# Aggregate Dynamics in Lumpy Economies

*Online Appendix: Not For Publication*

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February 15, 2019

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# A Definitions

Let $(\Omega, \mathcal{P}, \mathcal{F})$ be a probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t; t \geq 0)$ and let $\tilde{S}_t(\omega) \in \mathbb{R}^n$ be an stochastic process $\mathcal{F}_t$ measurable in this probability space. Here we state all the definitions used in the main text.

**Definition 1** (Stopping time)**.** *A function $\tau : \Omega \to [0, \infty]$ is called a stopping time with respect to $\mathcal{F}_t$ if $\{\omega; \tau(\omega) \leq t\} \in \mathcal{F}_t$ for all $t$.*

**Definition 2** (Strong Markov process for Itō diffusions)**.** *Let $f$ be a bounded Borel function on $\mathbb{R}^n$, $\tau$ a stopping time with respect to $\mathcal{F}_t$ and $\tau < \infty$ a.s.. Then a process has the Strong Markov property under $\tau$ if*

$$\mathbb{E}^x[f(X_{\tau+h})|\mathcal{F}_\tau] = \mathbb{E}^{X_\tau}[f(X_{\tau+h})] \quad \textit{for all } h > 0$$

*A process has the Strong Markov property if it has the Strong Markov property under all $\tau$.*

**Definition 3** (Well-defined stopping process)**.** *We say that $(\{X_t\}_t, \tau)$ is a well-defined stopping process if $\tau$ is bounded almost surely, or the following conditions hold:*

1. $\Pr^{\mathcal{P}}[\tau < \infty] = 1$.
2. $\Pr^{\mathcal{P}}[|X_\tau|] < \infty$.
3. $\lim_{t\to\infty} \mathbb{E}^{\mathcal{P}}\left[|X_t \mathbb{1}_{\{\tau>t\}}|\right] = 0$.

# B Extensions

For the following extensions, we maintain the assumption that the moments of the adjustment size can be written as:

$$g_m(x) = \mathbb{E}^{\hat{x}+x,S^{-x}}\left[(\hat{x}-\Delta x)^m\right] - \mathbb{E}^{\hat{x},S^{-x}}\left[(\hat{x}-\Delta x+x)^m\right] \tag{B.1}$$

In the proofs, we skip the steps that are identical to the main proofs in the Appendix.

## B.1 Characterization of the CIR for higher moments of the distribution

**Proposition B.1.** *Assume the uncontrolled state follows $d\tilde{x}_t = \sigma dB_t$, and the processes $\left(\left\{\int_0^t s x_s^m \, dB_s\right\}_t, \tau\right)$ are well-defined stopping processes for all $m$,*

- *To a first order, the transitional dynamics are given by*

$$\mathcal{A}_m(\delta) = \delta \times (\Gamma_m + \Theta_m - \mathcal{M}_m[x]\Theta_0) + o(\delta^2) \tag{B.2}$$

  *where the intensive and the extensive margins relate to ergodic moments as follows:*

$$\Gamma_m = m\mathbb{E}[x^{m-1}a]. \tag{B.3}$$

$$\Theta_m = \sum_{j=0}^{\infty} \theta_{m,j}\mathcal{M}_j[x] \quad \text{with} \quad \theta_{m,j} \equiv \frac{2}{\sigma^2(m+1)} \sum_{k\geq j}^{\infty} \frac{\hat{x}^{k-j}}{k!j!} \left[\frac{d^{k+1}g_{m+2}(x)/(m+2)}{dx^{k+1}} - \frac{d^k g_{m+1}(x)}{dx^k}\right]\Bigg|_{x=0}. \tag{B.4}$$

  - *If $\tau|S_t \sim \tau|S_t^{-x}$, then $g_m(x) = 0$ with $\theta(m,i) = 0$ $\forall m, i$.*

- *The reset state and the volatility are given by* (**??**) *and* (**??**) *evaluated at $\nu = 0$. The ergodic moments are given by*

$$\mathcal{M}_m[x] = \frac{2}{(m+1)(m+2)} \frac{\mathbb{E}\left[(\hat{x}-\Delta x)^{m+2} - \hat{x}^{m+2}\right]}{\mathbb{E}\left[\Delta x^2\right]}, \tag{B.5}$$

$$\mathcal{M}_{m,1}[x,a] = \frac{2}{(m+1)(m+2)} \left[\frac{\mathbb{E}^{\hat{S}}\left[(\hat{x}-\Delta x)^{m+2}\,\tau\right]}{\mathbb{E}^{\hat{S}}\left[\Delta x^2\right]} - \frac{\mathbb{E}[x^{m+2}]}{\sigma^2}\right], \tag{B.6}$$

## B.2 Characterization of the CIR with drift

**Proposition B.2.** *Assume the uncontrolled process follows $d\tilde{x}_t = \nu dt + \sigma dB_t$.*

- ***Aggregation:*** *To a first order, the CIR is given by*

$$\mathcal{A}_m(\delta) = \delta \times (\mathcal{Z}_m - \mathcal{M}_m[x]\Theta_0) + o(\delta^2) \tag{B.7}$$

  *where the intensive and extensive margin are given by*

$$\mathcal{Z}_m = \Theta_m + \Gamma_m - \frac{\sigma^2 m}{2\nu}\mathcal{Z}_{m-1} \tag{B.8}$$

$$\Gamma_m = \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \varphi_m^\Gamma(S_t)dt\right]}{\mathbb{E}^{\hat{S}}[\tau]} \quad ; \quad \varphi_m^\Gamma(S_t) = \frac{1}{\nu}\left(\mathbb{E}^S\left[x_\tau^m\right] - x_t^m\right) \tag{B.9}$$

$$\Theta_m = \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \varphi_m^\Theta(S_t)dt\right]}{\mathbb{E}^{\hat{S}}[\tau]} \quad ; \quad \varphi_m^\Theta(S_t) = \frac{1}{\nu}\left[\frac{\partial \mathbb{E}^S\left[x_\tau^{m+1}/(m+1)\right]}{\partial x} - \mathbb{E}^S\left[x_\tau^m\right]\right] \tag{B.10}$$

- ***Representation:***

$$\Gamma_m = m\mathbb{E}[x^{m-1}a] + \frac{\mathbb{1}_{\{m\geq 2\}}\sigma^2 m(m-1)}{2\nu}\mathcal{M}_{m-2,1}[x,a] \tag{B.11}$$

$$\Theta_m = \sum_{j=0}^{\infty}\theta_{m,j}\mathcal{M}_j[x] \quad \text{with} \quad \theta_{m,j} = \sum_{k\geq j}^{\infty}\frac{\hat{x}^{k-j}}{\nu k!j!}\left[\frac{d^{k+1}g_{m+1}(x)}{dx^{k+1}}/m+1 - \frac{d^k g_m(x)}{dx^k}\right]\Bigg|_{x=0}. \tag{B.12}$$

- ***Observability:*** *The reset state $\hat{x}$ and structural parameters $(\nu, \sigma)$ are recovered as*

$$\hat{x} = \mathbb{E}[\Delta x]\left(\frac{1-\mathbb{CV}^2[\tau]}{2}\right) + \frac{Cov[\tau, \Delta x]}{\mathbb{E}[\tau]}, \qquad \nu = -\frac{\mathbb{E}[\Delta x]}{\mathbb{E}[\tau]}, \qquad \sigma^2 = \frac{\mathbb{E}[\Delta x^2]}{\mathbb{E}[\tau]} + 2\nu\hat{x} \tag{B.13}$$

*and the ergodic moments are recovered as:*

$$\mathbb{E}[x^m] = \frac{\hat{x}^{m+1} - \mathbb{E}[(\hat{x}-\Delta x)^{m+1}]}{\mathbb{E}[\Delta x](m+1)} - \frac{\sigma^2 m}{2\nu}\mathbb{E}[x^{m-1}], \tag{B.14}$$

$$\mathcal{M}_{m,1}[x,a] = \frac{\mathbb{E}\left[\tau/\mathbb{E}[\tau]\,(\hat{x}-\Delta x)^{m+1}\right] - \mathbb{E}[x^{m+1}]}{\nu(m+1)} - \frac{\sigma^2 m}{2\nu}\mathbb{E}[x^{m-1}a] \tag{B.15}$$

*with initial conditions* $\mathcal{M}_1[x]=0$ *and* $\mathcal{M}_{0,1}[x,a] = \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]}$.

*Proof.* We start the proof from equation (**??**) in the Appendix, as all previous steps are identical.

**Aggregation.** The first order approximation to the CIR yields $\mathcal{A}'_m(0)$ equal to:

$$\underbrace{\int \mathbb{E}^S\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right]dF(S)}_{\mathcal{B}_m} - \mathbb{E}[x^m]\underbrace{\int \frac{\partial \mathbb{E}^S[\tau]}{\partial x}dF(S)}_{\Theta_0} + \underbrace{\int\left(\frac{\partial}{\partial x}\mathbb{E}^S\left[\int_0^\tau x_t^m dt\right] - \mathbb{E}^S\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right]\right)dF(S)}_{\mathcal{C}_m}. \tag{B.16}$$

To characterize the first term $\mathcal{B}_m \equiv \int \mathbb{E}^S\left[\int_0^\tau m x_t^{m-1}dt\right]dF(S)$, we apply Itō's lemma to $x_t^m$, integrate with respect to initial condition $S$, use the OST, divide by the drift, and rearrange:

$$\mathbb{E}^S\left[\int_0^\tau m x_t^{m-1}dt\right] = \underbrace{\frac{\mathbb{E}^S[x_\tau^m] - x_t^m}{\nu}}_{\varphi_m^\Gamma(S_t)} - \frac{\sigma^2 m}{2\nu}\mathbb{E}^S\left[\int_0^\tau (m-1)x_t^{m-2}dt\right].$$

Integrating both sides across all initial conditions, using the definition of $\Gamma_m$ in (B.9), and recognizing $\mathcal{B}_m$ and $\mathcal{B}_{m-1}$ we get:

$$\mathcal{B}_m = \Gamma_m - \frac{\sigma^2 m}{2\nu}\mathcal{B}_{m-1}, \qquad \Gamma_0 = 0 \tag{B.17}$$

To characterize $\mathcal{C}_m$, we use the previous expressions to compute its two terms separately:

$$\frac{\partial \mathbb{E}^S\left[\int_0^\tau x_t^m dt\right]}{\partial x} = \frac{1}{\nu}\left(\frac{\partial \mathbb{E}^S\left[x_\tau^{m+1}\right]/(m+1)}{\partial x} - x_t^m\right) - \frac{\sigma^2 m}{2\nu}\frac{\partial \mathbb{E}^S\left[\int_0^\tau x_t^{m-1}dt\right]}{\partial x} \tag{B.18}$$

$$\mathbb{E}^S\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right] = \frac{1}{\nu}\left(\mathbb{E}^S[x_\tau^m] - x_t^m\right) - \frac{\sigma^2 m}{2\nu}\mathbb{E}^S\left[\int_0^\tau \frac{\partial x_t^{m-1}}{\partial x}dt\right] \tag{B.19}$$

Subtracting the two previous equations,

$$\frac{\partial \mathbb{E}^S\left[\int_0^\tau x_t^m dt\right]}{\partial x} - \mathbb{E}^S\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right] = \underbrace{\frac{1}{\nu}\left(\frac{\partial \mathbb{E}^S\left[x_\tau^{m+1}\right]/(m+1)}{\partial x} - \mathbb{E}^S[x_\tau^m]\right)}_{\varphi_m^\Theta(S_t)} - \frac{\sigma^2 m}{2\nu}\left\{\frac{\partial \mathbb{E}^S\left[\int_0^\tau x_t^{m-1}dt\right]}{\partial x} - \mathbb{E}^S\left[\int_0^\tau \frac{\partial x_t^{m-1}}{\partial x}dt\right]\right\}$$

Integrating with the ergodic distribution and using the definition of $\Theta_m$ in (B.10) and recognizing $\mathcal{C}_m$ and $\mathcal{C}_{m-1}$ we get:

$$\mathcal{C}_m = \Theta_m - \frac{\sigma^2 m}{2\nu}\mathcal{C}_{m-1}, \qquad \mathcal{C}_{-1} = 0. \tag{B.20}$$

Define $\mathcal{Z}_m \equiv \mathcal{B}_m + \mathcal{C}_m$, which implies $\mathcal{Z}_m = \Gamma_m + \Theta_m - \frac{\sigma^2 m}{2\nu}\mathcal{Z}_{m-1}$. Combine the results in (B.16), (B.17) and (B.20) to obtain (B.7): $\mathcal{A}'_m(0) = (\mathcal{Z}_m - \mathcal{M}_m[x]\Theta_0)$.

Lastly, we corroborate that the expression $\int \frac{\partial \mathbb{E}^S[\tau]}{\partial x}dF(S)$ is equal to $\Theta_0$. By the OST, we have $\mathbb{E}^S[x_\tau] - x = \nu\mathbb{E}^S[\tau]$. Thus $\frac{\partial \mathbb{E}^S[\tau]}{\partial x} = \frac{1}{\nu}\left[\frac{\partial \mathbb{E}^S[x_\tau]}{\partial x} - 1\right]$. Substituting and using Auxiliary Theorem **??** we recover the expression for $\Theta_0$ in the definition of $\Theta_m$:

$$\Theta_0 \equiv \int \frac{\partial \mathbb{E}^S[\tau]}{\partial x}dF(S) = \int \frac{1}{\nu}\left[\frac{\partial \mathbb{E}^S[x_\tau]}{\partial x} - 1\right]dF(S)$$

**Representation for the intensive margin:** The proof for $\Gamma_m$ are easy extended to

$$\Gamma_m = \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \left[\mathbb{E}^{S_t}[x_\tau^m] - x_t^m\right]dt\right]}{\nu\mathbb{E}^{\hat{S}}[\tau]} = \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau [x_\tau^m - x_t^m]\,dt\right]}{\nu\mathbb{E}^{\hat{S}}[\tau]}$$

With similar steps as the case with no drift we have that

$$\frac{\mathbb{E}^{\hat{S}}\left[x_\tau^m \tau\right]}{\mathbb{E}^{\hat{S}}\left[\tau\right]} - \mathbb{E}[x^m] = \mathbb{1}_{\{m\geq 1\}}\nu m \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{m-1} t dt\right]}{\mathbb{E}^{\hat{S}}[\tau]} + \frac{\mathbb{1}_{\{m\geq 2\}}\sigma^2 m(m-1)}{2}\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{m-2} t dt\right]}{\mathbb{E}^{\hat{S}}[\tau]} \tag{B.21}$$

Thus

$$\Gamma_m = \mathbb{1}_{\{m\geq 1\}} m\mathbb{E}[x^{m-1}a] + \frac{\mathbb{1}_{\{m\geq 2\}}\sigma^2 m(m-1)}{2\nu}\mathcal{M}_{m-2,1}[x,a] \tag{B.22}$$

**Representation for the extensive margin:** The proofs for $\Theta_m$ are easily to extend to

$$\begin{aligned}
\Theta_m &= \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \left(\frac{\partial \mathbb{E}^S[x_\tau^{m+1}/(m+1)]}{\partial x} - \mathbb{E}^S[x_\tau^m]\right) dt\right]}{\nu \mathbb{E}^{\hat{S}}\left[\tau\right]} &\text{(B.23)}\\
&= \frac{1}{\nu}\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \left[\left[\frac{dg_{m+1}(y-\hat{x})}{dy}/(m+1) + \mathbb{E}^{\hat{x},S^{-x}}\left[(y-\hat{x}+x_\tau)^{m+1}\right]\right] - \left[g_m(y-\hat{x}) + \mathbb{E}^{\hat{x},S^{-x}}\left[(y-\hat{x}+x_\tau)^m\right]\right]\right] dt\right]}{\mathbb{E}^{\hat{S}}[\tau]}\\
&= \frac{1}{\nu}\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \left[\frac{dg_{m+1}(y-\hat{x})}{dy}/(m+1) - g_m(y-\hat{x})\right] dt\right]}{\mathbb{E}^{\hat{S}}[\tau]}\\
&= \frac{1}{\nu}\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \left[\sum_{j=0}^\infty \frac{d^j}{j! dx^j}\left[\frac{dg_{m+1}(x)}{dx}/(m+1) - g_m(x)\right]_{x=0}(y-\hat{x})^j\right] dt\right]}{\mathbb{E}^{\hat{S}}[\tau]}\\
&= \frac{1}{\nu}\sum_{j=0}^\infty\sum_{z=0}^j \frac{\left[\frac{d^{j+1}g_{m+1}(0)}{dx^{j+1}}/m+1 - \frac{d^j g_m(0)}{dx^j}\right]\hat{x}^z}{j!}\begin{pmatrix} j \\ z\end{pmatrix}\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{j-z} dt\right]}{\mathbb{E}^{\hat{S}}[\tau]}\\
&= \frac{1}{\nu}\sum_{j=0}^\infty\sum_{z=0}^j \frac{\left[\frac{d^{j+1}g_{m+1}(0)}{dx^{j+1}}/m+1 - \frac{d^j g_m(0)}{dx^j}\right]\hat{x}^z}{j!}\begin{pmatrix} j \\ z\end{pmatrix}\mathcal{M}_{j-z}[x]\\
&= \sum_{j=0}^\infty\sum_{z=0}^j \underbrace{\frac{1}{\nu}\frac{\left[\frac{d^{j+1}g_{m+1}(0)}{dx^{j+1}}/m+1 - \frac{d^j g_m(0)}{dx^j}\right]\hat{x}^z}{z!(j-z)!}}_{=\mathcal{H}_{m,j,z}}\mathcal{M}_{j-z}[x]\\
&= \sum_{h=0}^\infty \theta_{m,h}\mathcal{M}_h[x], \ \textit{with} \ \theta_{m,h} = \sum_{k\geq h}^\infty \mathcal{H}_{m,k,k-h}. &\text{(B.24)}
\end{aligned}$$

**Observation:** First, we characterize the structural parameters of the stochastic process.

- It is easy to show that $\frac{\mathbb{E}^{\hat{S}}[\Delta x]}{\mathbb{E}^{\hat{S}}[\tau]} = -\nu$.
- For characterizing $\sigma$ define $Y_t = x_t - \nu t$ with initial condition $Y_0 = \hat{x}$. With similar steps as the process without drift we have that

$$\sigma^2 = \frac{\mathbb{E}^{\hat{S}}\left[\Delta Y_\tau^2\right]}{\mathbb{E}^{\hat{S}}[\tau]} = \frac{\mathbb{E}^{\hat{S}}\left[(x_\tau - \nu\tau - x + x_0)^2\right]}{\mathbb{E}^{\hat{S}}[\tau]} = \frac{\mathbb{E}^{\hat{S}}\left[(\nu\tau + \Delta x)^2\right]}{\mathbb{E}^{\hat{S}}[\tau]} \tag{B.25}$$

  or equivalently

$$\sigma^2 = \frac{\mathbb{E}^{\hat{S}}[\Delta x^2]}{\mathbb{E}^{\hat{S}}[\tau]} + 2\nu\frac{\mathbb{E}^{\hat{S}}[\Delta x\tau]}{\mathbb{E}^{\hat{S}}[\tau]} + \nu^2\frac{\mathbb{E}^{\hat{S}}[\tau^2]}{\mathbb{E}^{\hat{S}}[\tau]} = \frac{\mathbb{E}^{\hat{S}}[\Delta x^2]}{\mathbb{E}[\tau]} - 2\frac{\mathbb{E}[\Delta x]\mathbb{E}[\Delta x\tau]}{\mathbb{E}[\tau]^2} + \frac{\mathbb{E}[\Delta x]^2\mathbb{E}[\tau^2]}{\mathbb{E}[\tau]^3}$$

  Applying the formula we get below we have the result.
- For the reset state $\hat{x}$, we apply Itō's lemma to $x_t^2$ to obtain $d(x_t^2) = 2x_t dx_t + (dx_t)^2 = \left(2\nu x_t + \sigma^2\right)dt + 2\sigma x_t dB_t$. Using the OST $\mathbb{E}^{\hat{S}}[\int_0^\tau x_s dB_s] = 0$. Moreover, given that $\mathbb{E}^{\hat{S}}[\int_0^\tau x_s ds] = \mathcal{M}_1[x]\mathbb{E}^{\hat{S}}[\tau] = 0$, we have that

$$\mathbb{E}^{\hat{S}}[x_\tau^2] = \hat{x}^2 + \sigma^2\mathbb{E}^{\hat{S}}[\tau] \tag{B.26}$$

  Completing squares

$$\mathbb{E}^{\hat{S}}[x_\tau^2] = \mathbb{E}^{\hat{S}}[(\hat{x} - (\hat{x} - x_\tau))^2] = \mathbb{E}^{\hat{S}}[\Delta x^2] - 2\hat{x}\mathbb{E}^{\hat{S}}[\Delta x] + (\hat{x})^2$$

Therefore

$$\hat{x} = \frac{1}{2\mathbb{E}^{\hat{S}}[\Delta x]}\left[\mathbb{E}^{\hat{S}}[\Delta x^2] - \sigma^2\mathbb{E}^{\hat{S}}[\tau]\right] \tag{B.27}$$

$$= \frac{1}{2\mathbb{E}^{\hat{S}}[\Delta x]}\left[\mathbb{E}^{\hat{S}}[\Delta x^2] - \left(\mathbb{E}^{\hat{S}}[\Delta x^2] + 2\frac{\mathbb{E}^{\hat{S}}[\Delta x]\mathbb{E}^{\hat{S}}[\Delta x\tau]}{\mathbb{E}^{\hat{S}}[\tau]} + \frac{\mathbb{E}^{\hat{S}}[\Delta x]^2\mathbb{E}^{\hat{S}}[\tau^2]}{\mathbb{E}^{\hat{S}}[\tau]^2}\right)\right] \tag{B.28}$$

$$= \frac{\mathbb{E}^{\hat{S}}[\Delta x\tau]}{\mathbb{E}^{\hat{S}}[\tau]} - \frac{\mathbb{E}^{\hat{S}}[\Delta x]\mathbb{E}^{\hat{S}}[\tau^2]}{2\mathbb{E}^{\hat{S}}[\tau]^2} \tag{B.29}$$

- For observability of ergodic moments of $x$, apply Itō's lemma to $x^{m+1}$ and get $dx_t^{m+1} = (m+1)x_t^m\nu dt + (m+1)x_t^m\sigma dB_t + \frac{\sigma^2}{2}m(m+1)x_t^{m-1}dt$. Integrating from 0 to $\tau$, using the OST to eliminate martingales, and rearranging:

$$\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^m dt\right] = \frac{1}{\nu(m+1)}\left(\mathbb{E}^{\hat{S}}[x_\tau^{m+1}] - \hat{x}^{m+1}\right) - \frac{\sigma^2}{2\nu}m\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{m-1}dt\right] \tag{B.30}$$

Substituting the equivalences $\mathbb{E}[x^m] = \mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^m dt\right]/\mathbb{E}^{\hat{S}}[\tau]$ and $\mathbb{E}^{\hat{S}}[\Delta x] = -\nu\mathbb{E}^{\hat{S}}[\tau]$ yields:

$$\mathbb{E}[x^m] = \frac{\hat{x}^{m+1} - \mathbb{E}^{\hat{S}}[(\hat{x}-\Delta x)^{m+1}]}{\mathbb{E}^{\hat{S}}[\Delta x](m+1)} - \frac{\sigma^2 m}{2\nu}\mathbb{E}[x^{m-1}], \qquad \mathcal{M}_1[x] = 0 \tag{B.31}$$

- For observability of ergodic moments of $x^m a$, where $a$ stand for the duration of the last action, we use Itō's lemma and the OST on $x_t^{m+1}t$:

$$\mathbb{E}^{\hat{S}}\left[\tau\left(\hat{x}-\Delta x\right)^{m+1}\right] = \mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{m+1}dt\right] + (m+1)\nu\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^m t dt\right] + \frac{\sigma^2 m(m+1)}{2}\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{m-1}t dt\right] \tag{B.32}$$

and therefore

$$\mathcal{M}_{m,1}[x,a] = \frac{\mathbb{E}^{\hat{S}}\left[\tau\left(\hat{x}-\Delta x\right)^{m+1}\right]}{\nu(m+1)\mathbb{E}^{\hat{S}}[\tau]} - \frac{\mathbb{E}[x^{m+1}]}{\nu(m+1)} - \frac{\sigma^2 m}{2\nu}\mathbb{E}[x^{m-1}a] \tag{B.33}$$

with initial condition $\mathcal{M}_{0,1}[x,a] = \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]}$.

