + Transversality
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Role of Uncertainty

- What are the "practical" complications introduced by uncertainty?
- Once we make sure the assumptions we need to solve the recursive problem hold, then not much.
- The expectation is just a weighed average of outcomes in different states:

$$V(x_t, z_t) = \max_{x_{t+1}=\pi(x_t,z_t)} \left\{ U(x_t, z_t, x_{t+1}) + \beta \sum_{j=1}^{N} pr\left[z_{t+1} = z_j \mid z_t\right] V(x_{t+1}, z_j) \right\}$$

- It is key the fact that the process is exogenous, and hence $pr\left[z_{t+1} = z_j \mid z_t\right]$ is not affected by the control variable x_{t+1}. This implies that:

$$\frac{\partial \mathbb{E}_t\left[V(x_{t+1}, z_{t+1})\right]}{\partial x_{t+1}} = \sum_{j=1}^{N} pr\left[z_{t+1} = z_j \mid z_t\right] \frac{\partial V(x_{t+1}, z_j)}{\partial x_{t+1}} = \mathbb{E}_t\left[\frac{\partial V(x_{t+1}, z_{t+1})}{\partial x_{t+1}}\right]$$

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Contraction Mapping Theorem (1): Motivation

- A map is a function that transforms functions into functions (rather than numbers).

$$\mathbb{T}\left(f\left(x\right)\right)=g\left(x\right)$$

- Bellman Equation can be written as map in value functions and policy rules. For any function W, define the map $\mathbb{T}$ as:

$$\mathbb{T}(W)=\max_{x_{t+1}\in G(x_t,z_t)} U(x_t,z_t,x_{t+1})+\beta\mathbb{E}_t[W(x_{t+1},z_{t+1},x_{t+2})]$$

- The solution is a fixed point of the mapping:

$$V^{*}=\mathbb{T}(V^{*})$$

- The Contraction Mapping Theorem ensures that you can find the fixed point with an iterative procedure.

Contraction Mapping Theorem (2): Example

- Example: household that maximizes intertemporal consumption:

$$\begin{aligned} V(a_t) &= \max_{c_t} u(c_t) + \beta \mathbb{E}_t [V(a_{t+1})] \\ s.t \;\; a_{t+1} &= Ra_t - c_t + y_t \end{aligned}$$

- Substitute restriction into the value function:

$$\begin{aligned} V(a_t) &= \max_{c_t} u(c_t) + \beta \mathbb{E}_t \left[V \left(\overbrace{Ra_t - c_t + y_t}^{a_{t+1}} \right) \right] \\ &= \mathbb{T}(V(Ra_t - c_t + y_t)) \end{aligned}$$

where the mapping is defined as:

$$\mathbb{T}(W(a)) \equiv \max_{c} u(c) + \beta \mathbb{E}_t [W(Ra - c + y)]$$

- For every value of a_t, finding optimal consumption c_t^*, is equivalent to finding the the fixed point of the mapping, where the function that maps into itself:

$$\mathbb{T}(V^*(a)) = V^*(a)$$

Contraction Mapping Theorem (3): Definition

- Definition: Let $(\mathcal{F}, ||\cdot||)$ be a metric space. An on-to map $T : \mathcal{F} \to \mathcal{F}$ is a **contraction map** iff there exists a number $\beta \in [0, 1)$ such that

$$\|\mathbb{T}f_1(x) - \mathbb{T}f_2(x)\| \leq \beta \|f_1(x) - f_2(x)\|, \qquad \forall f_1, f_2 \in \mathcal{F}$$

  i.e. functions $\mathbb{T}f_1(x)$ and $\mathbb{T}f_2(x)$ are 'closer' than $f_1(x)$ and $f_2(x)$.