□

## B.3 Characterization of the CIR for Different Initial Conditions

For this proof we focus in the case $m = I = 1$.

**Proposition B.3.** *Assume a function $\mathcal{G}(x,\delta)$ s.t.*

1. $\mathcal{G}(x,0) = x$
2. $\exists z > 0$ *s.t.* $\forall \epsilon \in (-z, z)$ *the function* $\mathcal{G}(\cdot,\epsilon)$ *is bijective.*
3. $\frac{\partial \mathcal{G}(\mathcal{G}^{-1}(y,0),0)}{\partial \delta} = -\sum_{i=0}^{I} \frac{\mathcal{G}_i y^i}{i!}$ *with* $\sum_i \mathcal{G}_i^2 = 1$.

*Then, up to first order, the CIR is given by:*

- ***Aggregation:***

$$\mathcal{A}_1^{\mathcal{G}}(\delta) = \delta \times \sum_{i=0}^{1} \frac{\mathcal{G}_i}{i!} \left(\Gamma_{1,i} + \Theta_{1,i}\right) + o(\delta^2) \tag{B.34}$$

$$\Gamma_{1,i} = \frac{\mathbb{E}\left[\int_0^\tau \varphi_{1,i}^{\Gamma}(S_t)dt\right]}{\mathbb{E}[\tau]} \quad ; \quad \varphi_{1,i}^{\Gamma}(S) = \frac{2}{\sigma^2}\left[\frac{i x^{i-1}\mathbb{1}_{\{i\geq 2\}}}{\sigma^2}\frac{\mathbb{E}^S\left[x_\tau^3\right] - x^3}{3} + x^i \frac{\mathbb{E}^S\left[x_\tau^{m+1}\right] - x^2}{\sigma^2}\right] \tag{B.35}$$

$$\Theta_{1,i} = \frac{\mathbb{E}\left[\int_0^\tau \varphi_{1,i}^{\Theta}(S_t)dt\right]}{\mathbb{E}[\tau]} \quad ; \quad \varphi_{1,i}^{\Theta}(S) = \frac{x^i}{\sigma^2}\left[\frac{\partial \mathbb{E}^S\left[x_\tau^{m+2}/(m+2)\right]}{\partial x} - \frac{\partial \mathbb{E}^S\left[x_\tau^{m+1}\right]}{\partial x}\right] \tag{B.36}$$

- ***Representation:***

$$\Gamma_{m,i} = (i+1)\mathcal{M}_{i,1}[x,a] \tag{B.37}$$

$$\Theta_{1,i} = \sum_{j=0}^{\infty} \theta_{1,j}\mathcal{M}_{j+i}[x] \tag{B.38}$$

*with $\theta_{m,j}$ defined in proposition* **??**.

***Observation:*** *Without change.*

*Proof.* Let us explain the assumptions on the function $\mathcal{G}$. First, if there is no change in the distribution ($\delta = 0$), then the initial distribution is equal to the steady state distribution $\mathcal{G}(x,0) = x$. Second, $\mathcal{G}(\cdot,\delta)$ is bijective in a small domain around $\delta$. The third assumption is that $\frac{\partial(\mathcal{G}^{-1}(y,0),0)}{\partial \delta}$ is differentiable for all orders, thus $\frac{\partial \mathcal{G}(\mathcal{G}^{-1}(y,0),0)}{\partial \delta} = -\sum_i \frac{\mathcal{G}_i y^i}{i!}$. Finally, since the perturbation can be re-scaled by the size of the shocks, we normalize the coefficients of the Taylor approximation such that their squares sum up to one: $\sum \mathcal{G}_i^2 = 1$. We focus on the case $m = 1$.

Most steps used in the proofs for the baseline case hold with only minor changes.

**Aggregation**: The main different comes at the moment of doing the Taylor approximation, since

$$\mathcal{A}_1^{\mathcal{G}}(\delta) = \int_x \left[\int_{S^{-x}} v^1(x, S^{-x}) dF(S^{-x}|x)\right] dF(\mathcal{G}^{-1}(x,\delta)) \tag{B.39}$$

$$= \int_x \left[\int_{S^{-x}} v^1(x, S^{-x}) dF(S^{-x}|x)\right] f(\mathcal{G}^{-1}(x,\delta))dx \tag{B.40}$$

$$= \delta \int_x v^1(x) f'(\mathcal{G}^{-1}(x,0)) \frac{\partial \mathcal{G}^{-1}(x,0)}{\partial \delta} dx + o(\delta^2) \tag{B.41}$$

Notice that if $y = \mathcal{G}(x,\delta)$, then $\frac{dy}{d\delta}\big|_{\delta=0} = \frac{\partial \mathcal{G}(x,0)}{\partial \delta} = \frac{\partial \mathcal{G}(\mathcal{G}^{-1}(y,0),0)}{\partial \delta}$ and by assumption

$$\mathcal{A}_1^{\mathcal{G}}(\delta) = -\delta \sum_i \frac{\mathcal{G}_i}{i!} \int \left[x^i v^m(x)\right] f'(\mathcal{G}^{-1}(x,0))dx \tag{B.42}$$

$$= -\delta \sum_i \frac{\mathcal{G}_i}{i!} \int \left[x^i v^1(x)\right] f'(\mathcal{G}^{-1}(x,0))dx \tag{B.43}$$

$$= \delta \sum_i \frac{\mathcal{G}_i}{i!} \int \frac{\partial}{\partial x}\left[x^i v^1(x)\right] dF(S) + o(\delta) \tag{B.44}$$

$$= \delta \sum_i \frac{\mathcal{G}_i}{i!} \left[\Gamma_{1,i} + \Theta_{1,i}\right] + o(\delta) \tag{B.45}$$

where we define

$$\Gamma_{1,i} \equiv \int \left( i x^{i-1} \mathbb{E}^S \left[ \int_0^\tau x_t dt \right] + x^i \mathbb{E}^S \left[ \int_0^\tau 1 dt \right] \right) dF(S)$$
$$\Theta_{1,i} \equiv \int x^i \left( \frac{\partial}{\partial x} \mathbb{E}^S \left[ \int_0^\tau x_t dt \right] - \mathbb{E}^S \left[ \int_0^\tau 1 dt \right] \right) dF(S)$$

With similar steps as in the main proof (a combination of Ito's lemma and the OST), we have that

$$\Gamma_{m,i} \equiv \frac{\mathbb{E}^{\hat{S}} \left[ \int_0^\tau \varphi^\Gamma_{m,i}(S_t) dt \right]}{\mathbb{E}^{\hat{S}} [\tau]} \qquad \Theta_{m,i} \equiv \frac{\mathbb{E}^{\hat{S}} \left[ \int_0^\tau \varphi^\Theta_{m,i}(S_t) dt \right]}{\mathbb{E}^{\hat{S}} [\tau]} \tag{B.46}$$

where

$$\varphi^\Gamma_{m,i}(S) = \frac{2}{\sigma^2} \left( \frac{i x^{i-1} \mathbb{1}_{\{i \geq 1\}}}{3\sigma^2} \left[ \mathbb{E}^S \left[ x_\tau^3 \right] - x^3 \right] + \frac{x^i}{\sigma^2} \left[ \mathbb{E}^S \left[ x_\tau^2 \right] - x^2 \right] \right)$$
$$\varphi^\Theta_{m,i}(S) = \frac{x^i}{\sigma^2} \left( \frac{\partial \mathbb{E}^S \left[ x_\tau^3 / 3 \right]}{\partial x} - \mathbb{E}^S \left[ x_\tau^2 \right] \right)$$

**Representation for the intensive margin**: Repeating the steps as in the main proof, it is easy to show that

$$\Gamma_{1,i} = \Gamma^1_{1,i} + \Gamma^2_{1,i} \tag{B.47}$$
$$\Gamma^1_{1,i} = \frac{\mathbb{1}_{\{i \geq 1\}}}{3\sigma^2} \frac{\mathbb{E}^{\hat{S}} \left[ \int_0^\tau i x_t^{i-1} \left[ x_\tau^3 - x_t^3 \right] dt \right]}{\mathbb{E}^{\hat{S}} [\tau]} \tag{B.48}$$
$$\Gamma^2_{1,i} = \frac{\mathbb{E}^{\hat{S}} \left[ \int_0^\tau x_t^i \left[ x_\tau^2 - x_t^2 \right] dt \right]}{\sigma^2 \mathbb{E}^{\hat{S}} [\tau]} \tag{B.49}$$

Notice that in the case $i = 0$ we have that

$$\Gamma_{1,0} = \Gamma^1_{1,0} + \Gamma^2_{1,0} = \Gamma^2_{1,0} = \mathcal{M}_1[a]. \tag{B.50}$$

Next, we characterize the case with $i = 1$. For $\Gamma^1_{1,1}$, we can use proposition (**??**) were we have shown that

$$\frac{\mathbb{E}^{\hat{S}} \left[ \int_0^\tau \left[ x_\tau^3 - x_t^3 \right] dt \right]}{\sigma^2 3 \mathbb{E}^{\hat{S}} [\tau]} 2 = 2\mathbb{C}ov[x, a], \tag{B.51}$$

Thus, $\Gamma^1_{1,1} = \mathbb{C}ov[x, a]$. Let us characterize the term $\Gamma^2_{1,i}$. Using the occupancy measure

$$\Gamma^2_{1,1} = \frac{\mathbb{E}^{\hat{S}} \left[ \int_0^\tau x_t \left[ x_\tau^2 - x_t^2 \right] dt \right]}{\sigma^2 \mathbb{E}^{\hat{S}} [\tau]} = \frac{\mathbb{E}^{\hat{S}} \left[ x_\tau^2 \int_0^\tau x_t dt \right]}{\sigma^2 \mathbb{E}^{\hat{S}} [\tau]} - \frac{\mathbb{E}^{\hat{S}} \left[ \int_0^\tau x_t^3 dt \right]}{\sigma^2 \mathbb{E}^{\hat{S}} [\tau]} = \frac{\mathbb{E}^{\hat{S}} \left[ x_\tau^2 \int_0^\tau x_t dt \right]}{\sigma^2 \mathbb{E}^{\hat{S}} [\tau]} - \frac{\mathcal{M}_3[x]}{\sigma^2} \tag{B.52}$$

To characterize the first term of the previous equation, we apply Ito's Lemma to $x_t^2 \int_0^t x_s ds$, and we have

$$d(x_t^2 \int_0^t x_s ds) = x_t^3 dt + 2x_t \int_0^t x_s ds dB_t + \frac{2\sigma^2}{2} \int_0^t x_s ds dt \tag{B.53}$$

Using the OTS and properties of the Ito's integral, we have that

$$\mathbb{E}^{\hat{S}} \left[ x_\tau^2 \int_0^\tau x_t dt \right] = \mathbb{E}^{\hat{S}} \left[ \int_0^\tau x_t^3 dt \right] + \sigma^2 \mathbb{E}^{\hat{S}} \left[ \int_0^\tau \int_0^t x_s ds dt \right]. \tag{B.54}$$

Using Fubini's for the Reimman integral

$$\mathbb{E}^{\hat{S}} \left[ \int_0^\tau \int_0^t x_s ds dt \right] = \mathbb{E}^{\hat{S}} \left[ \int_0^\tau x_s \int_{\tau-2}^\tau dt ds \right] = \mathbb{E}^{\hat{S}} \left[ \int_0^\tau x_s s ds \right], \tag{B.55}$$

and thus we have that

$$\Gamma^2_{1,1} = \mathbb{C}ov[x, a] \tag{B.56}$$

**Representation for the extensive margin**: For $\Theta_{m,i}$ we have that

$$\begin{aligned}
\Theta_{1,i} &= \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \varphi_m^{\Theta}(S_t)dt\right]}{\mathbb{E}^{\hat{S}}[\tau]} = \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^i \left[\frac{dg_3(x_t-\hat{x})}{dy}/(3) - g_2(x_t-\hat{x})\right] dt\right]}{\sigma^2\mathbb{E}^{\hat{S}}[\tau]} \\
&= \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^i \left[\sum_{j=0}^\infty \frac{d^j}{j!dx^j}\left[\frac{dg_3(x)}{dx}/(3) - g_2(x)\right]_{x=0}(x_t-\hat{x})^j\right] dt\right]}{\sigma^2\mathbb{E}^{\hat{S}}[\tau]} \\
&= \sum_{j=0}^\infty \sum_{z=0}^j \frac{\left[\frac{d^{j+1}g_3(0)}{dx^{j+1}}/3 - \frac{d^{j+1}g_2(0)}{dx^{j+1}}\right]\hat{x}^z}{j!} \begin{pmatrix} j \\ z \end{pmatrix} \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{j+i-z}dt\right]}{\sigma^2\mathbb{E}^{\hat{S}}[\tau]} \\
&= \sum_{j=0}^\infty \sum_{z=0}^j \frac{\left[\frac{d^{j+1}g_3(0)}{dx^{j+1}}/3 - \frac{d^{j}g_2(0)}{dx^{j}}\right]\hat{x}^z}{\sigma^2 j!} \begin{pmatrix} j \\ z \end{pmatrix} \mathcal{M}_{j+i-z}[x] \\
&= \sum_{j=0}^\infty \theta_{m,j}\mathcal{M}_{j+i}[x], \ \textit{with} \ \theta_{m,j} = \sum_{k\geq j}^\infty \frac{\left[\frac{d^{k+1}g_3(x)}{dx^{k+1}}/3 - \frac{d^k g_2(x)}{dx^k}\right]\Big|_{x=0}\hat{x}^{k-j}}{\sigma^2 k! j!}.
\end{aligned} \tag{B.57}$$

□

## B.4 Characterization of the CIR for mean-reverting process

**Proposition B.4.** *Assume that the uncontrolled state follows a mean-reverting Brownian motion with* $\rho < 0$*:*

$$d\tilde{x}_t = \rho \tilde{x}_t dt + \sigma dB_t, \qquad B_t \sim Wiener.$$

- ***Aggregation:*** *To a first order, the transitional dynamics are given by*

$$\mathcal{A}_m(\delta) = \delta \times \left( \mathcal{Z}_m - \mathcal{M}_m[x] \left[ \Theta_0 - \frac{2\rho}{\sigma^2} \mathcal{C}_2 \right] \right) + o(\delta^2) \tag{B.58}$$

*where the intensive* $\Gamma_m$ *and the extensive* $\Theta_m$ *margin components are:*

$$\begin{aligned}
\mathcal{Z}_m &= \Gamma_m + \Theta_m - \frac{2\rho}{\sigma^2(m+1)} \mathcal{Z}_{m+2} \\
\mathcal{C}_2 &= \int \frac{\partial}{\partial x} \mathbb{E}^S \left[ \int_0^\tau x_t^2 dt \right] - \mathbb{E}^S \left[ \int_0^\tau 2x_t dt \right] dF(S) \\
\Gamma_m &\equiv \frac{\mathbb{E}^{\hat{S}} \left[ \int_0^\tau \varphi_m^\Gamma(S_t) dt \right]}{\mathbb{E}^{\hat{S}}[\tau]} \quad with \quad \varphi_m^\Gamma(S) \equiv \frac{2}{\sigma^2(m+1)} \left( \mathbb{E}^S \left[ x_\tau^{m+1} \right] - x^{m+1} \right) \\
\Theta_m &\equiv \frac{\mathbb{E}^{\hat{S}} \left[ \int_0^\tau \varphi_m^\Theta(S_t) dt \right]}{\mathbb{E}^{\hat{S}}[\tau]} \quad with \quad \varphi_m^\Theta(S) \equiv \frac{2}{\sigma^2(m+1)} \left( \frac{\partial \mathbb{E}^S[x_\tau^{m+2}]/(m+2)}{\partial x} - \mathbb{E}^S \left[ x_\tau^{m+1} \right] \right)
\end{aligned}$$

- ***Representation:***

$$\begin{aligned}
\Gamma_m &= \frac{2}{\sigma^2(m+1)} \mathbb{E}[x^{m+1}] + m\mathbb{E}[x^{m-1}a] \\
\Theta_m &= \sum_{h=0}^{\infty} \theta_{m,h} \mathcal{M}_h[x], \quad \theta_{m,h} = \sum_{k \geq h}^{\infty} \frac{(-1)^{k-h}}{\rho} \frac{\hat{x}^{k-h}}{(k-h)!h!} \left[ \frac{d^{j+1} g_{m+2}(0)}{dx^{j+1}} / (m+2) - \frac{d^{j+1} g_{m+1}(0)}{dx^{j+1}} \right]
\end{aligned}$$

- ***Observation:*** *The reset state and structural parameters are recovered with a system of equations:*

$$\hat{x} = \frac{\mathbb{E}^{\hat{S}}[e^{-\rho\tau} \Delta x]}{\mathbb{E}^{\hat{S}}[e^{-\rho\tau}] - 1} \tag{B.59}$$

$$\frac{\sigma^2}{\rho} = 2 \frac{\hat{x}^2 - \mathbb{E}^{\hat{S}} \left[ e^{-2\rho\tau} (\hat{x} - \Delta x)^2 \right]}{\mathbb{E}^{\hat{S}} \left[ e^{-2\rho\tau} \right] - 1} \tag{B.60}$$

$$\mathbb{E}^{\hat{S}} \left[ erf \left( (\hat{x} - \Delta x) \sqrt{\frac{\rho}{\sigma^2}} \right) \right] = erf \left( \hat{x} \sqrt{\frac{\rho}{\sigma^2}} \right) \tag{B.61}$$

*where* $erf(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ *is the Gauss error function; and the ergodic moments are recovered as*

$$\mathbb{E}[x^m] = \frac{\mathbb{E}[(\hat{x} - \Delta x)^m - \hat{x}^m]}{\rho m \mathbb{E}[\tau]} - \frac{\sigma^2(m-1)}{2\rho} \mathcal{M}_{m-2}[x] \tag{B.62}$$

$$\mathbb{E}[x^m a] = \frac{\mathbb{E}[\tau(\hat{x} - \Delta x)^m]}{\mathbb{E}[\tau]} \frac{\sigma^2(m-1)}{2\rho} \mathbb{E}[x^{m-2}a] - \frac{\mathcal{M}_m[x]}{m\rho} \tag{B.63}$$

*with* $\mathcal{M}_0[x] = 1$*,* $\mathbb{E}[x] = 0$*,* $\mathbb{C}ov[x,a] = 0, \mathcal{M}_{0,1}[x,a] = \mathcal{M}_1[a]$*.*

*Proof.* **Aggregation** We start the proof from equation (**??**) in the Appendix, as all previous steps are identical. The first order approximation to the CIR yields $\mathcal{A}'_m(0)$ equal to:

$$\underbrace{\int \mathbb{E}^S \left[ \int_0^\tau \frac{\partial x_t^m}{\partial x} dt \right] dF(S)}_{\mathcal{B}_m} - \mathbb{E}[x^m] \underbrace{\int \frac{\partial \mathbb{E}^S[\tau]}{\partial x} dF(S)}_{\Theta_0} + \underbrace{\int \left( \frac{\partial}{\partial x} \mathbb{E}^S \left[ \int_0^\tau x_t^m dt \right] - \mathbb{E}^S \left[ \int_0^\tau \frac{\partial x_t^m}{\partial x} dt \right] \right) dF(S)}_{\mathcal{C}_m}. \tag{B.64}$$

To characterize the first term $\mathcal{B}_m \equiv \int \mathbb{E}^S \left[ \int_0^\tau m x_t^{m-1} dt \right] dF(S)$, we apply Itō's Lemma to $x_t^{m+1}$ to get $dx_t^{m+1} = \rho(m+1)x_t^{m+1}dt + \frac{\sigma^2}{2} m(m+1) x_t^{m-1} dt + \sigma(m+1) x_t^m dB_t$. Integrating both sides from 0 to $\tau$ and taking expectations with initial condition $S$, and applying the OST to eliminate martingales with zero initial condition, we get:

$$\frac{\mathbb{E}^S \left[ x_\tau^{m+1} \right] - x^{m+1}}{m+1} = \rho \mathbb{E}^S \left[ \int_0^\tau x_t^{m+1} dt \right] + \frac{\sigma^2}{2} \mathbb{E}^S \left[ \int_0^\tau m x_t^{m-1} dt \right]$$