- Why is this useful? Consider a sequence of functions $\{f_n(x)\}_{n=0}^{\infty}$ given as:

$$f_n(x) = \mathbb{T}f_{n-1}(x)$$

  If $\mathbb{T}$ is a contraction map, then

$$\|f_n(x) - f_{n-1}(x)\| = \|\mathbb{T}f_{n-1}(x) - \mathbb{T}f_{n-2}(x)\|$$
$$\leq \beta \|f_{n-1}(x) - f_{n-2}(x)\| \leq \|f_{n-1}(x) - f_{n-2}(x)\|$$

  $\Rightarrow$ **Functions in the sequence become closer and closer.**

Contraction Mapping Theorem (4): Theorem

Theorem 1

Let $(\mathcal{F}, ||\cdot||)$ be a complete metric space and $\mathbb{T}$ a contraction mapping. Then it has a **unique** *fixed point, $\mathbb{T}f^* = f^*$.*

Moreover, for any initial guess f_0, the sequence $f_n = \mathbb{T}f_{n-1}$ will converge to f^*.

$$f_n(x) \rightarrow_{n\rightarrow\infty} f^*(x)$$

- Why is it unique? Assume not, then $\exists\ f^* \neq \tilde{f}$ such that $\mathbb{T}f^* = f^*$, $\mathbb{T}\tilde{f} = \tilde{f}$.

$$0 < \left\| f^* - \tilde{f} \right\| = \left\| \mathbb{T}f^* - \mathbb{T}\tilde{f} \right\| \leq \beta \left\| f^* - \tilde{f} \right\| \Longrightarrow \beta \geq 1! \quad (\mathit{Contradition})$$

- Very useful in practice: take any arbitrary initial guess for V, iterate the Bellman equation until convergence.
- The key is to check that our Bellman equation is a contraction map!

Contraction Mapping Theorem (5): Blackwell Conditions

- In general, it is hard to prove a map is a contraction.
- But we have the following useful *sufficient conditions* (Blackwell):

(B1) **Monotonicity:**

$$f_1(x) \leq f_2(x) \text{ for all } x \implies \mathbb{T}f_1(x) \leq \mathbb{T}f_2(x) \text{ for all } x$$

(B2) **Discounting:** There exists a $\beta \in [0,1)$ such that, for any constant k and any function f, we have:

$$\mathbb{T}(f+k) \leq \mathbb{T}f + \beta k$$

- If map $\mathbb{T}$ satisfies (B1) and (B2), then $\mathbb{T}$ is a contraction.
- Let us check if a map $\mathbb{T}$ given by a Bellman Equation is a contraction.

Contraction Mapping Theorem (6): Bellman & Blackwell

- **Bellman and Monotonicity:**
  Suppose $V(a) \geq W(a)$ for all a. Then it must be that $\mathbb{T}V(a) \geq \mathbb{T}W(a)$,

$$\begin{aligned}
\mathbb{T}V(a) &= \max_c \{u(c) + \beta\mathbb{E}[V(Ra - c + y)]\} \\
&\geq u(c_W^*) + \beta\mathbb{E}[V(Ra - c_W^* + y)] \\
&\geq u(c_W^*) + \beta\mathbb{E}[W(Ra - c_W^* + y)] \\
&= \max_c \{u(c) + \beta\mathbb{E}[W(Ra - c_W^* + y)]\} = \mathbb{T}W(a)
\end{aligned}$$

- **Bellman and Discounting:**

$$\begin{aligned}
\mathbb{T}(V(a) + k) &= \max_c \{u(c) + \beta\mathbb{E}[V(Ra - c + y) + k]\} \\
&= \max_c \{u(c) + \beta\mathbb{E}[V(Ra - c + y)]\} + \beta k \\
&= \mathbb{T}V(X) + \beta k
\end{aligned}$$

- Conclusion: if $\beta < 1$ then $\mathbb{T}$ (Bellman Equation) is a contraction map.

Roadmap

Characterization (1): First Order Conditions

- Bellman Equation:

$$V(x,z) = \max_{x'=\pi(x,z)} U(x,z,x') + \beta\mathbb{E}[V(x',z')] \qquad \forall x \in X$$

  - By the above assumptions, the maximization is strictly concave and differentiable.
  - For interior solutions, the first order conditions (FOC) are necessary.

- The FOC with respect to control x':

$$D_{x'} U(x,z,x') + \beta D\mathbb{E}[V(x',z')] = 0$$

  where D denotes the gradient and $D_{x'}$ the gradient wrt the vector x'.

- How to evaluate $D\mathbb{E}[V(x',z')] = 0$?

Characterization (2): Envelope Conditions

- As we saw earlier, since the stochastic process is exogenous, we can exchange the derivative and the expectation:

$$D\mathbb{E}[V(x', z')] = \mathbb{E}[DV(x', z')]$$

- Now use $V(x,z) = U(x,z,x') + \beta\mathbb{E}[V(x',z')]$ to compute $DV(x,z)$:

$$\begin{aligned} DV(x,z) &= D_x U(x,z,x') + D_{x'} U(x,z,x') \frac{dx'}{dx} + \beta\mathbb{E}\left[DV(x',z')\frac{dx'}{dx}\right] \\ &= D_x U(x,z,x') + \underbrace{\{D_{x'} U(x,z,x') + \beta\mathbb{E}\left[DV(x',z')\right]\}}_{=0 \text{ by FOC}} \frac{dx'}{dx} \\ &= D_x U(x,z,x') \end{aligned}$$

- Intuition: V is maximized wrt to x' (small changes in x' do not affect it, envelope theorem).

Characterization (3): Euler Equation

- Back to the FOC:

$$D_{x'} U(x, z, x') + \beta \mathbb{E}[DV(x', z')] = 0$$

- The second term is the derivative we just computed with envelope condition, but one period forward:

$$\mathbb{E}[DV(x', z')] = \mathbb{E}[D_{x'} U(x', z', x'')]$$

- Substituting back:

$$D_{x'} U(x, z, x') + \beta \mathbb{E}[D_{x'} U(x', z', x'')] = 0$$

- This Euler equation characterizes implicitly the (unknown) optimal policy $\pi(x, z)$:

$$D_{x'} U(x, z, \pi(x, z)) + \beta \mathbb{E}[D_{x'} U(\pi(x, z), z', \pi(\pi(x, z), z'))] = 0$$

Characterization (4): Transversality Condition

- The Euler Equation establishes the optimality of the solution between two contiguous periods (one period deviations from optimal policy are not profitable).
- What about an infinite deviation?
- The solution must also satisfy the *transversality condition*:

$$\lim_{t \to \infty} \beta^t \ \mathbb{E}_t \left[D_x U(x_t^*, z, x_{t+1}^*) \right] \cdot x_t^* = 0$$

  - The discounted value of the x_t^* must approach zero at infinity.
  - Infinite-horizon equivalent of a "terminal" condition in the finite case, so that all wealth must be consumed by the end of period.

Characterization (5): One-dimensional case

- Suppose x, z and x' are real numbers.
- The First Order Condition is:

$$-\frac{\partial U(x,z,x')}{\partial x'} = \beta \mathbb{E}\left[\frac{\partial V(x',z')}{\partial x'}\right]$$

  - Indifference condition: Today's cost of increasing x' (i.e. capital tomorrow) has to be equal to its expected discounted marginal gain on future utility (i.e. profits).

- The Envelope Condition:

$$\frac{\partial V(x,z)}{\partial x} = \frac{\partial U(x,z,x')}{\partial x}$$

Characterization (5): One-dimensional case (cont...)

- Forward the envelope one period

$$\frac{\partial V(x', z')}{\partial x'} = \frac{\partial U(x', z', x'')}{\partial x'}$$

- The Euler Equation (substitute forwarded envelope into FOC):

$$-\frac{\partial U(x, z, x')}{\partial x'} = \beta \mathbb{E}\left[\frac{\partial U(x', z', x'')}{\partial x'}\right]$$

  - New indifference condition only between today and tomorrow (effect on future continuation value is second order, because of envelope theorem).

Characterization (5): One-dimensional case (cont...)