Solving for $\mathbb{E}^S\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right]$ and multiplying/dividing by $m+2$:

$$\mathbb{E}^S\left[\int_0^\tau m x_t^{m-1}dt\right] = \underbrace{\frac{2}{\sigma^2}\frac{\mathbb{E}^S\left[x_\tau^{m+1}\right]-x^{m+1}}{m+1}}_{\varphi_m^\Gamma(S)} - \frac{2\rho}{\sigma^2(m+2)}\mathbb{E}^S\left[\int_0^\tau (m+2)x_t^{m+1}dt\right] \tag{B.65}$$

Integrating both sides across with the initial distribution, and defining $\Gamma_m \equiv \int \frac{2}{\sigma^2}\frac{\mathbb{E}^S\left[x_\tau^{m+1}\right]-x^{m+1}}{m+1}dF(S)$, we get a recursive formula for $\mathcal{B}_m$:

$$\mathcal{B}_m = \Gamma_m - \frac{2\rho}{\sigma^2(m+2)}\mathcal{B}_{m+2} \tag{B.66}$$

To characterize the term $\mathcal{C}_m$, we focus separately on each of its terms. If we apply Itō's Lemma to $x_t^{m+2}$ we get $dx_t^{m+2} = \rho(m+2)x_t^{m+2}dt + \frac{\sigma^2}{2}(m+2)(m+1)x_t^m dt + \sigma(m+2)x_t^{m+2}dB_t$, and we can express it as

$$\mathbb{E}^S[x_\tau^{m+2} - x^{m+2}] = \rho(m+2)\mathbb{E}^S\left[\int_0^\tau x_t^{m+2}dt\right] + \frac{\sigma^2}{2}(m+2)(m+1)\mathbb{E}^S\left[\int_0^\tau x_t^m dt\right]$$

Solving for $\mathbb{E}^S\left[\int_0^\tau x_t^m dt\right]$ and taking the derivative with respect to initial condition $x$:

$$\frac{\partial}{\partial x}\mathbb{E}^S\left[\int_0^\tau x_t^m dt\right] = \frac{2}{\sigma^2(m+1)}\left(\frac{\partial\mathbb{E}^S[x_\tau^{m+2}]/(m+2)}{\partial x} - x^{m+1}\right) - \frac{2\rho}{\sigma^2(m+1)}\frac{\partial}{\partial x}\mathbb{E}^S\left[\int_0^\tau x_t^{m+2}dt\right] \tag{B.67}$$

Subtract (B.67) minus (B.65):

$$\begin{aligned}
&\frac{\partial}{\partial x}\mathbb{E}^S\left[\int_0^\tau x_t^m dt\right] - \mathbb{E}^S\left[\int_0^\tau m x_t^{m-1}dt\right] \\
=\;& \frac{2}{\sigma^2(m+1)}\left(\frac{\partial\mathbb{E}^S[x_\tau^{m+2}]/(m+2)}{\partial x} - x^{m+1}\right) - \frac{2\rho}{\sigma^2(m+1)}\frac{\partial}{\partial x}\mathbb{E}^S\left[\int_0^\tau x_t^{m+2}dt\right] \\
-\;& \frac{2}{\sigma^2}\frac{\mathbb{E}^S\left[x_\tau^{m+1}\right]-x^{m+1}}{m+1} + \frac{2\rho}{\sigma^2(m+2)}\mathbb{E}^S\left[\int_0^\tau (m+2)x_t^{m+1}dt\right] \\
=\;& \underbrace{\frac{2}{\sigma^2(m+1)}\left(\frac{\partial\mathbb{E}^S[x_\tau^{m+2}]/(m+2)}{\partial x} - \mathbb{E}^S\left[x_\tau^{m+1}\right]\right)}_{\varphi_m^\Theta(S_t)} - \frac{2\rho}{\sigma^2(m+1)}\left\{\frac{\partial}{\partial x}\mathbb{E}^S\left[\int_0^\tau x_t^{m+2}dt\right] - \mathbb{E}^S\left[\int_0^\tau (m+2)x_t^{m+1}dt\right]\right\} \\
-\;& \frac{2\rho}{\sigma^2(m+2)(m+1)}\mathbb{E}^S\left[\int_0^\tau (m+2)x_t^{m+1}dt\right]
\end{aligned}$$

Integrating with the ergodic distribution, we obtain:

$$\mathcal{C}_m = \Theta_m - \frac{2\rho}{\sigma^2(m+1)}\mathcal{C}_{m+2} - \frac{2\rho}{\sigma^2(m+1)(m+2)}\mathcal{B}_{m+2} \tag{B.68}$$

where $\Theta_m \equiv \int \frac{2}{\sigma^2(m+1)}\left(\frac{\partial\mathbb{E}^S[x_\tau^{m+2}]/(m+2)}{\partial x} - \mathbb{E}^S\left[x_\tau^{m+1}\right]\right)dF(S)$. Finally, let $\mathcal{Z}_m \equiv \mathcal{B}_m + \mathcal{C}_m$ which equals

$$\begin{aligned}
\mathcal{Z}_m &\equiv \mathcal{B}_m + \mathcal{C}_m \\
&= \Gamma_m + \Theta_m - \frac{2\rho}{\sigma^2}\left[\frac{\mathcal{B}_{m+2}}{(m+2)} + \frac{\mathcal{C}_{m+2}}{(m+1)} + \frac{\mathcal{B}_{m+2}}{(m+1)(m+2)}\right] \\
&= \Gamma_m + \Theta_m - \frac{2\rho}{\sigma^2(m+1)}\mathcal{Z}_{m+2}
\end{aligned}$$

Lastly, we find the expression for $\int \frac{\partial\mathbb{E}^S[\tau]}{\partial x}dF(S)$. By Ito's Lemma applied to $x_t^2$ and the OST, we have that

$$\mathbb{E}^S[x_\tau^2] - x^2 = 2\rho\mathbb{E}^S\left[\int_0^\tau x_t^2 dt\right] + \sigma^2\mathbb{E}^S[\tau]$$

Solving for $\mathbb{E}^S[\tau]$ and taking derivative with respect to the initial condition yields:

$$\frac{\partial\mathbb{E}^S[\tau]}{\partial x} = \frac{2}{\sigma^2}\left(\frac{\partial\mathbb{E}^S[x_\tau^2/2]}{\partial x} - x - \rho\frac{\partial}{\partial x}\mathbb{E}^S\left[\int_0^\tau x_t^2 dt\right]\right)$$

By the OST, we also have that $\mathbb{E}^S[x_\tau] - x = \rho\mathbb{E}^S\left[\int_0^\tau x_t dt\right]$, which can be substituted back

$$\frac{\partial \mathbb{E}^S[\tau]}{\partial x} = \frac{2}{\sigma^2}\left(\frac{\partial \mathbb{E}^S[x_\tau^2/2]}{\partial x} - \mathbb{E}^S[x_\tau] - \rho\left\{\frac{\partial}{\partial x}\mathbb{E}^S\left[\int_0^\tau x_t^2 dt\right] - \mathbb{E}^S\left[\int_0^\tau x_t dt\right]\right\}\right)$$

Integrating with the ergodic distribution, we get

$$\int \frac{\partial \mathbb{E}^S[\tau]}{\partial x} dF(S) = \Theta_0 - \frac{2\rho}{\sigma^2}\mathcal{C}_2$$

since the last two terms can be manipulated as $\int \frac{\partial}{\partial x}\mathbb{E}^S\left[\int_0^\tau x_t^2 dt\right] - \mathbb{E}^S\left[\int_0^\tau 2x_t dt\right] + \mathbb{E}^S\left[\int_0^\tau x_t dt\right] dF(S) = \mathcal{C}_2 + 0 = \mathcal{C}_2$.

**Representation for the intensive margin.** Start from the definition of $\Gamma_m$ and $\varphi_m^\Gamma(S)$ and repeat the steps as in previous proofs to reach:

$$\frac{\sigma^2(m+1)}{2}\mathbb{E}^{\hat{S}}[\tau]\Gamma_m = \mathbb{E}^{\hat{S}}\left[x_\tau^{m+1}\int_0^\tau dt\right] - \mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{m+1}dt\right] \tag{B.69}$$

We now characterize $\mathbb{E}^{\hat{S}}\left[x_\tau^{m+1}\int_0^\tau dt\right]$. Applying Ito's lemma followed by the OST to $Y_t^{m+1} \equiv x_t^{m+1}\int_0^t ds$

$$\begin{aligned} dY_t^{m+1} &= x_t^{m+1}dt + (m+1)\left[\rho x_t^{m+1} + \sigma x_t^m\right]\int_0^t ds dB_t + \frac{m(m+1)\sigma^2}{2}x_t^{m-1}\int_0^t ds dt \\ \mathbb{E}^{\hat{S}}\left[Y_\tau^{m+1}\right] &= \underbrace{\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{m+1}dt\right]}_{\mathbb{E}[x^{m+1}]\mathbb{E}^{\hat{S}}[\tau]} + \frac{m(m+1)\sigma^2}{2}\underbrace{\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{m-1}t dt\right]}_{\mathbb{E}[x^{m-1}a]\mathbb{E}^{\hat{S}}[\tau]} \end{aligned}$$

Substituting back into (B.69) and rearranging:

$$\frac{\sigma^2(m+1)}{2}\mathbb{E}^{\hat{S}}[\tau]\Gamma_m = \mathbb{E}[x^{m+1}]\mathbb{E}^{\hat{S}}[\tau] + \frac{m(m+1)\sigma^2}{2}\mathbb{E}[x^{m-1}a]\mathbb{E}^{\hat{S}}[\tau]$$

implying:

$$\Gamma_m = \frac{2}{\sigma^2(m+1)}\mathbb{E}[x^{m+1}] + m\mathbb{E}[x^{m-1}a] \tag{B.70}$$

**Representation for the extensive margin** Start from the definition of $g_m(x)$ in (B.1) evaluated at $x = y - \hat{x}$

$$g_m(y-\hat{x}) = \mathbb{E}^{y,S^{-x}}\left[x_\tau^m\right] - \mathbb{E}^{\hat{x},S^{-x}}\left[(x_\tau + y - \hat{x})^m\right] \tag{B.71}$$

and find expressions for the following objects:

$$\begin{aligned} \mathbb{E}^{y,S^{-x}}[x_\tau^{m+1}] &= g_{m+1}(y-\hat{x}) + \mathbb{E}^{\hat{x},S^{-x}}\left[(x_\tau + y - \hat{x})^{m+1}\right] \\ \frac{\partial \mathbb{E}^{y,S^{-x}}[x_\tau^{m+2}/(m+2)]}{\partial x} &= \frac{dg_{m+2}(y-\hat{x})}{dy}/(m+2) + \mathbb{E}^{\hat{x},S^{-x}}\left[(x_\tau + y - \hat{x})^{m+1}\right] \end{aligned}$$

Recover $\varphi_m^\Theta(y, S^{-x})$ by subtracting the two previous expressions and simplifying

$$\rho\varphi_m^\Theta(y, S^{-x}) = \frac{dg_{m+2}(y-\hat{x})}{dy}/(m+2) - g_{m+1}(y-\hat{x})$$

An infinite Taylor approximation around 0 yields:

$$\rho\varphi_m^\Theta(y, S^{-x}) = \sum_{j=0}^\infty \frac{d^j}{j!dx^j}\left[\frac{dg_{m+2}(x)}{dx}/(m+2) - g_{m+1}(x)\right]_{x=0}(y-\hat{x})^j = \sum_{j=0}^\infty \frac{1}{j!}\left[\frac{d^{j+1}g_{m+2}(0)}{dx^{j+1}}/(m+2) - \frac{d^j g_{m+1}(0)}{dx^j}\right](y-\hat{x})^j$$

Opening the binomial term as $(y-\hat{x})^j = \sum_{z=0}^j (-1)^z \frac{j!}{z!(j-z)!}\frac{\hat{x}^z}{j!}y^{j-z}$ and substituting back

$$\varphi_m^\Theta(y, S^{-x}) = \frac{1}{\rho}\sum_{j=0}^\infty\sum_{z=0}^j (-1)^z \frac{\hat{x}^z}{z!(j-z)!}\left[\frac{d^{j+1}g_{m+2}(0)}{dx^{j+1}}/(m+2) - \frac{d^j g_{m+1}(0)}{dx^j}\right]y^{j-z}$$

Finally, we find $\Theta_m$ by integrating over all initial conditions $y$ with the occupancy measure (note that the integral only affects the term $y^{j-z}$):

$$\Theta_m = \sum_{j=0}^{\infty}\sum_{z=0}^{j}\underbrace{\frac{1}{\rho}(-1)^z\frac{\hat{x}^z}{z!(j-z)!}\left[\frac{d^{j+1}g_{m+2}(0)}{dx^{j+1}}/(m+2)-\frac{d^{j+1}g_{m+1}(0)}{dx^{j+1}}\right]}_{\mathcal{H}_{m,j,z}}\underbrace{\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{j-z}dt\right]}{\mathbb{E}^{\hat{S}}[\tau]}}_{\mathcal{M}_{j-z}[x]}$$

$$= \sum_{j=0}^{\infty}\sum_{z=0}^{j}\mathcal{H}_{m,j,z}\mathcal{M}_{j-z}[x]$$

$$= \sum_{h=0}^{\infty}\theta_{m,h}\mathcal{M}_h[x], \qquad where \qquad \theta_{m,h}=\sum_{k\geq h}^{\infty}\mathcal{H}_{m,k,k-h}$$

**Observation:**

- To characterize the three structural parameters $(\hat{x},\rho,\sigma)$ we need three equations that are independent of the initial conditions. Apply Itō's Lemma to $Y_t=e^{-\rho t}x_t$ and obtain $dY_t=\sigma e^{\rho t}dB_t$. Integrating from 0 to $\tau$, taking expectations with initial condition $\hat{S}$, and using the OST we obtain $\mathbb{E}^{\hat{S}}[e^{-\rho\tau}(\hat{x}-\Delta x)]=\hat{x}$. Thus, we have that

$$\hat{x}=\frac{\mathbb{E}^{\hat{S}}[e^{-\rho\tau}\Delta x]}{\mathbb{E}^{\hat{S}}[e^{-\rho\tau}]-1} \tag{B.72}$$

  Apply Itō's Lemma to $Y_t=e^{-2\rho t}x_t^2$ and obtain $dY_t=2\sigma x_t e^{-2\rho t}dB_t+\sigma^2e^{-2\rho t}$. Following the same steps we obtain $\mathbb{E}^{\hat{S}}[e^{-2\rho\tau}x_\tau^2]=\hat{x}^2+\sigma^2\mathbb{E}^{\hat{S}}\left[\int_0^\tau e^{-2\rho t}dt\right]=\hat{x}^2+\frac{\sigma^2}{2\rho}\mathbb{E}^{\hat{S}}\left[\left(1-e^{-2\rho\tau}\right)\right]$. Thus

$$\frac{\sigma^2}{\rho}=2\frac{\hat{x}^2-\mathbb{E}^{\hat{S}}\left[e^{-2\rho\tau}(\hat{x}-\Delta x)^2\right]}{\mathbb{E}^{\hat{S}}\left[e^{-2\rho\tau}\right]-1} \tag{B.73}$$

  Define the error function $erf(x)\equiv\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$. Apply Ito's Lemma to $Y_t=erf\left(x_t\sqrt{\frac{\rho}{\sigma^2}}\right)$ to obtain With the same steps that include the OST, we find the third equation for the system:

$$\mathbb{E}^{\hat{S}}\left[erf\left((\hat{x}-\Delta x)\sqrt{\frac{\rho}{\sigma^2}}\right)\right]=erf\left(\hat{x}\sqrt{\frac{\rho}{\sigma^2}}\right) \tag{B.74}$$

- To characterize the ergodic moment $\mathcal{M}_m[x]$, we apply Itō's Lemma to $Y_t=x_t^m$ and obtain $dY_t=\rho m x_t^m dt+m\sigma x_t^{m-1}dB_t+\frac{\sigma^2 m(m-1)}{2}x_t^{m-2}dt$. Integrating from 0 to $\tau$ and taking expectations with initial condition $\hat{S}$,

$$\mathbb{E}[x_\tau^m]-\hat{x}^m = \rho m\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^m dt\right]+\frac{\sigma^2m(m-1)}{2}\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{m-2}dt\right]+m\sigma\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{m-1}dB_t\right]$$

  In the first two terms we substitute the relationship $\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^m dt\right]=\mathbb{E}[\tau]\mathcal{M}_m[x]$; the last term is equal to zero by the OST. Rearranging:

$$\mathcal{M}_m[x] = \frac{\mathbb{E}[(\hat{x}-\Delta x)^m-\hat{x}^m]}{\rho m\mathbb{E}[\tau]}-\frac{\sigma^2(m-1)}{2\rho}\mathcal{M}_{m-2}[x].$$

- To characterize the covariance with age $\mathcal{M}_{m,1}[x,a]$, we apply Itō's theorem to $Y_t=x_t^m t$ and obtain: $dY_t=x_t^m dt+\rho m t x_t^m dt+\frac{\sigma^2m(m-1)}{2}tx_t^{m-2}dt+\sigma m t x_t^{m-1}dB_t$. Integrating from 0 to $\tau$ and taking expectations with initial condition $\hat{S}$,

$$\mathbb{E}[\tau x_\tau^m] = \mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^m dt\right]+\rho m\mathbb{E}^{\hat{S}}\left[\int_0^\tau t x_t^m dt\right]+\frac{\sigma^2m(m-1)}{2}\mathbb{E}^{\hat{S}}\left[\int_0^\tau tx_t^{m-2}dt\right]+\sigma m\mathbb{E}^{\hat{S}}\left[\int_0^\tau tx_t^{m-1}dB_t\right]$$

  In the first three terms we substitute the relationships $\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^m dt\right]=\mathbb{E}[\tau]\mathcal{M}_m[x]$ and $\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^k t^l dt\right]=\mathbb{E}[\tau]\mathcal{M}_{k,l}[x,a]$ the last term is equal to zero by the OST:

$$\mathbb{E}[\tau x_\tau^m] = \mathbb{E}[\tau]\mathcal{M}_m[x]+\rho m\mathbb{E}[\tau]\mathcal{M}_{m,1}[x,a]+\frac{\sigma^2m(m-1)}{2}\mathbb{E}[\tau]\mathcal{M}_{m-2,1}[x,a]$$

  Rearranging and solving for $\mathcal{M}_{m,1}[x,a]$ we obtain a recursive representation for the moment:

$$\mathbb{E}[x^m a] = \frac{\mathbb{E}[\tau(\hat{x}-\Delta x)^m]}{\mathbb{E}[\tau]}\frac{\sigma^2(m-1)}{2\rho}\mathbb{E}[x^{m-2}a]-\frac{\mathcal{M}_m[x]}{m\rho}$$

□

# C Economic Frameworks With Small GE Effects

This section describes general equilibrium frameworks that feature small general equilibrium feedback into agents' policies, and therefore, the tools developed in this paper can be directly applied.

## C.1 Monetary Shocks: Golosov and Lucas (2007)

There is a representative consumer, a continuum of firms that operate in monopolistic competition, and a monetary authority. We study the transitional dynamics to steady state for an exogenously given initial condition of the distribution of idiosyncratic states.

**Money Supply** The economy is subject to monetary shocks, which we summary in the money supply $M_t$. The log money supply is assumed to follow a Brownian motion with drift $\mu_m$ and volatility $\sigma_m$,

$$d\log(M_t) = \mu_m dt + \sigma_m dB_t^m. \tag{C.75}$$

**Representative Household** The household has the following preferences over consumption $C_t$, labor $N_t$, and real money holding $M_t/P_t$, where $P_t$ is the aggregate price level and the future is discounted at rate $\rho > 0$:

$$\mathbb{E}_0\left[\int_0^\infty e^{-\rho t}\left(\frac{C_t^{1-\gamma}}{1-\gamma} - \alpha N_t + \log\left(\frac{M_t}{P_t}\right)\right)\, dt\right] \tag{C.76}$$

Consumption consists of a CES aggregator as in Woodford (2009), Midrigan (2011), and Álvarez and Lippi (2014) with demand elasticity $\eta$. The household has access to complete financial markets. The budget includes labor earnings $W_tN_t$, profits $\Pi_t$ from the ownership of all firms, and the opportunity cost of holding cash $R_tM_t$, where $R_t$ is the nominal interest rate. Let $D_t$ be the stochastic discount factor; with complete financial markets, the time-0 Arrow-Debreu budget constraint reads:

$$\mathbb{E}_0\left[\int_0^\infty D_t\left(P_tC_t + R_tM_t - W_tN_t - \Pi_t\right)dt\right] \le M_0. \tag{C.77}$$

The household chooses consumption, labor supply and money holdings to maximize (C.76) subject to (C.77). The household's first order conditions establish nominal wages as a proportion of the (constant) money stock $W_t = \alpha(\rho + \mu_m)M_t$; thus the stochastic process of money supply transfers one-to-one to the process of nominal wages.

**Monopolistic Firms** On the production side, there is a continuum of firms indexed by $z \in [0,1]$ who operate in a monopolistically competitive market. Each firm maximizes its expected stream of profits, discounted at $D_t$. It chooses a price and then satisfies all its demand. For every price change, it must pay a menu cost $\theta$ measured in units of labor. Production uses a linear technology with labor as its only input: producing $y_t(z)$ units requires $l_t(z) = y_t(z)A_t(z)$ units of labor, so that the marginal nominal cost is $A_t(z)W_t$. Given the consumer's demand $c_t(z)$ for product $z$, the instantaneous profit can be written as a function of prices:

$$\Pi_t(p,z) = c_t(p,z)\Big(p - A_t(z)W_t\Big). \tag{C.78}$$

The firm's problem is to choose a sequence of adjustment dates $(\tau_i(z))$ and reset prices $(p_i(z))$ that solve the following stopping-time problem:

$$\max_{\{\tau_i,p_i\}_{i=1}^\infty} \mathbb{E}_0\left[\sum_{i=0}^\infty D_{\tau_{i+1}}W_{\tau_{i+1}}\theta + \sum_{i=0}^\infty \int_{\tau_i}^{\tau_{i+1}} D_t\Pi_t(p_i,z)dt\right], \tag{C.79}$$

where $\Pi_t(p,z)$ satisfies (C.78) and $\tau_0 = 0$. Firm $z$'s log idiosyncratic cost $a_t(z) \equiv \ln A_t(z)$ evolves according to a diffusion process which is idiosyncratic and independent across $z$:

$$da_t(z) = \sigma dB_t^a(z), \tag{C.80}$$

We define firms' markups as $\mu_t(z) \equiv p_t(z)/(A_t(z)W_t)$ and aggregate markup as $\upmu_t$. With this definition we can rewrite the firm problem as a function of individual and aggregate markups:

$$\max_{\{\tau_i,\mu_{\tau_i}\}_{i=1}^\infty} \mathbb{E}_0\left[\sum_{i=0}^\infty e^{-\rho\tau_{i+1}}\theta + \sum_{i=0}^\infty \int_{\tau_i}^{\tau_{i+1}} e^{-\rho t}\upmu_t^{\eta-1/\sigma}\mu_t(z)^{-\eta}(\mu_t(z)-1)dt\right], \tag{C.81}$$

where idiosyncratic markups has the following stochastic process

$$\log(\mu_t(z)) = \log\left(\mu_{\tau_i}(z)\right) - \mu_m(t-\tau_i) - \sigma_m(B_t^m - B_{\tau_i}^m) - \sigma(B_t^a - B_{\tau_i}^a) \quad \forall t \in [\tau_i, \tau_{i+1}] \tag{C.82}$$

**Steady state equilibrium** Given the exogenous stochastic processes for idiosyncratic productivity ($B^a_t(z)$), an equilibrium is defined by a set of stochastic processes for (i) consumption strategies $c_t(z)$, labor supply $N_t$, and money holdings $M_t$ for the household, (ii) markup policies functions $\mu_t(z)$ for the firms, (iii) prices $P_t$, $W_t$, $D_t$, and (iv) a fixed distribution over firm states $F(\mu)$ such that the household and the firms optimize, markets clear at each date, and the distribution is consistent with actions.

In a steady state equilibrium with no money shocks, the price index, nominal wages, and nominal interest rates grow at rate $\mu_m$, as there is no aggregate uncertainty. Nominal expenditure is constant and equal to the nominal wage, and by market clearing, aggregate output equals aggregate consumption and also the real wage, which in turn is equal to the inverse of the aggregate markup $Y = \mu^{-1/\sigma}$.

**Transition Dynamics for Aggregate Markups** Fix an initial distribution of markups—this initial distribution could be the outcome of an one-time unanticipated change in the money supply. Proposition 7 in Álvarez and Lippi (2014) shows that steady-state decision rules, as derived in partial equilibrium from solving (C.81) with steady state aggregate markups $\mu = \eta/(\eta-1)$, provide an accurate approximation of the policies during the transition in a general equilibrium. Thus solving

$$V(\mu) = \max_{\{\tau_i,\mu_i\}_{i=1}^{\infty}} \mathbb{E}_0 \left[ \sum_{i=0}^{\infty} e^{-\rho\tau_{i+1}}\theta + \sum_{i=0}^{\infty} \int_{\tau_i}^{\tau_{i+1}} e^{-\rho t} \left(\frac{\eta}{\eta-1}\right)^{\eta-1/\sigma} \mu_t(z)^{-\eta}(\mu_t(z)-1)dt \right] \tag{C.83}$$

provides a good approximation of the firm policy. We can conclude that the CIR of this general equilibrium model can be approximate with the partial equilibrium policies.

## C.2 Real Exchange Dynamics: Blanco and Cravino (2018)

The world economy consists of two symmetric countries, $i$ and $n$, each inhabited by a government, a monetary authority, a representative household, a producer of final goods and a continuum of monopolistic intermediate producers indexed by $\omega \in [0,2]$.

**Representative Household** Each household in country $i$ has preferences given by

$$U_i = \mathbb{E}_0 \left[ \sum_{t=0} \beta^t e^{\eta^\beta_t} \left[ \eta^C_{i,t} \log(C_{i,t}) - N_{i,t} \right] \right] \tag{C.84}$$

where $C_{i,t}$ and $N_{i,t}$ denote consumption and labor, $\eta^C_{i,t}$ is a taste shock to the utility of consumption and $\eta^\beta_t$ is a symmetric discount factor shock. The time 0 intertemporal budget constraint is:

$$\mathbb{E}_0 \left[ \sum_{t=0} D_{i,t} \left( P_{i,t}C_{i,t} - W_{i,t}N_{i,t} - \Upsilon_{i,t} - T_{i,t} \right) \right] = 0 \tag{C.85}$$

Here $P_{i,t}$, $W_{i,t}$, $\Upsilon_{i,t}$, and $T_{i,t}$ respectively denote the price of consumption, nominal wages, firm's profits, and government transfers, all denominated in the currency of country $i$. $D_{i,t}$ is the time 0 local-currency price of an Arrow security that pays one unit of the local currency at time $t$. By definition, the nominal exchange rate satisfies: $\epsilon_{in,t} = \frac{\epsilon_{in,0}D_{i,t}}{D_{n,t}}$.

**Monetary and Fiscal Policy** As in Carvalho and Nechio (2011), we define the monetary policy in an implicit way, and assume that aggregate nominal expenditures in each country are exogenous and given by $Z_{i,t} = P_{i,t}C_{i,t}$. Government expenditures $G_t$ are also exogenous, and taxes $T_{i,t}$ balance the budget every period. This is without loss of generality, since the Ricardian equivalence holds.

**Final goods producer** The final good in country $i$, $Y_{i,t}$, is produced according to:

$$Y_{i,t} = \left[ \mu^{\frac{1}{\xi}} Y_{ii,t}^{\frac{\xi-1}{\xi}} + [1-\mu]^{\frac{1}{\xi}} Y_{ni,t}^{\frac{\xi-1}{\xi}} \right]^{\frac{\xi}{\xi-1}}, \quad where \tag{C.86}$$

$$Y_{ni,t} = \left[ \int [Y_{ni,t}(\omega)/E_{in,t}(\omega)]^{\frac{\eta-1}{\eta}} d\omega \right]^{\frac{\eta}{\eta-1}}, \qquad Y_{ii,t} = \left[ \int [Y_{ii,t}(\omega)E_{in,t}(\omega)]^{\frac{\eta-1}{\eta}} d\omega \right]^{\frac{\eta}{\eta-1}}$$

Here $Y_{ni,t}(\omega)$ denotes the quantity of intermediate good $\omega$ produced in country $n$ and consumed in country $i$, $\mu$ is the share of domestic goods in absorption in the symmetric steady state, and $\xi$ and $\eta$ are the elasticities of substitution between domestic and foreign goods and across varieties, respectively. $E_{in,t}(\omega)$ is a quality shock as in Woodford (2009).

**Intermediate Good Producer** Intermediate producers behave as monopolistic competitors and set prices. Importantly, producers set prices in the currency of the country where they sell. The probability that a producer has the option to change its price in any period is given by $1-\theta_p$. The production function for intermediate goods is

$$Y_{ni,t}(\omega) = E_t(\omega) N_{ni,t}(\omega). \tag{C.87}$$

Here $E_t(\omega)$ and $N_{ni,t}(\omega)$ denote the idiosyncratic productivity and labor input, respectively.

Let $S_t = (n_t, a_t)$ be the idiosyncratic state of the firm composed by a Poisson counter with arrival rate $\lambda$, $n_t$, and the time since the last adjustment, $a_t$, following the stochastic process $da_t = dt$. The initial conditions after an adjustment are given by $\hat{S} = (0,0)$. Define the stopping time $\tau = \inf_t\{t \geq 0 : a_t \geq T \ or \ n_t \geq 1\}$. The profit maximizing price for an intermediate producer that gets to adjust prices satisfies

$$\hat{p}_{in,t} = \arg\max_{p_{in,t}} \mathbb{E}\left[\int_t^{t+\tau} \frac{D_s}{D_t}\left[\frac{p_{in,t}}{E_{in,s}} - \frac{W_{i,s}}{A_{in,s}}\right] Y_{n,t}\left(\frac{A_{in,t}p_{in,t}}{P_{in,s}}\right)^{-\eta}\left(\frac{P_{in,t}}{P_{n,s}}\right)^{-\xi}\right]. \tag{C.88}$$

Here $\frac{D_s}{D_t}$ is country i's nominal discount factor between dates $t$ and $t+s$, $E_{in,s}$ is the nominal exchange rate, expressed in units of currency $n$ per-unit of currency $i$. $P_{in}$ is the price level of the goods produced in $i$ and consumption in $n$ and $P_n$ is the price level in $n$ country.

**Stochastic process for aggregate shocks** $\eta^C_{i,t}$, $\eta^C_{n,t}$, $Z_{i,t}$ and $Z_{n,t}$ follow a Brownian process with no drift in logs. We leave without specification the stochastic process for government expenditures, $G_{i,t}$ and $G_{n,t}$, and discount factor shocks, $\eta^\beta_t$.

**Equilibrium** An equilibrium is a set of allocations for the households $\{C_{i,t}, W_{i,t}(h)\}_{\forall i,t}$, production decisions for final good producers $\{Y_{i,t}, Y_{in,t}, \{Y_{in}(\omega)\}_\omega\}_{\forall n,t}$, and price policy functions for intermediate producers $\{\bar{P}_{in,t}\}_{\forall i,n,t}$, such that given prices: (i) households maximize (C.84) subject to (C.85); (ii) final good producers minimize cost according to equations (C.86); (iii) intermediate producers maximize profits according to equation (C.88); and (iv) labor and goods markets clear.

**Nominal and real exchange rates** The assumption of complete markets and the process for nominal exchange rate implies that $d\log(E_{in,t}) = dZ_t$, where $Z_{i,t}$ is a Weiner process given by $dZ_t = d\log(\eta^C_{i,t}) + d\log(Z_{n,t}) - \left(d\log(\eta^C_{n,t}) + d\log(Z_{i,t})\right)$. Additionally, the complete market assumption and the labor-leisure condition implies that $d\log(W_{i,t}) = d\log(Z_{i,t}) - d\log(\eta^C_{i,t})$ and $d\log(W_{n,t}) = d\log(Z_{n,t}) - d\log(\eta^C_{n,t})$. Define aggregate markups as:

$$\mu_{in,t} = \log\left(\frac{\eta}{\eta-1}\frac{A_{in,t}}{W_i E_{in}}\right) \ ; \ \mu_{ii,t} = \log\left(\frac{\eta}{\eta-1}\frac{A_{ii,t}}{W_i}\right)$$
$$\mu_{ni,t} = \log\left(\frac{\eta}{\eta-1}\frac{A_{ni,t}E_{in}}{W_n}\right) \ ; \ \mu_{nn,t} = \log\left(\frac{\eta}{\eta-1}\frac{A_{nn,t}}{W_n}\right)$$

Under the observation that multiplicative term in the profits function are only second order, up to a first order, we have that the optimality conditions over the reset markup $\hat{\mu}_{in}$ are given by

$$0 = \mathbb{E}\left[\int_t^{t+\tau} \mu_{in,s} ds\right], \tag{C.89}$$

and thus, independent of general equilibrium feedback. Finally, define $rer$ as the log deviation from the steady state of the real exchange. Then real exchange dynamics are given by

$$rer_{in,t} = p_i + e_{in,t} - p_n = \mu\mu_{ii} + (1-\mu)\mu_{ni} - ((1-\mu)\mu_{in} + \mu\mu_{nn}),$$

where we used that $w_i + e_{in,t} - w_{n,t} = 0$.

# D Lumpy Investment: Proofs

This section provides additional steps for characterizing the investment model in Section 2.

## D.1 Characterization of Equilibrium Transition Dynamics

We skip the description of the environment for this case.

**Optimal policy.** In equilibrium, due to linear preferences in consumption, the time-zero Arrow-Debreu price is $Q_t = Q_0 e^{-\rho t}$. Since the law of motion of the price system is independent of the distribution of firms, the firms' state only depends on the idiosyncratic states $K$ and $E$. Let $V(K,E)$ the present discounted value of the optimal plan with the state $(K,E)$ measure in time $t$ consumption units.

Define the inaction region as $\mathcal{R} = \{K, E : \underline{K}(E) < K < \overline{K}(E)\}$. For all $(K,E) \in \mathcal{R}$, $V(\cdot)$ satisfies the HJB given by

$$(\rho+\lambda)\,V(K,E) = E^{1-\alpha}K^{\alpha} - \psi K \frac{\partial V(K,E)}{\partial K} + \frac{\sigma^2 E^2}{2}\frac{\partial^2 V(K,E)}{\partial E^2} + (\mu + \frac{\sigma^2}{2})E\frac{\partial V(K,E)}{\partial E} + \dots \tag{D.1}$$

$$\dots + \lambda \mathbb{E}_{\xi}\left[\max\left\{\max_{K^*} V(K^*,E) - \xi E - (K^* - K), V(K,E)\right\}\right], \tag{D.2}$$

together with the value matching conditions:

$$V(\underline{K}(E),E) = \max_{K^*} V(K^*,E) - \kappa E - (K^* - \underline{K}(E))$$
$$V(\overline{K}(E),E) = \max_{K^*} V(K^*,E) - \kappa E - (K^* - \overline{K}(E)), \tag{D.3}$$

and the smooth pasting conditions:

$$1 = \left.\frac{\partial V(K,E)}{\partial K}\right|_{K=\underline{K}(E)} = \left.\frac{\partial V(K,E)}{\partial K}\right|_{K=\overline{K}(E)} \tag{D.4}$$

$$\frac{\partial V(K^*,E)}{\partial E} - \kappa = \left.\frac{\partial V(K,E)}{\partial E}\right|_{K=\underline{K}(E)} = \left.\frac{\partial V(K,E)}{\partial E}\right|_{K=\overline{K}(E)}. \tag{D.5}$$

For additional details over the HJB equations, value matching and smooth pasting conditions see Oksendal (2007) and Baley and Blanco (2019). The next proposition shows that we can consider the normalized capital $\tilde{K} \equiv \frac{K}{E}$ as a state.

**Proposition D.1.** *Let $V$ the solution of* (D.2) *to* (D.5). *Also define the total drift $\nu \equiv -(\psi+\mu)$ and the adjusted discount $\tilde{\rho} \equiv \rho + \lambda - \mu - \frac{\sigma^2}{2}$. Then $V(K,E) = Ev\left(\frac{K}{E}\right)$ where $v$ satisfies the following HJB, value matching and smooth-pasting conditions:*

$$\tilde{\rho} v\left(\tilde{K}\right) = \tilde{K}^{\alpha} - (\psi + \mu + \frac{\sigma^2}{2})\tilde{K} v'\left(\tilde{K}\right) + \frac{\sigma^2}{2}\tilde{K}^2 v''\left(\tilde{K}\right) + \lambda \mathbb{E}_{\xi}\left[\max\left\{\max_{\tilde{K}^*} v(\tilde{K}^*) - \xi - (\tilde{K}^* - \tilde{K}), v(\tilde{K})\right\}\right] \tag{D.6}$$

$$v\left(\underline{\tilde{K}}\right) - \underline{\tilde{K}} = v\left(\tilde{K}^*\right) - \kappa - \tilde{K}^* \tag{D.7}$$

$$v\left(\overline{\tilde{K}}\right) - \overline{\tilde{K}} = v\left(\tilde{K}^*\right) - \kappa - \tilde{K}^*, \tag{D.8}$$

$$v'\left(\underline{\tilde{K}}\right) = v'\left(\overline{\tilde{K}}\right) = v'\left(\tilde{K}^*\right) = 1. \tag{D.9}$$

*Proof.* We guess and verify that $V(K,E) = Ev\left(\frac{K}{E}\right)$. Note that the following relations hold:

$$\frac{\partial V(K,E)}{\partial K} = v'\left(\frac{K}{E}\right), \qquad \frac{\partial V(K,E)}{\partial E} = v\left(\frac{K}{E}\right) - \frac{K}{E}v'\left(\frac{K}{E}\right), \qquad \frac{\partial^2 V(K,E)}{\partial E^2} = \frac{K^2}{E^3}v''\left(\frac{K}{E}\right). \tag{D.10}$$

Using these results we have that (D.2) can be written as

$$(\rho+\lambda)Ev\left(\frac{K}{E}\right) = E\left(\frac{K}{E}\right)^{\alpha} - \psi\frac{K}{E}v'\left(\frac{K}{E}\right) + \frac{\sigma^2}{2}\left(\frac{K}{E}\right)^2 v''\left(\frac{K}{E}\right) + (\mu+\frac{\sigma^2}{2})\left[v\left(\frac{K}{E}\right) - \frac{K}{E}v'\left(\frac{K}{E}\right)\right]$$
$$+\lambda E\mathbb{E}_{\xi}\left[\max\left\{\max_{K^*} v\left(\frac{K^*}{E}\right) - \xi - \left(\frac{K^*}{E} - \frac{K}{E}\right), v\left(\frac{K}{E}\right)\right\}\right]$$

Thus the HJB is given by

$$\tilde{\rho} v\left(\tilde{K}\right) = \tilde{K}^{\alpha} - (\psi + \mu + \frac{\sigma^2}{2})\tilde{K} v'\left(\tilde{K}\right) + \frac{\sigma^2}{2}\tilde{K}^2 v''\left(\tilde{K}\right) + \lambda \mathbb{E}_{\xi}\left[\max\left\{\max_{\tilde{K}^*} v(\tilde{K}^*) - \xi - (\tilde{K}^* - \tilde{K}), v(\tilde{K})\right\}\right]. \tag{D.11}$$

The value-matching conditions can be expressed as

$$v\left(\underline{\tilde{K}}\right) = \max_{\tilde{K}^*} v\left(\tilde{K}^*\right) - \kappa - (\tilde{K}^* - \underline{\tilde{K}})$$
$$v\left(\overline{\tilde{K}}\right) = \max_{\tilde{K}^*} v\left(\tilde{K}^*\right) - \kappa - (\tilde{K}^* - \overline{\tilde{K}}). \tag{D.12}$$

and the smooth pasting conditions as

$$v'\left(\underline{\tilde{K}}\right) = v'\left(\overline{\tilde{K}}\right) = 1 \tag{D.13}$$

and the smooth pasting conditions for the productivity are given by

$$v\left(\tilde{K}^*\right) - \tilde{K}^* v'\left(\tilde{K}^*\right) - \kappa = v\left(\underline{\tilde{K}}\right) - \underline{\tilde{K}} v'\left(\underline{\tilde{K}}\right)$$
$$v\left(\tilde{K}^*\right) - \tilde{K}^* v'\left(\tilde{K}^*\right) - \kappa = v\left(\overline{\tilde{K}}\right) - \overline{\tilde{K}} v'\left(\overline{\tilde{K}}\right). \tag{D.14}$$

Using the optimality condition we have that $v'\left(\tilde{K}^*\right) = 1$ and using the value matching (D.13) we have

$$v\left(\tilde{K}^*\right) - \tilde{K}^* - \kappa = v\left(\underline{\tilde{K}}\right) - \underline{\tilde{K}}$$
$$v\left(\tilde{K}^*\right) - \tilde{K}^* - \kappa = v\left(\overline{\tilde{K}}\right) - \overline{\tilde{K}}. \tag{D.15}$$

Notice that these smooth pasting conditions are redundant due to (D.12). □

The next proposition characterize the stopping policy

**Proposition D.2.** *Let $w(k)$ and $\{\underline{k}, k^*, \overline{k}\}$ satisfy the following system of differential equation*

$$\tilde{\rho} w(k) = e^{\alpha k} - (\psi + \mu) w'(k) + \frac{\sigma^2}{2} w''(k) + \lambda \mathbb{E}_\xi \left[\max\left\{w(k^*) - \xi - (e^{k^*} - e^k), w(k)\right\}\right] \tag{D.16}$$
$$w(\underline{k}) - e^{\underline{k}} = w(k^*) - \kappa - e^{k^*} \tag{D.17}$$
$$w(\overline{k}) - e^{\overline{k}} = w(k^*) - \kappa - e^{k^*}, \tag{D.18}$$
$$w'(\underline{k}) = e^{\underline{k}} \ ; \ w'(\overline{k}) = e^{\overline{k}} \ ; \ w'(k^*) = e^{k^*}. \tag{D.19}$$

*Then given the initial $log(K_0/E_0) = k_0$ the optimal stopping policy is given by*

$$\tau(k_0) = \inf_t \{t \geq 0 : k_t \notin [\underline{k}, \overline{k}] \ o \ N_t^{k_t} - N_0^{k_0} = 1\}, \tag{D.20}$$

*where $dk_t = -(\delta + \mu)dt + \sigma dB_t$ with initial condition $k_0$ and $N_t^{k_t}$ is a Poisson process with arrival rate $\lambda(k) = \lambda G(w(k^*) - w(k) - (e^{k^*} - e^k))$*

*Proof.* We depart from the solution of the optimal policy given by (D.6) to (D.9). Doing a guess and verify $v\left(\tilde{K}\right) = w(\log(\tilde{K}))$, it is easy to see that $w$ satisfies (D.16) to (D.19). Therefore the stopping policy is given by

$$\tau(k_0) = \inf_t \{t \geq 0 : k_t \notin [\underline{k}, \overline{k}] \ o \ (N_t - N_0 = 1 \ and \ \xi \leq w(k^*) - w(k) - (e^{k^*} - e^k))\}, \tag{D.21}$$

or equivalently, (D.20). □

# E Additional proofs

## E.1 Relationship between kurtosis and age in drift-less time-dependent models

**Lemma 1.** *If the uncontrolled state follows $d\tilde{x}_t = \sigma dB_t$, and $\tau$ is independent of $x$, then the following equalities hold:*

$$\frac{\mathbb{E}[x^2]}{\sigma^2} = \mathcal{M}_1[a] \tag{E.22}$$
$$\mathbb{K}ur\left[\Delta x\right] = 3\left(1 + \mathbb{CV}^2[\tau]\right). \tag{E.23}$$