- Finally, the Transversality Condition.
- Suppose the last period is $t = T$, we choose x_{T+1} to max $\beta^T U(x_T, z_T, x_{T+1})$
- Because of potential corner, the FOC reads:

$$\beta^T \frac{\partial U(x_T, z_T, x_{T+1})}{\partial x_{T+1}} x_{T+1} = 0$$

  - Either an interior solution is optimal $\left(\frac{\partial U(x_T, z_T, x_{T+1})}{\partial x_{T+1}} = 0\right)$ or we go to a corner solution $x_{T+1} = 0$.

- When $T \to \infty$, we take the limit:

$$\lim_{T\to\infty} \beta^T \underbrace{\frac{\partial U(x_T, z_T, x_{T+1})}{\partial x_{T+1}}}_{\text{use Euler}} x_{T+1} = \lim_{T\to\infty} \beta^{T+1} \frac{\partial U(x_{T+1}, z_{T+1}, x_{T+2})}{\partial x_{T+1}} x_{T+1} = 0$$

Characterization (6): Our previous example

- Bellman:

$$V(a,y) = \max_{a'} \quad u(Ra + y - a') + \beta \mathbb{E}\left[V(a', y')\right]$$

- FOC:

$$\frac{\partial u(c)}{\partial c} = \beta \mathbb{E}\left[\frac{\partial V(a', y')}{\partial a'}\right]$$

- Envelope:

$$\frac{\partial V(a,y)}{\partial a} = R\frac{\partial u(c)}{\partial c} \qquad \Longrightarrow_{forward} \qquad \frac{\partial V(a',y')}{\partial a'} = R\frac{\partial u(c')}{\partial c'}$$

- Euler = FOC + Forward Envelope

$$\frac{\partial u(c)}{\partial c} = \beta R \mathbb{E}\left[\frac{\partial u(c')}{\partial c'}\right]$$

- Transversality

$$\lim_{t\to\infty} \beta^t \, \frac{\partial u(c_t)}{\partial c_t} a_t = 0$$

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Solution Methods

- The Bellman equation is a functional equation.
- How to Solve Functional Equations? No general way, several approaches.
- Closed form: Guess a functional form (often same form as U) with undetermined coefficients and verify.
- Numerical dynamic programming:
  a) Value function iteration.
  b) Policy function iteration (Howard improvement algorithm).
  c) Projection methods (approximate policy with polynomials).

Guess and Verify (Undetermined Coefficients)

- In some special cases, one can obtain closed form solutions.
- Example: Stochastic growth with Log utility and Cobb-Douglas production.
  - Consider the problem:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \left(\sum_{t=0}^{\infty} \beta^t \ln c_t \right)$$

$$st : k_{t+1} = \theta_t k_t^{\alpha} - c_t, \qquad k_0, \theta_0 \text{ given}$$

$$\log(\theta_t) \sim \ iid(0, \sigma^2)$$

  - In this case k_t (state), k_{t+1} (control) and $c_t = \theta_t k_t^{\alpha} - k_{t+1}$
- Bellman Equation:

$$V(k_t, \theta_t) = \max_{k_{t+1}} \ \ln(\theta_t k_t^{\alpha} - k_{t+1}) + \beta \mathbb{E}_t [V(k_{t+1}, \theta_{t+1})]$$

Guess and Verify (Undetermined Coefficients)

- First Order Condition:

$$\frac{1}{\theta_t k_t^{\alpha} - k_{+1}} = \beta \mathbb{E}_t \left[\frac{\partial V\left(k_{t+1}, \theta_{t+1}\right)}{\partial k_{t+1}} \right]$$

- Envelope condition:

$$\frac{\partial V\left(k_t, \theta_t\right)}{\partial k_t} = \frac{\alpha \theta_t k_t^{\alpha-1}}{\theta_t k_t^{\alpha} - k_{t+1}}$$

- Forwarding by one period:

$$\mathbb{E}_t \left[\frac{\partial V\left(k_{t+1}, \theta_{t+1}\right)}{\partial k_{t+1}} \right] = \mathbb{E}_t \left[\frac{\alpha \theta_{t+1} k_{t+1}^{\alpha-1}}{\theta_{t+1} k_{t+1}^{\alpha} - k_{t+2}} \right]$$