*Proof.* First, let us show that (E.22). By Auxiliary Theorem **??**, we have that $\frac{\mathbb{E}[x^2]}{\sigma^2} = \frac{\mathbb{E}[\int_0^\tau x_t^2 dt]}{\mathbb{E}[\tau]}$ and we have that

$$\frac{\mathbb{E}[x^2]}{\sigma^2} = \frac{\mathbb{E}[\int_0^\tau B_t^2 dt]}{\mathbb{E}[\tau]} = \frac{\int_0^\infty \mathbb{E}\left[B_t^2 I(t \leq \tau)\right] dt}{\mathbb{E}[\tau]} = \frac{\int_0^\infty \mathbb{E}\left[B_t^2\right] \mathbb{E}\left[I(t \leq \tau)\right] dt}{\mathbb{E}[\tau]} = \mathbb{E}\left[\int_0^\tau t dt\right] = \mathcal{M}_1[a], \tag{E.24}$$

where in the last step we used independence of the stopping time. Let us show (E.23). Start from the equivalence $-\Delta x = \sigma B_\tau$, raise to the 4-th power and take expectations to obtain $\mathbb{E}[\Delta x^4] = \sigma^4 \mathbb{E}[B_\tau^4]$. By the independence assumption of the stopping time and the normal distribution of the Brownian motions, $\mathbb{E}[B_\tau^4] = 3\mathbb{E}[\tau^2]$. Now using this result, we express the kurtosis of $\Delta x$ as follows:

$$\mathbb{K}ur\,[\Delta x] = \frac{\mathbb{E}[\Delta x^4]}{\mathbb{E}[\Delta x^2]^2} = \frac{3\sigma^4 \mathbb{E}[\tau^2]}{\sigma^4 \mathbb{E}[\tau]^2} = \frac{3(\mathbb{V}[\tau] + \mathbb{E}[\tau]^2)}{\mathbb{E}[\tau]^2} = 3(1 + \mathbb{CV}^2[\tau]) \tag{E.25}$$

□

## E.2 Proof of Example 1 with no Drift

This subsection of the appendix proofs the theorem in example **??** in the case of no drift since it is less involve—then, we generalized in the case with drift. We use the following identities shown in subsection in the Online Appendix.

$$\tilde{\lambda} = \frac{\lambda}{\sigma^2} \tag{E.26}$$

$$\xi_1 = -\sqrt{2\tilde{\lambda}} \quad ; \quad \xi_2 = \sqrt{2\tilde{\lambda}} \tag{E.27}$$

$$v_2(\hat{x}) = \frac{-e^{\xi_1 \hat{x}} \left[\overline{\alpha}_2 \kappa_2^2(\underline{x}) - \underline{\alpha}_2 \kappa_2^2(\overline{x})\right] - e^{\xi_2 \hat{x}} \left[\underline{\alpha}_1 \kappa_2^2(\overline{x}) - \overline{\alpha}_1 \kappa_2^2(\underline{x})\right] + \kappa_2^2(\hat{x})}{\lambda} \tag{E.28}$$

$$v_1(x) = \frac{-e^{\xi_1 x} \left[\overline{\alpha}_2 \kappa_1^1(\underline{x}) - \underline{\alpha}_2 \kappa_1^1(\overline{x})\right] - e^{\xi_2 x} \left[\underline{\alpha}_1 \kappa_1^1(\overline{x}) - \overline{\alpha}_1 \kappa_1^1(x)\right] + \kappa_1^1(x)}{\lambda} \tag{E.29}$$

$$\overline{\alpha}_1 = \frac{e^{-\sqrt{2\tilde{\lambda}}\overline{x}}}{e^{\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})} - e^{-\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})}} \quad ; \quad \overline{\alpha}_2 = \frac{e^{\sqrt{2\tilde{\lambda}}\overline{x}}}{e^{\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})} - e^{-\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})}} \tag{E.30}$$

$$\underline{\alpha}_1 = \frac{e^{-\sqrt{2\tilde{\lambda}}\underline{x}}}{e^{\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})} - e^{-\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})}} \quad ; \quad \underline{\alpha}_2 = \frac{e^{\sqrt{2\tilde{\lambda}}\underline{x}}}{e^{\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})} - e^{-\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})}} \tag{E.31}$$

$$\kappa_1^1(x) = x, \quad ; \quad \kappa_2^2(x) = \frac{1}{\tilde{\lambda}} + x^2 \tag{E.32}$$

$$\mathbb{E}^{\hat{x}}\left[\int_0^\tau e^{\xi_1 x_t} dt\right] = \frac{-e^{\xi_1 \hat{x}}(e^{\xi_1 \underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_1 \overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2 \hat{x}}\left(e^{\xi_1 \overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1 \underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_1 \hat{x}}\hat{x}}{-\sigma^2 \xi_1} \tag{E.33}$$

$$\mathbb{E}^{\hat{x}}\left[\int_0^\tau e^{\xi_2 x_t} dt\right] = \frac{-e^{\xi_1 \hat{x}}(e^{\xi_2 \underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_2 \overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2 \hat{x}}\left(e^{\xi_2 \overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2 \underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_2 \hat{x}}\hat{x}}{-\sigma^2 \xi_2} \tag{E.34}$$

$$\hat{x} = e^{-\sqrt{2\tilde{\lambda}}\hat{x}}\left[\overline{\alpha}_2 \underline{x} - \underline{\alpha}_2 \overline{x}\right] + e^{\sqrt{2\tilde{\lambda}}\hat{x}}\left[\underline{\alpha}_1 \overline{x} - \overline{\alpha}_1 x\right] \tag{E.35}$$

**Proposition E.3.** *Assume $\nu = 0$ and $H(\xi) = 1$ for all $\xi \in [0, \kappa]$ in the model presented in Section **??**. Then the CIR is given by*

$$\mathcal{A}_1(\delta) = \delta \frac{\mathbb{E}[x^2]}{\sigma^2} + o(\delta^2) \tag{E.36}$$

*Proof.* Define the following objects

$$LHS = \left.\frac{d\mathcal{A}_1(\delta)}{d\delta}\right|_{\delta=0} \tag{E.37}$$

$$RHS = \frac{\mathbb{E}[x^2]}{\sigma^2} \tag{E.38}$$

We need to show that $LHS = RHS$. We divide the proof in 4 steps.

**Step 1:** Let us operate with the LHS and the RHS with the occupancy measure. The $LHS$ and the $RHS$ are given by

$$LHS = \left.\frac{d\mathcal{A}_1(\delta)}{d\delta}\right|_{\delta=0} = \int_{\underline{x}}^{\overline{x}} v_1'(x) f(x) = \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t) dt\right]}{\mathbb{E}^{\hat{x}}\left[\tau\right]} \tag{E.39}$$

$$RHS = \frac{\mathbb{E}[x^2]}{\sigma^2} = \frac{v_2(\hat{x})}{\sigma^2 \mathbb{E}^{\hat{x}}\left[\tau\right]}. \tag{E.40}$$

Therefore $LHS = RHS$ iff $\frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t)dt\right]}{\mathbb{E}^{\hat{x}}[\tau]} = \frac{v_2(\hat{x})}{\sigma^2 \mathbb{E}^{\hat{x}}[\tau]}$, or equivalently $\lambda \mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t)dt\right]\lambda = \lambda \frac{v_2(\hat{x})}{\sigma^2}$ (i.e. $\lambda \mathbb{E}^{\hat{x}}\left[\tau\right] LHS = \lambda \mathbb{E}^{\hat{x}}\left[\tau\right] RHS$ ).

**Step 2:** Now, let us simply $\lambda\mathbb{E}^{\hat{x}}[\tau]\, LHS = \lambda\mathbb{E}^{\hat{x}}[\tau]\, RHS$. Defining Now we will show that the left hand side is equal to the right hand side. Define

$$\mathcal{T}_1 = \frac{-e^{\sqrt{2\bar{\lambda}}\hat{x}}(\overline{\alpha}_2 - \underline{\alpha}_2) - e^{-\sqrt{2\bar{\lambda}}\hat{x}}(\underline{\alpha}_1 - \overline{\alpha}_1) + 1}{\lambda}, \tag{E.41}$$

then we have that

$$\begin{aligned}
&= \lambda\mathbb{E}^{\hat{x}}[\tau]\, RHS \\
&= \frac{\lambda v_2(\hat{x})}{\sigma^2} \\
&= \frac{\frac{-e^{-\sqrt{2\bar{\lambda}}\hat{x}}(\overline{\alpha}_2 - \underline{\alpha}_2) - e^{\sqrt{2\bar{\lambda}}\hat{x}}(\underline{\alpha}_1 - \overline{\alpha}_1) + 1}{\bar{\lambda}} - e^{-\sqrt{2\bar{\lambda}}\hat{x}}(\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2) - e^{\sqrt{2\bar{\lambda}}\hat{x}}(\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2) + \hat{x}^2}{\sigma^2} \\
&= \mathcal{T}_1 + \frac{-e^{-\sqrt{2\bar{\lambda}}\hat{x}}(\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2) - e^{\sqrt{2\bar{\lambda}}\hat{x}}(\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2) + \hat{x}^2}{\sigma^2}
\end{aligned} \tag{E.42}$$

$$\begin{aligned}
&= \lambda\mathbb{E}^{\hat{x}}[\tau]\, LHS \\
&= \lambda\mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t)dt\right] \\
&= -\xi_1\mathbb{E}^{\hat{x}}\left[\int_0^\tau e^{\xi_1 x_t}dt\right](\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) - \xi_2\mathbb{E}^{\hat{x}}\left[\int_0^\tau e^{\xi_2 x_t}dt\right](\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x}) + \mathbb{E}^{\hat{x}}[\tau] \\
&= \frac{\left(-e^{-\sqrt{2\bar{\lambda}}\hat{x}}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\sqrt{2\bar{\lambda}}\hat{x}}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{-\sqrt{2\bar{\lambda}}\hat{x}}\hat{x}\right)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x})}{\sigma^2}\cdots \\
&\cdots + \frac{\left(-e^{-\sqrt{2\bar{\lambda}}\hat{x}}(e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\sqrt{2\bar{\lambda}}\hat{x}}\left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\sqrt{2\bar{\lambda}}\hat{x}}\hat{x}\right)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})}{\sigma^2} + \mathcal{T}_1.
\end{aligned} \tag{E.43}$$

Simplifying the $\sigma^2$ and $\mathcal{T}_1$ from both sides, we have

$$\begin{aligned}
&= \lambda\mathbb{E}^{\hat{x}}[\tau]\, RHS \\
&= -e^{-\sqrt{2\bar{\lambda}}\hat{x}}(\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2) - e^{\sqrt{2\bar{\lambda}}\hat{x}}(\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2) + \hat{x}^2
\end{aligned} \tag{E.44}$$

$$\begin{aligned}
&= \lambda\mathbb{E}^{\hat{x}}[\tau]\, LHS \\
&= \left(-e^{-\sqrt{2\bar{\lambda}}\hat{x}}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\sqrt{2\bar{\lambda}}\hat{x}}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{-\sqrt{2\bar{\lambda}}\hat{x}}\hat{x}\right)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x})\cdots \\
&\cdots + \left(-e^{-\sqrt{2\bar{\lambda}}\hat{x}}(e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\sqrt{2\bar{\lambda}}\hat{x}}\left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\sqrt{2\bar{\lambda}}\hat{x}}\hat{x}\right)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x}).
\end{aligned} \tag{E.45}$$

Therefore $LHS = RHS$ iff equation (E.44) is equal to equation (E.45).

**Step 3:** Using the definition of $\hat{x} = e^{-\sqrt{2\tilde{\lambda}}\hat{x}}\left[\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}\right] + e^{\sqrt{2\tilde{\lambda}}\hat{x}}\left[\underline{\alpha}_1\overline{x} - \overline{\alpha}_1 x\right]$ we can operate in the LHS and the RHS in equations (E.44) and (E.45).

$$\begin{aligned}
&= \lambda\mathbb{E}^{\hat{x}}\left[\tau\right] RHS,\\
&= -e^{-\sqrt{2\tilde{\lambda}}\hat{x}}(\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2) - e^{\sqrt{2\tilde{\lambda}}\hat{x}}(\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2) + \hat{x}^2
\end{aligned} \tag{E.46}$$

$$\begin{aligned}
&= \lambda\mathbb{E}^{\hat{x}}\left[\tau\right] LHS,\\
&= -e^{-\sqrt{2\tilde{\lambda}}\hat{x}}\left((e^{-\sqrt{2\tilde{\lambda}}\underline{x}}\underline{x}\overline{\alpha}_2 - e^{-\sqrt{2\tilde{\lambda}}\overline{x}}\overline{x}\underline{\alpha}_2)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + (e^{\sqrt{2\tilde{\lambda}}\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\sqrt{2\tilde{\lambda}}\overline{x}}\overline{x}\underline{\alpha}_2)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right)\dots\\
&\dots - e^{\sqrt{2\tilde{\lambda}}\hat{x}}\left(\left(e^{-\sqrt{2\tilde{\lambda}}\overline{x}}\overline{x}\underline{\alpha}_1 - e^{-\sqrt{2\tilde{\lambda}}\underline{x}}\underline{x}\overline{\alpha}_1\right)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + \left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right)\dots\\
&\dots + \hat{x}\left[e^{-\sqrt{2\tilde{\lambda}}\hat{x}}(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + e^{\sqrt{2\tilde{\lambda}}\hat{x}}(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right]\\
&= -e^{-\sqrt{2\tilde{\lambda}}\hat{x}}\left((e^{-\sqrt{2\tilde{\lambda}}\underline{x}}\underline{x}\overline{\alpha}_2 - e^{-\sqrt{2\tilde{\lambda}}\overline{x}}\overline{x}\underline{\alpha}_2)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + (e^{\sqrt{2\tilde{\lambda}}\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\sqrt{2\tilde{\lambda}}\overline{x}}\overline{x}\underline{\alpha}_2)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right)\dots\\
&\dots - e^{\sqrt{2\tilde{\lambda}}\hat{x}}\left(\left(e^{-\sqrt{2\tilde{\lambda}}\overline{x}}\overline{x}\underline{\alpha}_1 - e^{-\sqrt{2\tilde{\lambda}}\underline{x}}\underline{x}\overline{\alpha}_1\right)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + \left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right) + \hat{x}^2,
\end{aligned} \tag{E.47}$$

canceling $\hat{x}^2$ from both sizes and multiplying and dividing the LHS by $\mathcal{T}_2 = e^{\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})} - e^{-\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})}$ we have that

$$\begin{aligned}
&= \lambda\mathbb{E}^{\hat{x}}\left[\tau\right] RHS,\\
&= -e^{-\sqrt{2\tilde{\lambda}}\hat{x}}(\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2) - e^{\sqrt{2\tilde{\lambda}}\hat{x}}(\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2)
\end{aligned} \tag{E.48}$$

$$\begin{aligned}
&= \lambda\mathbb{E}^{\hat{x}}\left[\tau\right] LHS,\\
&= -e^{-\sqrt{2\tilde{\lambda}}\hat{x}}\mathcal{T}_2\left((\underline{\alpha}_1\underline{x}\overline{\alpha}_2 - \overline{\alpha}_1\overline{x}\underline{\alpha}_2)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + (\underline{\alpha}_2\underline{x}\overline{\alpha}_2 - \overline{\alpha}_2\overline{x}\underline{\alpha}_2)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right)\dots\\
&\dots - e^{\sqrt{2\tilde{\lambda}}\hat{x}}\mathcal{T}_2\left((\overline{\alpha}_1\overline{x}\underline{\alpha}_1 - \underline{\alpha}_1\underline{x}\overline{\alpha}_1)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + (\overline{\alpha}_2\overline{x}\underline{\alpha}_1 - \underline{\alpha}_2\underline{x}\overline{\alpha}_1)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right).
\end{aligned} \tag{E.49}$$

Therefore $LHS = RHS$ iff equation (E.48) is equal to equation (E.49).

**Step 4:** Now we show that (E.48) is equal to equation (E.49). By definition of $\overline{\alpha}_2$, $\underline{\alpha}_1$, $\overline{\alpha}_1$ and $\underline{\alpha}_2$, we have that

$$\overline{\alpha}_2\underline{\alpha}_1 - \overline{\alpha}_1\underline{\alpha}_2 = \frac{e^{\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})} - e^{-\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})}}{(e^{\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})} - e^{-\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})})^2} = \mathcal{T}_2^{-1}. \tag{E.50}$$

Using this result we have that

$$\begin{aligned}
&= (\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2)\\
&= \mathcal{T}_2\left(\underline{x}^2\overline{\alpha}_2\left(\underline{\alpha}_1\overline{\alpha}_2 - \overline{\alpha}_1\underline{\alpha}_2\right) - \overline{x}^2\underline{\alpha}_2\left(\underline{\alpha}_1\overline{\alpha}_2 - \overline{\alpha}_1\underline{\alpha}_2\right)\right)\\
&= \mathcal{T}_2\left(\underline{x}\overline{x}\left(-\underline{\alpha}_1\overline{\alpha}_2\underline{\alpha}_2 - \overline{\alpha}_1\underline{\alpha}_2\overline{\alpha}_2\right) + \underline{\alpha}_1\underline{x}^2\overline{\alpha}_2^2 + \overline{\alpha}_1\overline{x}^2\underline{\alpha}_2^2\right)\dots\\
&\dots + \mathcal{T}_2\left(\underline{x}\overline{x}\left(\underline{\alpha}_1\overline{\alpha}_2\underline{\alpha}_2 + \overline{\alpha}_1\underline{\alpha}_2\overline{\alpha}_2\right) - \overline{\alpha}_1\underline{x}^2\overline{\alpha}_2\underline{\alpha}_2 - \underline{\alpha}_1\overline{x}^2\underline{\alpha}_2\overline{\alpha}_2\right)\\
&= \mathcal{T}_2\left((\underline{\alpha}_1\underline{x}\overline{\alpha}_2 - \overline{\alpha}_1\overline{x}\underline{\alpha}_2)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + (\underline{\alpha}_2\underline{x}\overline{\alpha}_2 - \overline{\alpha}_2\overline{x}\underline{\alpha}_2)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right)
\end{aligned} \tag{E.51}$$

and also

$$\begin{aligned}
&= (\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2)\\
&= \mathcal{T}_2\left(\underline{\alpha}_1\overline{x}^2\left(\underline{\alpha}_1\overline{\alpha}_2 - \overline{\alpha}_1\underline{\alpha}_2\right) - \overline{\alpha}_1\underline{x}^2\left(\underline{\alpha}_1\overline{\alpha}_2 - \overline{\alpha}_1\underline{\alpha}_2\right)\right)\\
&= \mathcal{T}_2\left(\underline{x}\overline{x}\left(\overline{\alpha}_2\overline{\alpha}_1\underline{\alpha}_1 + \underline{\alpha}_2\underline{\alpha}_1\overline{\alpha}_1\right) - \underline{\alpha}_2\underline{x}^2\overline{\alpha}_1\underline{\alpha}_1 - \overline{\alpha}_2\overline{x}^2\underline{\alpha}_1\overline{\alpha}_1\right)\dots\\
&\dots + \mathcal{T}_2\left(\underline{x}\overline{x}\left(-\overline{\alpha}_2\overline{\alpha}_1\underline{\alpha}_1 - \underline{\alpha}_2\underline{\alpha}_1\overline{\alpha}_1\right) + \overline{\alpha}_1^2\underline{x}^2\underline{\alpha}_2 + \underline{\alpha}_1^2\overline{x}^2\overline{\alpha}_2\right)\\
&= \mathcal{T}_2\left((\overline{\alpha}_1\overline{x}\underline{\alpha}_1 - \underline{\alpha}_1\underline{x}\overline{\alpha}_1)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + (\overline{\alpha}_2\overline{x}\underline{\alpha}_1 - \underline{\alpha}_2\underline{x}\overline{\alpha}_1)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right).
\end{aligned} \tag{E.52}$$

Combining from equations (E.48) to (E.52) we have that

$$0 = LHS - RHS \iff \tag{E.53}$$

$$0 = -e^{-\sqrt{2\tilde{\lambda}}\hat{x}}\left(\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2 - (\mathcal{T}_2\left((\underline{\alpha}_1\underline{x}\overline{\alpha}_2 - \overline{\alpha}_1\overline{x}\underline{\alpha}_2)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + (\underline{\alpha}_2\underline{x}\overline{\alpha}_2 - \overline{\alpha}_2\overline{x}\underline{\alpha}_2)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right))\right)\dots$$
$$- e^{\sqrt{2\tilde{\lambda}}\hat{x}}\left(\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2 - (\mathcal{T}_2\left((\overline{\alpha}_1\overline{x}\underline{\alpha}_1 - \underline{\alpha}_1\underline{x}\overline{\alpha}_1)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + (\overline{\alpha}_2\overline{x}\underline{\alpha}_1 - \underline{\alpha}_2\underline{x}\overline{\alpha}_1)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right))\right) \iff \tag{E.54}$$

$$0 = -e^{-\sqrt{2\tilde{\lambda}}\hat{x}}\left(\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2 - \left(\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2\right)\right)\dots$$
$$- e^{\sqrt{2\tilde{\lambda}}\hat{x}}\left(\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2 - \left(\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2\right)\right) \iff \tag{E.55}$$

$$0 = 0 \tag{E.56}$$

Thus, we have shown the result. □

## E.3 Proof for Example 1 with drift

This subsection in the Online appendix proofs the theorem in example **??**. We use the following identities shown in section F.1 in the Online Appendix.

$$\tilde{\nu} = -\frac{\psi + \mu}{\sigma^2} \quad , \quad \nu = -(\psi + \mu) \tag{E.57}$$

$$\tilde{\lambda} = \frac{\lambda}{\sigma^2}\ ;\ \tilde{\lambda}(\varphi) = \frac{\lambda - \varphi}{\sigma^2} \tag{E.58}$$

$$\xi_1 = -\tilde{\nu} - \sqrt{\tilde{\nu}^2 + 2\tilde{\lambda}}\ ;\ \xi_1(\varphi) = -\tilde{\nu} - \sqrt{\tilde{\nu}^2 + 2\tilde{\lambda}(\varphi)} \tag{E.59}$$

$$\xi_2 = -\tilde{\nu} + \sqrt{\tilde{\nu}^2 + 2\tilde{\lambda}}\ ;\ \xi_2(\varphi) = -\tilde{\nu} + \sqrt{\tilde{\nu}^2 + 2\tilde{\lambda}(\varphi)} \tag{E.60}$$

$$\overline{\alpha}_1 = \frac{e^{\xi_1\overline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}}\ ;\ \overline{\alpha}_1(\varphi) = \frac{e^{\xi_1(\varphi)\overline{x}}}{e^{\xi_1(\varphi)\underline{x}+\xi_2(\varphi)\overline{x}} - e^{\xi_2(\varphi)\underline{x}+\xi_1(\varphi)\overline{x}}} \tag{E.61}$$

$$\overline{\alpha}_2 = \frac{e^{\xi_2\overline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}}\ ;\ \overline{\alpha}_2(\varphi) = \frac{e^{\xi_2(\varphi)\overline{x}}}{e^{\xi_1(\varphi)\underline{x}+\xi_2(\varphi)\overline{x}} - e^{\xi_2(\varphi)\underline{x}+\xi_1(\varphi)\overline{x}}} \tag{E.62}$$

$$\underline{\alpha}_1 = \frac{e^{\xi_1\underline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}}\ ;\ \underline{\alpha}_1(\varphi) = \frac{e^{\xi_1(\varphi)\underline{x}}}{e^{\xi_1(\varphi)\underline{x}+\xi_2(\varphi)\overline{x}} - e^{\xi_2(\varphi)\underline{x}+\xi_1(\varphi)\overline{x}}} \tag{E.63}$$

$$\underline{\alpha}_2 = \frac{e^{\xi_2\underline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}}\ ;\ \underline{\alpha}_2(\varphi) = \frac{e^{\xi_2(\varphi)\underline{x}}}{e^{\xi_1(\varphi)\underline{x}+\xi_2(\varphi)\overline{x}} - e^{\xi_2(\varphi)\underline{x}+\xi_1(\varphi)\overline{x}}} \tag{E.64}$$

$$\kappa_j^m(x) = \sum_{i=0}^{j}(x)^m\frac{m!}{i!}\left[\frac{\xi_1 + \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1 - \xi_2}(\xi_1 x)^{i-m} + \frac{\xi_2 + \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2 - \xi_1}(\xi_2 x)^{i-m}\right] \tag{E.65}$$

$$\kappa_1^1(x) = \frac{\tilde{\nu}}{\tilde{\lambda}} + x \tag{E.66}$$

$$\kappa_1^1(x,\varphi) = \frac{\tilde{\nu}}{\tilde{\lambda}(\varphi)} + x \tag{E.67}$$

$$\kappa_2^2(x) = 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + \frac{1}{2\tilde{\lambda}}\right] + 2\frac{\tilde{\nu}}{\tilde{\lambda}}x + x^2 \tag{E.68}$$

$$v_2(\hat{x}) = \frac{-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_2^2(\underline{x}) - \underline{\alpha}_2\kappa_2^2(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_2^2(\overline{x}) - \overline{\alpha}_1\kappa_2^2(\underline{x})\right] + \kappa_2^2(\hat{x})}{\lambda} \tag{E.69}$$

$$v_1(x) = \frac{-e^{\xi_1 x}\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right] - e^{\xi_2 x}\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(x)\right] + \kappa_1^1(x)}{\lambda} \tag{E.70}$$

$$\hat{x} = -\frac{\tilde{\nu}}{\tilde{\lambda}} + e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right] + e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(\underline{x})\right] \tag{E.71}$$

$$\mathbb{E}^{\hat{x}}\left[\int_0^\tau e^{\xi_1 x_t}dt\right] = -\frac{-e^{\xi_1\hat{x}}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_1\hat{x}}\hat{x}}{\sigma^2(\tilde{\nu} + \xi_1)} \tag{E.72}$$

$$\mathbb{E}^{\hat{x}}\left[\int_0^\tau e^{\xi_2 x_t}dt\right] = -\frac{-e^{\xi_1\hat{x}}(e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_2\hat{x}}\hat{x}}{\sigma^2(\tilde{\nu} + \xi_2)} \tag{E.73}$$

$$h_1(\varphi) = \frac{-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\kappa_1^1(\underline{x},\varphi) - \underline{\alpha}_2(\varphi)\kappa_1^1(\overline{x},\varphi)\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\kappa_1^1(\overline{x},\varphi) - \overline{\alpha}_1(\varphi)\kappa_1^1(\underline{x},\varphi)\right] + \overline{\kappa}_1^1(\hat{x})}{\lambda - \varphi} \tag{E.74}$$

**Proposition E.4.** *Assume $H(\xi) = 1$ for all $\xi \in [0, \overline{\kappa}]$ in the model presented in Section* **??**. *Then the CIR is given by*

$$\mathcal{A}_1(\delta) = \delta \frac{\mathbb{E}[x^2] - \nu \mathbb{C}ov[x, a]}{\sigma^2} \tag{E.75}$$

*Proof.* We proceed with the same steps as in the case with no drift. Define the following object

$$LHS = \left. \frac{d\mathcal{A}_1(\delta)}{d\delta} \right|_{\delta=0} \tag{E.76}$$

$$RHS = \frac{\mathbb{E}[x^2] - \nu \mathbb{C}ov[x, a]}{\sigma^2}. \tag{E.77}$$

For showing that the $LHS = RHS$, we divide the proof in 5 steps.

**Step 1:** Let us operate with the LHS and the RHS with the occupancy measure and the moment generating function for age. Using the occupancy measure and the moment generating function for age, we have that

$$\mathbb{C}ov[x, a] = \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t t dt\right]}{\mathbb{E}^{\hat{x}}[\tau]} = \frac{\left. \frac{\partial \mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t e^{\varphi t} dt\right]}{\partial \varphi} \right|_{\varphi=0}}{\mathbb{E}^{\hat{x}}[\tau]} = \frac{h_1'(0)}{\mathbb{E}^{\hat{x}}[\tau]}. \tag{E.78}$$

The $LHS$ and the $RHS$ are given by

$$LHS = \left. \frac{d\mathcal{A}_1(\delta)}{d\delta} \right|_{\delta=0} = \int_{\underline{x}}^{\overline{x}} v_1'(x) f(x) = \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t) dt\right]}{\mathbb{E}^{\hat{x}}[\tau]}$$

$$RHS = \frac{\mathbb{E}[x^2]}{\sigma^2} = \frac{v_2(\hat{x}) - \nu h_1'(0)}{\sigma^2 \mathbb{E}^{\hat{x}}[\tau]}. \tag{E.79}$$

Therefore $LHS = RHS$ iff $\frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t) dt\right]}{\mathbb{E}^{\hat{x}}[\tau]} = \frac{v_2(\hat{x}) - \nu h_1'(0)}{\sigma^2 \mathbb{E}^{\hat{x}}[\tau]}$, or equivalently $\sigma^2 \lambda \mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t) dt\right] = \lambda v_2(\hat{x}) - \lambda \nu h_1'(0)$.

**Step 2:** This steps writes

$$\sigma^2 \lambda \mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t) dt\right] = \mathcal{K}_1 - \tilde{\nu} \mathcal{K}_2, \tag{E.80}$$

where

$$\begin{aligned} \mathcal{K}_1 = & \left[ -e^{\xi_1 \hat{x}} (e^{\xi_1 \underline{x}} \underline{x} \overline{\alpha}_2 - e^{\xi_1 \overline{x}} \overline{x} \underline{\alpha}_2) - e^{\xi_2 \hat{x}} \left( e^{\xi_1 \overline{x}} \overline{x} \underline{\alpha}_1 - e^{\xi_1 \underline{x}} \underline{x} \overline{\alpha}_1 \right) + e^{\xi_1 \hat{x}} \hat{x} \right] \left[ \overline{\alpha}_2 \kappa_1^1(\underline{x}) - \underline{\alpha}_2 \kappa_1^1(\overline{x}) \right] \dots \\ \dots + & \left[ -e^{\xi_1 \hat{x}} (e^{\xi_2 \underline{x}} \underline{x} \overline{\alpha}_2 - e^{\xi_2 \overline{x}} \overline{x} \underline{\alpha}_2) - e^{\xi_2 \hat{x}} \left( e^{\xi_2 \overline{x}} \overline{x} \underline{\alpha}_1 - e^{\xi_2 \underline{x}} \underline{x} \overline{\alpha}_1 \right) + e^{\xi_2 \hat{x}} \hat{x} \right] \left[ \underline{\alpha}_1 \kappa_1^1(\overline{x}) - \overline{\alpha}_1 \kappa_1^1(x) \right] + \sigma^2 \mathbb{E}^{\hat{x}}[\tau] \end{aligned} \tag{E.81}$$

$$\begin{aligned} \mathcal{K}_2 = & \frac{-e^{\xi_1 \hat{x}} (e^{\xi_1 \underline{x}} \underline{x} \overline{\alpha}_2 - e^{\xi_1 \overline{x}} \overline{x} \underline{\alpha}_2) - e^{\xi_2 \hat{x}} \left( e^{\xi_1 \overline{x}} \overline{x} \underline{\alpha}_1 - e^{\xi_1 \underline{x}} \underline{x} \overline{\alpha}_1 \right) + e^{\xi_1 \hat{x}} \hat{x}}{\tilde{\nu} + \xi_1} \left[ \overline{\alpha}_2 \kappa_1^1(\underline{x}) - \underline{\alpha}_2 \kappa_1^1(\overline{x}) \right] \dots \\ \dots + & \frac{-e^{\xi_1 \hat{x}} (e^{\xi_2 \underline{x}} \underline{x} \overline{\alpha}_2 - e^{\xi_2 \overline{x}} \overline{x} \underline{\alpha}_2) - e^{\xi_2 \hat{x}} \left( e^{\xi_2 \overline{x}} \overline{x} \underline{\alpha}_1 - e^{\xi_2 \underline{x}} \underline{x} \overline{\alpha}_1 \right) + e^{\xi_2 \hat{x}} \hat{x}}{\tilde{\nu} + \xi_2} \left[ \underline{\alpha}_1 \kappa_1^1(\overline{x}) - \overline{\alpha}_1 \kappa_1^1(x) \right]. \end{aligned} \tag{E.82}$$

Operating over $\sigma^2\lambda\mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t)dt\right]$ we have that

$$
\begin{aligned}
&= \sigma^2\lambda\mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t)dt\right]\\
&= -\xi_1\sigma^2\mathbb{E}^{\hat{x}}\left[\int_0^\tau e^{\xi_1 x_t}dt\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right] - \xi_2\sigma^2\mathbb{E}^{\hat{x}}\left[\int_0^\tau e^{\xi_2 x_t}dt\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(x)\right] + \sigma^2\mathbb{E}^{\hat{x}}\left[\tau\right]\\
&= \frac{\xi_1}{(\tilde{\nu}+\xi_1)}\left[-e^{\xi_1\hat{x}}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_1\hat{x}}\hat{x}\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right]\dots\\
&\dots + \frac{\xi_2}{(\tilde{\nu}+\xi_2)}\left[-e^{\xi_1\hat{x}}(e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_2\hat{x}}\hat{x}\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(x)\right] + \sigma^2\mathbb{E}^{\hat{x}}[\tau]\\
&= [1 + \frac{\xi_1}{(\tilde{\nu}+\xi_1)} - 1]\left[-e^{\xi_1\hat{x}}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_1\hat{x}}\hat{x}\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right]\dots\\
&\dots + [1 + \frac{\xi_2}{(\tilde{\nu}+\xi_2)} - 1]\left[-e^{\xi_1\hat{x}}(e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_2\hat{x}}\hat{x}\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(x)\right] + \sigma^2\mathbb{E}^{\hat{x}}[\tau]\\
&= \left[-e^{\xi_1\hat{x}}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_1\hat{x}}\hat{x}\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right]\dots\\
&\dots + \left[-e^{\xi_1\hat{x}}(e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_2\hat{x}}\hat{x}\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(x)\right] + \sigma^2\mathbb{E}^{\hat{x}}[\tau]\dots\\
&\dots - \frac{\tilde{\nu}}{\tilde{\nu}+\xi_1}\left[-e^{\xi_1\hat{x}}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_1\hat{x}}\hat{x}\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right]\dots\\
&\dots - \frac{\tilde{\nu}}{\tilde{\nu}+\xi_2}\left[-e^{\xi_1\hat{x}}(e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_2\hat{x}}\hat{x}\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(x)\right] \qquad \text{(E.83)}\\
&= \mathcal{K}_1 - \tilde{\nu}\mathcal{K}_2 \qquad \text{(E.84)}
\end{aligned}
$$

where $\mathcal{K}_1$ and $\mathcal{K}_2$ are defined in equations (E.81) and (E.82).

**Step 3:** This step characterizes $\mathcal{K}_1$ equal to

$$\mathcal{K}_1 = v_2(\hat{x})\lambda - \left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\mathbb{E}^{\hat{x}}[\tau]\lambda. \qquad \text{(E.85)}$$

Define $\mathcal{T}_2 = \overline{\alpha}_2\underline{\alpha}_1 - \overline{\alpha}_1\underline{\alpha}_2$. Using the definition of $\kappa_1^1(x)$, $\hat{x}$, and operating over $\mathcal{K}_1$ we have that

$$
\begin{aligned}
\mathcal{K}_1 &= \left[-e^{\xi_1\hat{x}}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_1\hat{x}}\hat{x}\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right]\dots\\
&\dots + \left[-e^{\xi_1\hat{x}}(e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_2\hat{x}}\hat{x}\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(x)\right] + \sigma^2\mathbb{E}^{\hat{x}}[\tau]\\
&= \hat{x}^2 + \hat{x}\frac{\tilde{\nu}}{\tilde{\lambda}} - \frac{\tilde{\nu}}{\tilde{\lambda}}e^{\xi_1\hat{x}}\mathcal{T}_2\left[\underline{\alpha}_1\underline{x}\overline{\alpha}_2 - \overline{\alpha}_1\overline{x}\underline{\alpha}_2\right)\left[\overline{\alpha}_2 - \underline{\alpha}_2\right] + \left(\underline{\alpha}_2\underline{x}\overline{\alpha}_2 - \overline{\alpha}_2\overline{x}\underline{\alpha}_2\right)\left[\underline{\alpha}_1 - \overline{\alpha}_1\right]]\dots\\
&\dots - \frac{\tilde{\nu}}{\tilde{\lambda}}e^{\xi_2\hat{x}}\mathcal{T}_2\left[(\overline{\alpha}_1\overline{x}\underline{\alpha}_1 - \underline{\alpha}_1\underline{x}\overline{\alpha}_1)\left[\overline{\alpha}_2 - \underline{\alpha}_2\right] + (\overline{\alpha}_2\overline{x}\underline{\alpha}_1 - \underline{\alpha}_2\underline{x}\overline{\alpha}_1)\left[\underline{\alpha}_1 - \overline{\alpha}_1\right]\right]\dots\\
&\dots - e^{\xi_1\hat{x}}\mathcal{T}_2\left[\underline{\alpha}_1\underline{x}\overline{\alpha}_2 - \overline{\alpha}_1\overline{x}\underline{\alpha}_2\right)\left[\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}\right] + \left(\underline{\alpha}_2\underline{x}\overline{\alpha}_2 - \overline{\alpha}_2\overline{x}\underline{\alpha}_2\right)\left[\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x}\right]]\dots\\
&\dots - e^{\xi_2\hat{x}}\mathcal{T}_2\left[(\overline{\alpha}_1\overline{x}\underline{\alpha}_1 - \underline{\alpha}_1\underline{x}\overline{\alpha}_1)\left[\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}\right] + (\overline{\alpha}_2\overline{x}\underline{\alpha}_1 - \underline{\alpha}_2\underline{x}\overline{\alpha}_1)\left[\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x}\right]\right] + \sigma^2\mathbb{E}^{\hat{x}}[\tau]\\
&= \hat{x}^2 + \hat{x}\frac{\tilde{\nu}}{\tilde{\lambda}} - \frac{\tilde{\nu}}{\tilde{\lambda}}e^{\xi_1\hat{x}}\mathcal{T}_2\left[\underline{x}\overline{\alpha}_2(\underline{\alpha}_1\overline{\alpha}_2 - \underline{\alpha}_2\overline{\alpha}_1) - \underline{x}\overline{\alpha}_2(\overline{\alpha}_2\overline{\alpha}_1 - \overline{\alpha}_1\underline{\alpha}_2)\right]\dots\\
&\dots - \frac{\tilde{\nu}}{\tilde{\lambda}}e^{\xi_2\hat{x}}\mathcal{T}_2\left[\overline{x}\underline{\alpha}_1(\underline{\alpha}_1\overline{\alpha}_2 - \underline{\alpha}_2\overline{\alpha}_1) - \overline{x}\underline{\alpha}_1(\overline{\alpha}_2\overline{\alpha}_1 - \overline{\alpha}_1\underline{\alpha}_2)\right] - e^{\xi_1\hat{x}}\mathcal{T}_2\left(\underline{x}^2\overline{\alpha}_2\left(\underline{\alpha}_1\overline{\alpha}_2 - \overline{\alpha}_1\underline{\alpha}_2\right) - \overline{x}^2\underline{\alpha}_2\left(\underline{\alpha}_1\overline{\alpha}_2 - \overline{\alpha}_1\underline{\alpha}_2\right)\right)\dots\\
&\dots - e^{\xi_2\hat{x}}\mathcal{T}_2\left(\underline{\alpha}_1\overline{x}^2\left(\underline{\alpha}_1\overline{\alpha}_2 - \overline{\alpha}_1\underline{\alpha}_2\right) - \overline{\alpha}_1\underline{x}^2\left(\underline{\alpha}_1\overline{\alpha}_2 - \overline{\alpha}_1\underline{\alpha}_2\right)\right) + \sigma^2\mathbb{E}^{\hat{x}}[\tau]\\
&= \hat{x}^2 + \hat{x}\frac{\tilde{\nu}}{\tilde{\lambda}} - \frac{\tilde{\nu}}{\tilde{\lambda}}\left[e^{\xi_1\hat{x}}\left[\underline{x}\overline{\alpha}_2 - \overline{x}\underline{\alpha}_2\right] + e^{\xi_2\hat{x}}\left[\overline{x}\underline{\alpha}_1 - \underline{x}\overline{\alpha}_1\right]\right] + \left[-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2\right]\right]\dots\\
&\dots + \sigma^2\frac{-e^{\sqrt{2\tilde{\lambda}}\hat{x}}(\overline{\alpha}_2 - \underline{\alpha}_2) - e^{-\sqrt{2\tilde{\lambda}}\hat{x}}(\underline{\alpha}_1 - \overline{\alpha}_1) + 1}{\lambda}. \qquad \text{(E.86)}
\end{aligned}
$$

Next we operate over the previous equations to show (E.85). We use the that the mean state is zero $\mathcal{M}_1[x] = 0$, thus

$$v_1(\hat{x}) \equiv -e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_1\kappa_1^1(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(\underline{x})\right] + \kappa_1^1(\hat{x}) = 0. \qquad \text{(E.87)}$$

Using this result we have that

$$
\begin{aligned}
\mathcal{K}_1 &= \hat{x}^2 + \hat{x}\frac{\tilde{\nu}}{\tilde{\lambda}} - \frac{\tilde{\nu}}{\tilde{\lambda}}\left[e^{\xi_1\hat{x}}\left[\underline{x}\overline{\alpha}_2 - \overline{x}\underline{\alpha}_2\right] + e^{\xi_2\hat{x}}\left[\overline{x}\underline{\alpha}_1 - \underline{x}\overline{\alpha}_1\right]\right] + \left[-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2\right]\right] + \dots \\
&\quad + \sigma^2\frac{-e^{\xi_1\hat{x}}(\overline{\alpha}_2 - \underline{\alpha}_2) - e^{\xi_2\hat{x}}(\underline{\alpha}_1 - \overline{\alpha}_1) + 1}{\lambda}\dots \\
&= \hat{x}^2 + \hat{x}\frac{\tilde{\nu}}{\tilde{\lambda}}2 + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + \frac{1}{2\tilde{\lambda}}\right] - 2\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 - \hat{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\dots \\
&\quad \dots - \left[e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\left[\underline{x}^2 + \underline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}2 + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + \frac{1}{2\tilde{\lambda}}\right]\right] - \underline{\alpha}_2\left[\overline{x}^2 + \overline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}2 + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + \frac{1}{2\tilde{\lambda}}\right]\right]\right]\right]\dots \\
&\quad \dots - \left[e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\left[\overline{x}^2 + \overline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}2 + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + \frac{1}{2\tilde{\lambda}}\right]\right] - \overline{\alpha}_1\left[\underline{x}^2 + \underline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}2 + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + \frac{1}{2\tilde{\lambda}}\right]\right]\right]\right]\dots \\
&\quad \dots + \left[e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\left[\underline{x}\frac{\tilde{\nu}}{\tilde{\lambda}} + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\right]\right] - \underline{\alpha}_2\left[\overline{x}\frac{\tilde{\nu}}{\tilde{\lambda}} + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\right]\right]\right]\right]\dots \\
&\quad \dots + \left[e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\left[\overline{x}\frac{\tilde{\nu}}{\tilde{\lambda}} + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\right]\right] - \overline{\alpha}_1\left[\underline{x}\frac{\tilde{\nu}}{\tilde{\lambda}} + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\right]\right]\right]\right] \\
&= \kappa_2^2(\hat{x}) - e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_2^2(\underline{x}) - \underline{\alpha}_2\kappa_2^2(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_2^2(\overline{x}) - \overline{\alpha}_1\kappa_2^2(\underline{x})\right] - 2\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 - \hat{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\dots \\
&\quad \dots + \left[e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\left[\underline{x}\frac{\tilde{\nu}}{\tilde{\lambda}} + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\right]\right] - \underline{\alpha}_2\left[\overline{x}\frac{\tilde{\nu}}{\tilde{\lambda}} + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\right]\right]\right]\right]\dots \\
&\quad \dots + \left[e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\left[\overline{x}\frac{\tilde{\nu}}{\tilde{\lambda}} + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\right]\right] - \overline{\alpha}_1\left[\underline{x}\frac{\tilde{\nu}}{\tilde{\lambda}} + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\right]\right]\right]\right] \\
&= \lambda v_2(\hat{x}) - 2\frac{\tilde{\nu}}{\tilde{\lambda}}\left(-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_1\kappa_1^1(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(\underline{x})\right] + \kappa_1^1(\hat{x})\right) + \hat{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\dots \\
&\quad \dots + \left[e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\left[-\underline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\right] - \underline{\alpha}_2\left[-\overline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\right]\right]\right]\dots \\
&\quad \dots + \left[e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\left[-\overline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\right] - \overline{\alpha}_1\left[-\underline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\right]\right]\right] \\
&= \lambda v_2(\hat{x}) - 2\frac{\tilde{\nu}}{\tilde{\lambda}}v_1(\hat{x}) + \hat{x}\frac{\tilde{\nu}}{\tilde{\lambda}} + \left[e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\left[-\underline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\right] - \underline{\alpha}_2\left[-\overline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\right]\right]\right] + \left[e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\left[-\overline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\right] - \overline{\alpha}_1\left[-\underline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\right]\right]\right]\dots \\
&= \lambda v_2(\hat{x}) + \frac{\tilde{\nu}}{\tilde{\lambda}}\left(-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_1\kappa_1^1(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(\underline{x})\right] + \kappa_1^1(\hat{x})\right)\dots \\
&\quad \dots + \frac{\tilde{\nu}}{\tilde{\lambda}}\left(-\frac{\tilde{\nu}}{\tilde{\lambda}}(\kappa_0^0(\hat{x}) - e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_0^0(\underline{x}) - \underline{\alpha}_1\kappa_0^0(\overline{x})\right]) - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_0^0(\overline{x}) - \overline{\alpha}_1\kappa_0^0(\underline{x})\right]\right)\dots \\
&= \lambda v_2(\hat{x}) - \left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\mathbb{E}^{\hat{x}}[\tau]\lambda.
\end{aligned}
\tag{E.88}
$$

Therefore, we have the result (E.85).