- Substituting back in FOC:

$$\frac{1}{\theta_t k_t^{\alpha} - k_{t+1}} = \mathbb{E}_t \left[\frac{\alpha \beta \theta_{t+1} k_{t+1}^{\alpha-1}}{\theta_{t+1} k_{t+1}^{\alpha} - k_{t+2}} \right]$$

Guess and Verify (Undetermined Coefficients)

- We guess the value function as a log-linear function of the states:

$$V\left(k_t, \theta_t\right) = v_1 + v_2 \log k_t + v_3 \log \theta_t$$

- Since $\log(\theta_t) \sim \ iid(0, \sigma^2)$, the guess implies that:

$$\mathbb{E}_t\left[V\left(k_{t+1}, \theta_{t+1}\right)\right] = v_1 + v_2 \log k_{t+1}$$

- Substituting the guess in the Bellman Equation:

$$V\left(k_t, \theta_t\right) = \max_{k_{t+1}} \ \ln\left(\theta_t k_t^{\alpha} - k_{t+1}\right) + \beta v_1 + \beta v_2 \log k_{t+1}$$

Guess and Verify (Undetermined Coefficients)

- Bellman Equation:

$$V(k_t, \theta_t) = \max_{k_{t+1}} \ln(\theta_t k_t^{\alpha} - k_{t+1}) + \beta v_1 + \beta v_2 \log k_{t+1}$$

- FOC:

$$-\frac{1}{\theta_t k_t^{\alpha} - k_{t+1}} + \frac{\beta v_2}{k_{t+1}} = 0 \qquad \Longrightarrow \quad k_{t+1} = \frac{\beta v_2}{1 + \beta v_2} \theta_t k_t^{\alpha}$$

- Substitute the solution into the value function:

$$V(k_t, \theta_t) = \ln\left(\theta_t k_t^{\alpha} - \frac{\beta v_2}{1 + \beta v_2} \theta_t k_t^{\alpha}\right) + \beta v_1 + \beta v_2 \log \frac{\beta v_2}{1 + \beta v_2} \theta_t k_t^{\alpha}$$

- Rearrange as follows (For homework verify this claim):

$$V(k_t, \theta_t) = \text{constant} + (1 + \beta v_2) \ln(\theta_t k_t^{\alpha})$$

and conclude that:

$$V(k_t, \theta_t) = \text{constant} + \underbrace{\alpha(1 + \beta v_2)}_{v_2} \ln k_t + \underbrace{(1 + \beta v_2)}_{v_3} \ln \theta_t$$

Guess and Verify (Undetermined Coefficients)

- If the guess is right, then the following equations must have a solution

$$\begin{aligned} \alpha\left(1+\beta v_2\right) &= v_2 \\ 1+\beta v_2 &= v_3 \end{aligned}$$

- Solving for v_2 and v_3 we obtain:

$$v_2 = \frac{\alpha}{1-\alpha\beta}, \qquad v_3 = \frac{1}{1-\alpha\beta}$$

- Hence the policy function:

$$k_{t+1} = \left[1+\frac{1}{\beta v_2}\right]^{-1} \theta_t k_t^{\alpha} = \alpha\beta\theta_t k_t^{\alpha}$$

- And the value function:

$$V\left(k_t, \theta_t\right) = \text{constant} + \frac{\alpha}{1-\alpha\beta} \ln k_t + \frac{1}{1-\alpha\beta} \ln \theta_t$$

Homework: Autocorrelated shocks

- Assume now that the productivity shocks θ_t are autocorrelated.

$$\log \theta_t = \gamma \log \theta_{t-1} + \varepsilon_t$$

  where ε_t is $iid(0, \sigma_\varepsilon^2)$ and $\gamma < 1$.

- Verify that the guess $V(k_t, \theta_t) = v_1 + v_2 \log k_t + v_3 \log \theta_t$ is still correct.

- Compute the optimal policy and the value function in this case.

- How does the semi-elasticity of the value function with respect to θ change with the persistence parameter γ?