**Step 4:** This step writes $\mathcal{K}_2$ equal to

$$
\mathcal{K}_2 = \lambda\sigma^2 h_1'(0) - \sigma^2\frac{\nu}{\lambda}\mathbb{E}^{\hat{x}}[\tau] + \mathcal{C} \tag{E.89}
$$

$$
\begin{aligned}
\mathcal{C} &= \sigma^2 e^{\xi_1\hat{x}}\left[\frac{d\overline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi) - \frac{d\underline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi)\right]_{\varphi=0} + \sigma^2 e^{\xi_2\hat{x}}\left[\frac{d\underline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi) - \frac{d\overline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi)\right]_{\varphi=0}\dots \\
&\quad + \frac{\mathcal{T}_2}{\tilde{\nu} - \xi_1}\left[-e^{\xi_1\hat{x}}(\underline{\alpha}_1\underline{x}\overline{\alpha}_2 - \overline{\alpha}_1\underline{\alpha}_2\overline{x}) - e^{\xi_2\hat{x}}\left(\overline{\alpha}_1\overline{x}\underline{\alpha}_1 - \underline{\alpha}_1\underline{x}\overline{\alpha}_1\right)\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right]\dots \\
&\quad + \frac{\mathcal{T}_2}{\tilde{\nu} - \xi_2}\left[-e^{\xi_1\hat{x}}(\underline{\alpha}_2\underline{x}\overline{\alpha}_2 - \overline{\alpha}_2\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(\overline{\alpha}_2\overline{x}\underline{\alpha}_1 - \underline{\alpha}_2\underline{x}\overline{\alpha}_1\right)\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(x)\right],
\end{aligned}
\tag{E.90}
$$

and in the next step we show that (E.90) is equal to zero.

First, we characterize $h_1'(0)$. Given

$$h_1(\varphi) = \frac{-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\kappa_1^1(\underline{x},\varphi) - \underline{\alpha}_2(\varphi)\kappa_1^1(\overline{x},\varphi)\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\kappa_1^1(\overline{x},\varphi) - \overline{\alpha}_1(\varphi)\kappa_1^1(\underline{x},\varphi)\right] + \overline{\kappa}_1^1(\hat{x},\varphi)}{\lambda - \varphi}, \tag{E.91}$$

and that

$$\frac{de^{\xi_1(\varphi)\hat{x}}}{d\varphi} = -\frac{e^{\xi_1(\varphi)\hat{x}}\hat{x}}{\sigma^2(\tilde{\nu} + \xi_1(\varphi))} \tag{E.92}$$

$$\frac{de^{\xi_2(\varphi)\hat{x}}}{d\varphi} = -\frac{e^{\xi_2(\varphi)\hat{x}}\hat{x}}{\sigma^2(\tilde{\nu} + \xi_2(\varphi))} \tag{E.93}$$

$$\frac{d\kappa_1^1(x,\varphi)}{d\varphi} = \frac{\nu}{(\lambda-\varphi)^2} \tag{E.94}$$

we have that

$$\begin{aligned}
h_1'(\varphi) = {} & \frac{-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\kappa_1^1(\underline{x},\varphi) - \underline{\alpha}_2(\varphi)\kappa_1^1(\overline{x},\varphi)\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\kappa_1^1(\overline{x},\varphi) - \overline{\alpha}_1(\varphi)\kappa_1^1(\underline{x},\varphi)\right] + \overline{\kappa}_1^1(\hat{x},\varphi)}{(\lambda-\varphi)^2}\dots \\
& \dots - \frac{-e^{\xi_1(\varphi)\hat{x}}\hat{x}\frac{1}{\sigma^2(\tilde{\nu}+\xi_1(\varphi))}\left[\overline{\alpha}_2(\varphi)\kappa_1^1(\underline{x},\varphi) - \underline{\alpha}_2(\varphi)\kappa_1^1(\overline{x},\varphi)\right] - e^{\xi_2(\varphi)\hat{x}}\hat{x}\frac{1}{\sigma^2(\tilde{\nu}+\xi_2(\varphi))}\left[\underline{\alpha}_1(\varphi)\kappa_1^1(\overline{x},\varphi) - \overline{\alpha}_1(\varphi)\kappa_1^1(\underline{x},\varphi)\right]}{(\lambda-\varphi)}\dots \\
& \dots + \frac{-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\frac{d\kappa_1^1(\underline{x},\varphi)}{d\varphi} - \underline{\alpha}_2(\varphi)\frac{d\kappa_1^1(\overline{x},\varphi)}{d\varphi}\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\frac{d\kappa_1^1(\overline{x},\varphi)}{d\varphi} - \overline{\alpha}_1(\varphi)\frac{d\kappa_1^1(\underline{x},\varphi)}{d\varphi}\right] + \frac{d\overline{\kappa}_1^1(\hat{x})}{d\varphi}}{\lambda-\varphi}\dots \\
& \dots + \frac{-e^{\xi_1(\varphi)\hat{x}}\left[\frac{d\overline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi) - \frac{d\underline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi)\right] - e^{\xi_2\hat{x}}\left[\frac{d\underline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi) - \frac{d\overline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi)\right]}{\lambda-\varphi} \\
= {} & \frac{-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\kappa_1^1(\underline{x},\varphi) - \underline{\alpha}_2(\varphi)\kappa_1^1(\overline{x},\varphi)\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\kappa_1^1(\overline{x},\varphi) - \overline{\alpha}_1(\varphi)\kappa_1^1(\underline{x},\varphi)\right] + \overline{\kappa}_1^1(\hat{x})}{(\lambda-\varphi)^2}\dots \\
& \dots - \frac{-e^{\xi_1(\varphi)\hat{x}}\hat{x}\frac{1}{\sigma^2(\tilde{\nu}+\xi_1(\varphi))}\left[\overline{\alpha}_2(\varphi)\kappa_1^1(\underline{x},\varphi) - \underline{\alpha}_2(\varphi)\kappa_1^1(\overline{x},\varphi)\right] - e^{\xi_2\hat{x}}\hat{x}\frac{1}{\sigma^2(\tilde{\nu}+\xi_2(\varphi))}\left[\underline{\alpha}_1(\varphi)\kappa_1^1(\overline{x},\varphi) - \overline{\alpha}_1(\varphi)\kappa_1^1(\underline{x},\varphi)\right]}{(\lambda-\varphi)}\dots \\
& \dots + \frac{\nu}{(\lambda-\varphi)^2}\frac{-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi) - \underline{\alpha}_2(\varphi)\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi) - \overline{\alpha}_1(\varphi)\right] + 1}{\lambda-\varphi}\dots \\
& \dots - \frac{e^{\xi_1(\varphi)\hat{x}}\left[\frac{d\overline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi) - \frac{d\underline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi)\right]}{\lambda-\varphi} - \frac{e^{\xi_2\hat{x}}\left[\frac{d\underline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi) - \frac{d\overline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi)\right]}{\lambda-\varphi}.
\end{aligned} \tag{E.95}$$

Evaluating equations (E.95) at zero

$$\begin{aligned}
& = h_1'(0) \\
& = \frac{v_1(\hat{x})}{\lambda} - \hat{x}\left(\frac{\frac{-e^{\xi_1\hat{x}}}{\sigma^2(\tilde{\nu}+\xi_1)}\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right] - \frac{e^{\xi_2\hat{x}}\hat{x}}{\sigma^2(\tilde{\nu}+\xi_2)}\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(\underline{x})\right]}{\lambda}\right)\dots \\
& \dots - \frac{\nu}{\lambda^2}\frac{-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2 - \underline{\alpha}_2\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1 - \overline{\alpha}_1\right] + 1}{\lambda}\dots \\
& \dots + \left.\frac{e^{\xi_1(\varphi)\hat{x}}\left[\frac{d\overline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi) - \frac{d\underline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi)\right]}{\lambda-\varphi}\right|_{\varphi=0} - \left.\frac{e^{\xi_2\hat{x}}\left[\frac{d\underline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi) - \frac{d\overline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi)\right]}{\lambda-\varphi}\right|_{\varphi=0} \\
& = -\hat{x}\left(\frac{\frac{-e^{\xi_1\hat{x}}}{\sigma^2(\tilde{\nu}+\xi_1)}\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right] - \frac{e^{\xi_2\hat{x}}\hat{x}}{\sigma^2(\tilde{\nu}+\xi_2)}\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(\underline{x})\right]}{\lambda}\right) + \frac{\nu}{(\lambda)^2}\mathbb{E}^{\hat{x}}[\tau]\dots \\
& \dots - \left.\frac{e^{\xi_1(\varphi)\hat{x}}\left[\frac{d\overline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi) - \frac{d\underline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi)\right]}{\lambda-\varphi}\right|_{\varphi=0} - \left.\frac{e^{\xi_2\hat{x}}\left[\frac{d\underline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi) - \frac{d\overline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi)\right]}{\lambda-\varphi}\right|_{\varphi=0}.
\end{aligned} \tag{E.96}$$

Therefore we have

$$
\begin{aligned}
&= h_1'(0)\lambda\sigma^2\\
&= -\hat{x}\left[-\frac{e^{\xi_1\hat{x}}}{(\tilde{\nu}+\xi_1)}\left[\overline{\alpha}_2\kappa_1^1(\underline{x})-\underline{\alpha}_2\kappa_1^1(\overline{x})\right]-\frac{e^{\xi_2\hat{x}}\hat{x}}{(\tilde{\nu}+\xi_2)}\left[\underline{\alpha}_1\kappa_1^1(\overline{x})-\overline{\alpha}_1\kappa_1^1(\underline{x})\right]\right]+\frac{\nu}{\lambda}\sigma^2\mathbb{E}^{\hat{x}}[\tau]\dots\\
&-\sigma^2\ e^{\xi_1(\varphi)\hat{x}}\left[\frac{d\overline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi)-\frac{d\underline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi)\right]\Bigg|_{\varphi=0}-\sigma^2\ e^{\xi_2\hat{x}}\left[\frac{d\underline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi)-\frac{d\overline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi)\right]\Bigg|_{\varphi=0}\dots\\
&\dots-\frac{\mathcal{T}_2}{\tilde{\nu}+\xi_1}\left[-e^{\xi_1\hat{x}}(\underline{\alpha}_1\underline{x}\overline{\alpha}_2-\overline{\alpha}_1\underline{\alpha}_2\overline{x})-e^{\xi_2\hat{x}}\left(\overline{\alpha}_1\overline{x}\underline{\alpha}_1-\underline{\alpha}_1\underline{x}\overline{\alpha}_1\right)\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x})-\underline{\alpha}_2\kappa_1^1(\overline{x})\right]\dots\\
&\dots-\frac{\mathcal{T}_2}{\tilde{\nu}+\xi_2}\left[-e^{\xi_1\hat{x}}(\underline{\alpha}_2\underline{x}\overline{\alpha}_2-\overline{\alpha}_2\overline{x}\underline{\alpha}_2)-e^{\xi_2\hat{x}}\left(\overline{\alpha}_2\overline{x}\underline{\alpha}_1-\underline{\alpha}_2\underline{x}\overline{\alpha}_1\right)\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x})-\overline{\alpha}_1\kappa_1^1(x)\right]\dots\\
&\dots+\frac{\mathcal{T}_2}{\tilde{\nu}+\xi_1}\left[-e^{\xi_1\hat{x}}(\underline{\alpha}_1\underline{x}\overline{\alpha}_2-\overline{\alpha}_1\underline{\alpha}_2\overline{x})-e^{\xi_2\hat{x}}\left(\overline{\alpha}_1\overline{x}\underline{\alpha}_1-\underline{\alpha}_1\underline{x}\overline{\alpha}_1\right)\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x})-\underline{\alpha}_2\kappa_1^1(\overline{x})\right]\dots\\
&\dots+\frac{\mathcal{T}_2}{\tilde{\nu}\xi_2}\left[-e^{\xi_1\hat{x}}(\underline{\alpha}_2\underline{x}\overline{\alpha}_2-\overline{\alpha}_2\overline{x}\underline{\alpha}_2)-e^{\xi_2\hat{x}}\left(\overline{\alpha}_2\overline{x}\underline{\alpha}_1-\underline{\alpha}_2\underline{x}\overline{\alpha}_1\right)\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x})-\overline{\alpha}_1\kappa_1^1(x)\right]\\
&= -\hat{x}\left[-\frac{e^{\xi_1\hat{x}}}{\tilde{\nu}+\xi_1}\left[\overline{\alpha}_2\kappa_1^1(\underline{x})-\underline{\alpha}_2\kappa_1^1(\overline{x})\right]-\frac{e^{\xi_2\hat{x}}\hat{x}}{\tilde{\nu}+\xi_2}\left[\underline{\alpha}_1\kappa_1^1(\overline{x})-\overline{\alpha}_1\kappa_1^1(\underline{x})\right]\right]+\frac{\nu}{\lambda}\sigma^2\mathbb{E}^{\hat{x}}[\tau]-\mathcal{C}\dots\\
&\dots+\frac{\mathcal{T}_2}{\tilde{\nu}+\xi_1}\left[-e^{\xi_1\hat{x}}(\underline{\alpha}_1\underline{x}\overline{\alpha}_2-\overline{\alpha}_1\underline{\alpha}_2\overline{x})-e^{\xi_2\hat{x}}\left(\overline{\alpha}_1\overline{x}\underline{\alpha}_1-\underline{\alpha}_1\underline{x}\overline{\alpha}_1\right)\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x})-\underline{\alpha}_2\kappa_1^1(\overline{x})\right]\dots\\
&\dots+\frac{\mathcal{T}_2}{\tilde{\nu}+\xi_2}\left[-e^{\xi_1\hat{x}}(\underline{\alpha}_2\underline{x}\overline{\alpha}_2-\overline{\alpha}_2\overline{x}\underline{\alpha}_2)-e^{\xi_2\hat{x}}\left(\overline{\alpha}_2\overline{x}\underline{\alpha}_1-\underline{\alpha}_2\underline{x}\overline{\alpha}_1\right)\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x})-\overline{\alpha}_1\kappa_1^1(x)\right]
\end{aligned}
\tag{E.97}
$$

Using the definition of $\mathcal{K}_2$ in equation (E.82) and equation (E.97), we have equation (E.89). Using equations (E.80), (E.89) and (E.89) , we have that

$$= \sigma^2\lambda\mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t)dt\right] \tag{E.98}$$

$$= \mathcal{K}_1-\tilde{\nu}\mathcal{K}_2 \tag{E.99}$$

$$= (v_2(\hat{x})\lambda-\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\mathbb{E}^{\hat{x}}[\tau]\lambda)-\tilde{\nu}(\lambda\sigma^2h_1'(0)-\sigma^2\frac{\nu}{\lambda}\mathbb{E}^{\hat{x}}[\tau]+\mathcal{C}) \tag{E.100}$$

$$= v_2(\hat{x})\lambda-\nu h_1'(0)-\tilde{\nu}\mathcal{C} \tag{E.101}$$

**Step 5:** This step shows that $\mathcal{C}=0$, where $\mathcal{C}$ is defined as

$$
\begin{aligned}
\mathcal{C} &= \sigma^2e^{\xi_1\hat{x}}\left[\frac{d\overline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi)-\frac{d\underline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi)\right]_{\varphi=0}+\sigma^2e^{\xi_2\hat{x}}\left[\frac{d\underline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi)-\frac{d\overline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi)\right]_{\varphi=0}\dots\\
&\dots-\frac{\mathcal{T}_2}{\tilde{\nu}+\xi_1}\left[-e^{\xi_1\hat{x}}(\underline{\alpha}_1\underline{x}\overline{\alpha}_2-\overline{\alpha}_1\underline{\alpha}_2\overline{x})-e^{\xi_2\hat{x}}\left(\overline{\alpha}_1\overline{x}\underline{\alpha}_1-\underline{\alpha}_1\underline{x}\overline{\alpha}_1\right)\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x})-\underline{\alpha}_2\kappa_1^1(\overline{x})\right]\dots\\
&\dots-\frac{\mathcal{T}_2}{\tilde{\nu}+\xi_2}\left[-e^{\xi_1\hat{x}}(\underline{\alpha}_2\underline{x}\overline{\alpha}_2-\overline{\alpha}_2\overline{x}\underline{\alpha}_2)-e^{\xi_2\hat{x}}\left(\overline{\alpha}_2\overline{x}\underline{\alpha}_1-\underline{\alpha}_2\underline{x}\overline{\alpha}_1\right)\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x})-\overline{\alpha}_1\kappa_1^1(x)\right].
\end{aligned}
\tag{E.102}
$$

First, let us the derivative of the $\underline{\alpha}_i$ and $\overline{\alpha}_i$

$$\sigma^2\frac{d\overline{\alpha}_2(\varphi)}{d\varphi}=\overline{\alpha}_2(\varphi)\mathcal{T}_2\left[\frac{\underline{\alpha}_1(\varphi)\overline{\alpha}_2(\varphi)\underline{x}-\overline{\alpha}_1(\varphi)\underline{\alpha}_2(\varphi)\overline{x}}{\tilde{\nu}+\xi_1}+\frac{\underline{\alpha}_1(\varphi)\overline{\alpha}_2(\varphi)\overline{x}-\overline{\alpha}_1(\varphi)\underline{\alpha}_2(\varphi)\underline{x}}{\tilde{\nu}+\xi_2}\right]-\frac{\overline{\alpha}_2(\varphi)\overline{x}}{\tilde{\nu}+\xi_2} \tag{E.103}$$

$$\sigma^2\frac{d\overline{\alpha}_1(\varphi)}{d\varphi}=\overline{\alpha}_1(\varphi)\mathcal{T}_2\left[\frac{\underline{\alpha}_1(\varphi)\overline{\alpha}_2(\varphi)\underline{x}-\overline{\alpha}_1(\varphi)\underline{\alpha}_2(\varphi)\overline{x}}{\tilde{\nu}+\xi_1}+\frac{\underline{\alpha}_1(\varphi)\overline{\alpha}_2(\varphi)\overline{x}-\overline{\alpha}_1(\varphi)\underline{\alpha}_2(\varphi)\underline{x}}{\tilde{\nu}+\xi_2}\right]-\frac{\overline{\alpha}_1(\varphi)\overline{x}}{\tilde{\nu}+\xi_1} \tag{E.104}$$

$$\sigma^2\frac{d\underline{\alpha}_2(\varphi)}{d\varphi}=\underline{\alpha}_2(\varphi)\mathcal{T}_2\left[\frac{\underline{\alpha}_1(\varphi)\overline{\alpha}_2(\varphi)\underline{x}-\overline{\alpha}_1(\varphi)\underline{\alpha}_2(\varphi)\overline{x}}{\tilde{\nu}+\xi_1}+\frac{\underline{\alpha}_1(\varphi)\overline{\alpha}_2(\varphi)\overline{x}-\overline{\alpha}_1(\varphi)\underline{\alpha}_2(\varphi)\underline{x}}{\tilde{\nu}+\xi_2}\right]-\frac{\underline{\alpha}_2(\varphi)\underline{x}}{\tilde{\nu}+\xi_2} \tag{E.105}$$

$$\sigma^2\frac{d\underline{\alpha}_1(\varphi)}{d\varphi}=\underline{\alpha}_1(\varphi)\mathcal{T}_2\left[\frac{\underline{\alpha}_1(\varphi)\overline{\alpha}_2(\varphi)\underline{x}-\overline{\alpha}_1(\varphi)\underline{\alpha}_2(\varphi)\overline{x}}{\tilde{\nu}+\xi_1}+\frac{\underline{\alpha}_1(\varphi)\overline{\alpha}_2(\varphi)\overline{x}-\overline{\alpha}_1(\varphi)\underline{\alpha}_2(\varphi)\underline{x}}{\tilde{\nu}+\xi_2}\right]-\frac{\underline{\alpha}_1(\varphi)\underline{x}}{\tilde{\nu}+\xi_1}. \tag{E.106}$$

Using the previous derivatives in the first part of equation (E.102) we have that

$$
\begin{aligned}
&= \sigma^2 e^{\xi_1 \hat{x}} \left[ \frac{d\overline{\alpha}_2(\varphi)}{d\varphi} \kappa_1^1(\underline{x},\varphi) - \frac{d\underline{\alpha}_2(\varphi)}{d\varphi} \kappa_1^1(\overline{x},\varphi) \right]_{\varphi=0} + \sigma^2 e^{\xi_2 \hat{x}} \left[ \frac{d\underline{\alpha}_1(\varphi)}{d\varphi} \kappa_1^1(\overline{x},\varphi) - \frac{d\overline{\alpha}_1(\varphi)}{d\varphi} \kappa_1^1(\underline{x},\varphi) \right]_{\varphi=0} \\
&= \frac{\mathcal{T}_2}{\tilde{\nu}+\xi_1} e^{\xi_1 \hat{x}} \left[ \left(\underline{\alpha}_1 \overline{\alpha}_2 \underline{x} - \overline{\alpha}_1 \underline{\alpha}_2 \overline{x}\right) \overline{\alpha}_2 \kappa_1^1(\underline{x}) - \left(\underline{\alpha}_1 \overline{\alpha}_2 \underline{x} - \overline{\alpha}_1 \underline{\alpha}_2 \overline{x}\right) \underline{\alpha}_2 \kappa_1^1(\overline{x}) \right] \dots \\
&\dots + \frac{\mathcal{T}_2}{\tilde{\nu}+\xi_1} e^{\xi_2 \hat{x}} \left[ \left(\underline{\alpha}_1 \overline{\alpha}_2 \underline{\alpha}_1 \underline{x} - \overline{\alpha}_1 \underline{\alpha}_2 \overline{x} \underline{\alpha}_1 - \underline{\alpha}_1 \underline{x}/\mathcal{T}_2\right) \kappa_1^1(\overline{x}) - \left(\underline{\alpha}_1 \overline{\alpha}_2 \underline{x} \overline{\alpha}_1 - \overline{\alpha}_1 \underline{\alpha}_2 \overline{x} \overline{\alpha}_1 - \overline{\alpha}_1 \overline{x}/\mathcal{T}_2\right) \kappa_1^1(\underline{x}) \right] \dots \\
&\dots + \frac{\mathcal{T}_2}{\tilde{\nu}+\xi_2} \; e^{\xi_1 \hat{x}} \left[ \left(\underline{\alpha}_1 \overline{\alpha}_2 \overline{x} - \overline{\alpha}_1 \underline{\alpha}_2 \underline{x} + \overline{\alpha}_2 \overline{x}/\mathcal{T}_2\right) \kappa_1^1(\underline{x}) - \left(\underline{\alpha}_1 \overline{\alpha}_2 \overline{x} - \overline{\alpha}_1 \underline{\alpha}_2 \underline{x} + \underline{\alpha}_2 \underline{x}/\mathcal{T}_2\right) \kappa_1^1(\overline{x}) \right] \dots \\
&\dots + \frac{\mathcal{T}_2}{\tilde{\nu}+\xi_2} \left[ e^{\xi_2 \hat{x}} \left[ \left(\overline{\alpha}_2 \overline{x} \underline{\alpha}_1 - \underline{\alpha}_2 \underline{x} \overline{\alpha}_1\right) \underline{\alpha}_1 \kappa_1^1(\overline{x}) - \left(\overline{\alpha}_2 \overline{x} \underline{\alpha}_1 - \underline{\alpha}_2 \underline{x} \overline{\alpha}_1\right) \overline{\alpha}_1 \kappa_1^1(\underline{x}) \right] \right] \dots \\
&= \frac{\mathcal{T}_2}{\tilde{\nu}+\xi_1} e^{\xi_1 \hat{x}} \left(\underline{\alpha}_1 \overline{\alpha}_2 \underline{x} - \overline{\alpha}_1 \underline{\alpha}_2 \overline{x}\right) \left[ \overline{\alpha}_2 \kappa_1^1(\underline{x}) - \underline{\alpha}_2 \kappa_1^1(\overline{x}) \right] - \frac{\mathcal{T}_2}{\tilde{\nu}+\xi_1} e^{\xi_2 \hat{x}} \left(\overline{\alpha}_1 \overline{x} \underline{\alpha}_1 - \underline{\alpha}_1 \underline{x} \overline{\alpha}_1\right) \left[ \overline{\alpha}_2 \kappa_1^1(\underline{x}) - \underline{\alpha}_2 \kappa_1^1(\overline{x}) \right] \dots \\
&\dots + \frac{\mathcal{T}_2}{\tilde{\nu}+\xi_2} e^{\xi_1 \hat{x}} \left(\underline{\alpha}_2 \underline{x} \overline{\alpha}_2 - \overline{\alpha}_2 \overline{x} \underline{\alpha}_2\right) \left[ \underline{\alpha}_1 \kappa_1^1(\overline{x}) - \overline{\alpha}_1 \kappa_1^1(\underline{x}) \right] - \frac{\mathcal{T}_2}{\tilde{\nu}+\xi_2} e^{\xi_2 \hat{x}} \left(\overline{\alpha}_2 \overline{x} \underline{\alpha}_1 - \underline{\alpha}_2 \underline{x} \overline{\alpha}_1\right) \left[ \underline{\alpha}_1 \kappa_1^1(\overline{x}) - \overline{\alpha}_1 \kappa_1^1(\underline{x}) \right].
\end{aligned}
\tag{E.107}
$$

Using the previous equation we have the equality. □

## E.4 Proof of Example 2

**Proposition E.5.** *Assume a random Ss model with $\lambda = 0$ and $\sigma^2 = 0$, then*

$$\Theta_1 = \theta_{1,0}\mathcal{M}_0[x] = -T/2. \tag{E.108}$$

*Proof.* Assume the Ss model with lower Ss band $\bar{x}$ and upper Ss band $\hat{x}$. Define the following objects:

$$g_m(x) = \mathbb{E}^{\hat{x}+x}\left[(\hat{x}-\Delta x)^m\right] - \mathbb{E}^{\hat{x}}\left[(\hat{x}-\Delta x + x)^m\right] \tag{E.109}$$

and

$$\theta_{1,j} \equiv \frac{1}{\nu}\sum_{k\geq j}^{\infty} \frac{\hat{x}^{k-j}}{k!j!}\left[\frac{d^{k+1}g_2(0)/2}{dx^{k+1}} - \frac{d^k g_1(0)}{dx^k}\right], \tag{E.110}$$

then we need to show that

$$\Theta_1 = \sum_{j=0}^{\infty}\theta_{1,j}\mathcal{M}_[x] = -\frac{T}{2}. \tag{E.111}$$

where $T$ is the time between adjustments. Applying the definition of $g_m(x)$, we have that

$$g_m(x) = \bar{x}(T)^m - (\bar{x}(T)+x)^m, \tag{E.112}$$

Note that $\nu T + \hat{x} = \bar{x}$. Then, by Proposition **??**, we have that $\hat{x} = \frac{\mathbb{E}[\Delta x]}{2} = \frac{-\nu T}{2}$ and $\bar{x}(T) = \frac{\nu T}{2}$. Therefore,

$$\frac{d^k g_m(x)}{dx^k} = -\frac{m!}{(m-k)!}\left(\left(\frac{\nu T}{2}\right)+x\right)^{m-k}\mathbb{1}_{\{m-k\geq 0\}} \quad \forall k\geq 1 \tag{E.113}$$

Thus, for all $k>1$ we have that

$$\frac{d^1 g_2(0)/2}{dx^1} - \frac{d^0 g_1(0)}{dx^0} = -2\left(\frac{\nu T}{2}\right)/2 - \left[\left(\frac{\nu T}{2}\right) - \left(\frac{-\nu T}{2}\right)\right] = \frac{-\nu T}{2} \tag{E.114}$$

$$\frac{d^2 g_2(0)/2}{dx^2} - \frac{d^1 g_1(0)}{dx^1} = -2/2 - (-1) = 0 \tag{E.115}$$

$$\frac{d^{k+1} g_2(0)/2}{dx^{k+1}} - \frac{d^k g_1(0)}{dx^k} = 0 - 0 = 0 \tag{E.116}$$

(E.117)

Thus,

$$\theta_{1,0} = \frac{\hat{x}^0\left(\left(\frac{-\nu T}{2}\right)\right)}{\nu} + \frac{1}{\nu}\sum_{k\geq 1}^{\infty}\frac{\hat{x}^{k-j}}{k!j!}0 = -T/2, \qquad \textit{and} \qquad \theta_{1,j} = \frac{1}{\nu}\sum_{k\geq 1}^{\infty}\frac{\hat{x}^{k-j}}{k!j!}0 = 0.\ \forall j>0, \tag{E.118}$$

Since $\mathcal{M}_0[x] = 1$, we obtain (E.111). □

# F Application of the three properties

This section solves numerically and analytically two models: the random Ss model presented in **??** with $H(\xi) = 1$ for all $\xi \in [0, \overline{k}]$ and the Calvo-Taylor model. We use these model to obtain intuitions of the theory develop in this paper.

## F.1 Lumpy Investment in the Random Ss Model

Before we compute all the endogenous objects in this model, we define a set of function and parameters

$$\tilde{\nu} = -\frac{\nu}{\sigma^2} \; ; \; \nu = -\psi + \mu \tag{F.1}$$

$$\tilde{\lambda} = \frac{\lambda}{\sigma^2} \; ; \; \tilde{\lambda}(\varphi) = \frac{\lambda - \varphi}{\sigma^2} \tag{F.2}$$

$$\xi_1 = -\tilde{\nu} - \sqrt{\tilde{\mu}^2 + 2\tilde{\lambda}} \; ; \; \xi_1(\varphi) = -\tilde{\nu} - \sqrt{\tilde{\mu}^2 + 2\tilde{\lambda}(\varphi)} \tag{F.3}$$

$$\xi_2 = -\tilde{\nu} + \sqrt{\tilde{\mu}^2 + 2\tilde{\lambda}} \; ; \; \xi_2(\varphi) = -\tilde{\nu} + \sqrt{\tilde{\mu}^2 + 2\tilde{\lambda}(\varphi)} \tag{F.4}$$

$$\overline{\alpha}_1 = \frac{e^{\xi_1 \overline{x}}}{e^{\xi_1 \underline{x} + \xi_2 \overline{x}} - e^{\xi_2 \underline{x} + \xi_1 \overline{x}}} \; ; \; \overline{\alpha}_1(\varphi) = \frac{e^{\xi_1(\varphi) \overline{x}}}{e^{\xi_1(\varphi) \underline{x} + \xi_2(\varphi) \overline{x}} - e^{\xi_2(\varphi) \underline{x} + \xi_1(\varphi) \overline{x}}} \tag{F.5}$$

$$\overline{\alpha}_2 = \frac{e^{\xi_2 \overline{x}}}{e^{\xi_1 \underline{x} + \xi_2 \overline{x}} - e^{\xi_2 \underline{x} + \xi_1 \overline{x}}} \; ; \; \overline{\alpha}_2(\varphi) = \frac{e^{\xi_2(\varphi) \overline{x}}}{e^{\xi_1(\varphi) \underline{x} + \xi_2(\varphi) \overline{x}} - e^{\xi_2(\varphi) \underline{x} + \xi_1(\varphi) \overline{x}}} \tag{F.6}$$

$$\underline{\alpha}_1 = \frac{e^{\xi_1 \underline{x}}}{e^{\xi_1 \underline{x} + \xi_2 \overline{x}} - e^{\xi_2 \underline{x} + \xi_1 \overline{x}}} \; ; \; \underline{\alpha}_1(\varphi) = \frac{e^{\xi_1(\varphi) \underline{x}}}{e^{\xi_1(\varphi) \underline{x} + \xi_2(\varphi) \overline{x}} - e^{\xi_2(\varphi) \underline{x} + \xi_1(\varphi) \overline{x}}} \tag{F.7}$$

$$\underline{\alpha}_2 = \frac{e^{\xi_2 \underline{x}}}{e^{\xi_1 \underline{x} + \xi_2 \overline{x}} - e^{\xi_2 \underline{x} + \xi_1 \overline{x}}} \; ; \; \underline{\alpha}_2(\varphi) = \frac{e^{\xi_2(\varphi) \underline{x}}}{e^{\xi_1(\varphi) \underline{x} + \xi_2(\varphi) \overline{x}} - e^{\xi_2(\varphi) \underline{x} + \xi_1(\varphi) \overline{x}}} \tag{F.8}$$

$$\kappa_j^m(x) = \sum_{i=0}^{j} (x)^m \frac{m!}{i!} \left[ \frac{\xi_1 - \xi_1 \xi_2 \frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1 - \xi_2} (\xi_1 x)^{i-m} + \frac{\xi_2 - \xi_1 \xi_2 \frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2 - \xi_1} (\xi_2 x)^{i-m} \right] \tag{F.9}$$

$$\overline{\kappa}_j^m(x) = \sum_{i=0}^{j} (x)^m \frac{m!}{i!} \left[ \frac{\xi_1 - \xi_1 \xi_2 \frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1 - \xi_2} (-\xi_1 x)^{i-m} + \frac{\xi_2 - \xi_1 \xi_2 \frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2 - \xi_1} (-\xi_2 x)^{i-m} \right] \tag{F.10}$$

$$\kappa_j^m(x, \varphi) = \sum_{i=0}^{j} (x)^m \frac{m!}{i!} \left[ \frac{\xi_1(\varphi) + \xi_1(\varphi)\xi_2(\varphi) \frac{\tilde{\mu}}{\tilde{\lambda}(\varphi)}}{\xi_1(\varphi) - \xi_2(\varphi)} (\xi_1(\varphi) x)^{i-m} + \frac{\xi_2(\varphi) + \xi_1(\varphi)\xi_2(\varphi) \frac{\tilde{\mu}}{\tilde{\lambda}(\varphi)}}{\xi_2(\varphi) - \xi_1(\varphi)} (\xi_2(\varphi) x)^{i-m} \right]. \tag{F.11}$$

With these objects we can characterize the policy function and the cross-sectional distribution of investment rates.

**Policy.** Following Section D, the policy function for the Ss bands and the reset state before the normalization solve the HJB, value matching, and smooth pasting conditions defined in **??**. Let us use the notation $(\underline{k}, \hat{k}, \overline{k})$ to denote the lower Ss band, the reset state and the upper Ss band before the re-normalization, respectively. Then $(\underline{k}, \hat{k}, \overline{k})$ are the solution to

$$\tilde{\rho} v(k) = e^{\alpha k} + \lambda \left( v(\hat{k}) - (e^{\hat{k}} - e^{k}) \right) + \nu v'(k) + \frac{\sigma^2}{2} v''(k) \qquad \forall k \in \left(\underline{k}, \overline{k}\right) \tag{F.12}$$

$$v(\underline{k}) - e^{\underline{k}} = v(\overline{k}) - e^{\overline{k}} = v\left(\hat{k}\right) - \kappa - e^{\hat{k}} \tag{F.13}$$

$$v'(\underline{k}) = e^{\underline{k}}, \qquad v'(\overline{k}) = e^{\overline{k}}, \qquad v'\left(\hat{k}\right) = e^{\hat{k}}. \tag{F.14}$$

with $\tilde{\rho} \equiv \rho + \lambda - \mu$.

**Proposition F.1.** *In the random Ss model the firm's policies are given by*

$$\tau = \inf\{t \geq 0 : k_t \notin [\underline{k}, \overline{k}] \text{ or } N_t \geq 1\} \tag{F.15}$$

*where the Ss bands and the reset state with* $\underline{k} < \hat{k} < \overline{k}$ *are defined implicitly as the solution to the system*

$$v'(\underline{k}|\underline{k}, \hat{k}, \overline{k}) = e^{\underline{k}} \quad v'(\hat{k}|\underline{k}, \hat{k}, \overline{k}) = e^{\hat{k}} \quad v'(\overline{k}|\underline{k}, \hat{k}, \overline{k}) = e^{\overline{k}} \tag{F.16}$$

*where* $v(x|\underline{k}, \hat{k}, \overline{k})$ *is given by*

$$v(k|\underline{k}, \hat{k}, \overline{k}) = A(\lambda_2) e^{\lambda_1 k} + A(\lambda_1) e^{\lambda_2 k} + C(\alpha) e^{\alpha k} + C(1) \lambda e^{k} + E \tag{F.17}$$

*with roots*

$$\lambda_1 = -\tilde{\nu} - \sqrt{\tilde{\nu}^2 + \frac{2\tilde{\rho}}{\sigma^2}} \quad ; \quad \lambda_2 = -\tilde{\nu} + \sqrt{\tilde{\nu}^2 + \frac{2\tilde{\rho}}{\sigma^2}} \tag{F.18}$$

*and coefficients computed as:*

$$C(\alpha) = \left(\tilde{\rho} - \alpha\tilde{\nu} - \frac{\sigma^2}{2}\alpha^2\right)^{-1} \tag{F.19}$$

$$B(k) = -\kappa - \left(e^{\hat{k}} - e^{k}\right) + C(\alpha)\left(e^{\alpha\hat{k}} - e^{\alpha k}\right) + \lambda C(1)\left(e^{\hat{k}} - e^{k}\right) \tag{F.20}$$

$$A(\lambda) = \frac{B(\underline{k})\left(e^{\lambda\overline{k}} - e^{\lambda\hat{k}}\right) - B(\overline{k})\left(e^{\lambda\underline{k}} - e^{\lambda\hat{k}}\right)}{D} \tag{F.21}$$

$$E = \frac{\lambda}{\tilde{\rho} - \lambda}\left(A_1 e^{\lambda_1\hat{k}} + A_2 e^{\lambda_2\hat{k}} + C(\alpha)e^{\alpha\hat{k}} + C(1)\lambda\left(e^{\hat{k}} - e^{\hat{k}}\right)\right) \tag{F.22}$$

$$D = \left(e^{-\lambda_1\underline{k}} - e^{-\lambda_1\hat{k}}\right)\left(e^{-\lambda_2\overline{k}} - e^{-\lambda_2\hat{k}}\right) - \left(e^{-\lambda_2\underline{k}} - e^{-\lambda_2\hat{k}}\right)\left(e^{-\lambda_1\overline{k}} - e^{-\lambda_1\hat{k}}\right) \tag{F.23}$$

*Proof.* The homogenous solution of (F.13) is given by

$$v^h(k) = A_1 e^{\lambda_1 k} + A_2 e^{\lambda_2 k}, \tag{F.24}$$

where the roots $\lambda_1$ and $\lambda_2$ are given by (F.18). Using the method of undetermined coefficients to find the non-homogenous solution, we have that

$$v(k) = A_1 e^{\lambda_1 k} + A_2 e^{\lambda_2 k} + C(\alpha)e^{\alpha k} + C(1)\lambda e^{k} + E \tag{F.25}$$

$$C(\alpha) = \left(\tilde{\rho} - \alpha\tilde{\nu} - \frac{\sigma^2}{2}\alpha^2\right)^{-1} \tag{F.26}$$

$$E = \frac{\lambda}{\tilde{\rho} - \lambda}\left(A_1 e^{\lambda_1\hat{k}} + A_2 e^{\lambda_2\hat{k}} + C(\alpha)e^{\alpha\hat{k}} + C(1)e^{\hat{k}} - e^{\hat{k}}\right). \tag{F.27}$$

The value matching conditions in (F.13) imply

$$A_1\left(e^{\lambda_1\underline{k}} - e^{\lambda_1\hat{k}}\right) + A_2\left(e^{\lambda_2\underline{k}} - e^{\lambda_2\hat{k}}\right) = -\kappa - \left(e^{\hat{k}} - e^{\underline{k}}\right) + C(\alpha)\left(e^{\alpha\hat{k}} - e^{\alpha\underline{k}}\right) + \lambda C(1)\left(e^{\hat{k}} - e^{\underline{k}}\right) =: B(\underline{k}) \tag{F.28}$$

$$A_1\left(e^{\lambda_1\overline{k}} - e^{\lambda_1\hat{k}}\right) + A_2\left(e^{\lambda_2\overline{k}} - e^{\lambda_2\hat{k}}\right) = -\kappa - \left(e^{\hat{k}} - e^{\overline{k}}\right) + C(\alpha)\left(e^{\alpha\hat{k}} - e^{\alpha\overline{k}}\right) + \lambda C(1)\left(e^{\hat{k}} - e^{\overline{k}}\right) =: B(\overline{k}), \tag{F.29}$$

and using Cramer's rule to solve this system we have that

$$A_1 = \frac{B(\underline{k})\left(e^{\lambda_2\overline{k}} - e^{\lambda_2\hat{k}}\right) - B(\overline{k})\left(e^{\lambda_2\underline{k}} - e^{\lambda_2\hat{k}}\right)}{D} \tag{F.30}$$

$$A_2 = -\frac{B(\underline{k})\left(e^{\lambda_1\overline{k}} - e^{\lambda_1\hat{k}}\right) + B(\overline{k})\left(e^{\lambda_1\underline{k}} - e^{\lambda_1\hat{k}}\right)}{D} \tag{F.31}$$

with the determinant

$$D = \left(e^{-\lambda_1\underline{k}} - e^{-\lambda_1\hat{k}}\right)\left(e^{-\lambda_2\overline{k}} - e^{-\lambda_2\hat{k}}\right) - \left(e^{-\lambda_1\overline{k}} - e^{-\lambda_1\hat{k}}\right)\left(e^{-\lambda_2\underline{k}} - e^{-\lambda_2\hat{k}}\right). \tag{F.32}$$

□

**Re-normalization** We normalize the policy function in both models to work with centralized moments. We redefine the reset state $\hat{x}$ and the Ss bands $\underline{x}, \overline{x}$ in the following way

$$(\underline{x}, \hat{x}, \overline{x}) = ((\underline{k} - \mathcal{M}_1[k], \hat{k} - \mathcal{M}_1[k], \overline{k} - \mathcal{M}_1[k])), \tag{F.33}$$

where $\mathcal{M}_1[k]$ is the average capital-gap under the firms' policies $(\underline{k}, \hat{k}, \overline{k})$. It is easy to check that under the re-normalized policy $\mathcal{M}_1[x] = 0$.

**Inputs for the verification of observability and representation.** In order to verify all the main proposition in the main text, first we compute several object:

$$\mathbb{E}^{\hat{x}}[\Delta x^m] \ ; \ \mathbb{E}^{\hat{x}}[x_\tau^m] \ ; \ \mathbb{E}^{\hat{x}}[e^{\varphi\tau}\Delta x^m] \ ; \ \mathbb{E}[x^m] \ ; \ \mathbb{E}[x^m a] \ ; \ f(x) \ ; \ \mathbb{E}[e^{\lambda_i x}] \tag{F.34}$$

We divide the computation of each object in a separate proposition to order the exposition.

**Proposition F.2.** *The following relations hold*

$$\mathbb{E}^{\hat{x}}[x_\tau^m] = -e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_{m-1}^m(\underline{x}) - \underline{\alpha}_2\kappa_{m-1}^m(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_{m-1}^m(\overline{x}) - \overline{\alpha}_1\kappa_{m-1}^m(\underline{x})\right] + \kappa_m^m(\hat{x}) \tag{F.35}$$

$$\mathbb{E}^{\hat{x}}[\Delta x^m] = -e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\overline{\kappa}_{m-1}^m(\hat{x}-\underline{x}) - \underline{\alpha}_2\overline{\kappa}_{m-1}^m(\hat{x}-\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\overline{\kappa}_{m-1}^m(\hat{x}-\overline{x}) - \overline{\alpha}_1\overline{\kappa}_{m-1}^m(\hat{x}-\underline{x})\right] + \overline{\kappa}_0^m(0) \tag{F.36}$$

*Proof.* The function $v_m(x) = \mathbb{E}^x[x_\tau^m]$ satisfies the HJB

$$0 = \nu v_m'(x) + \frac{\sigma^2}{2}v_m''(x) + \lambda\left(x^m - v_m(x)\right) \tag{F.37}$$

with the border conditions $v_m(\underline{x}) = \underline{x}^m$ and $v_m(\overline{x}) = \overline{x}^m$. The homogenous solution is given by

$$v^h(x) = A_1e^{\xi_1 x} + A_2e^{\xi_2 x}, \tag{F.38}$$

where $\xi_i$ are defined in (F.3) and (F.4).To compute the non-homogenous solution let us use a guess and verify $v^{nh}(x) = \sum_{i=0}^m a_{i,m}x^i$. Then

$$0 = \tilde{\nu}\left(\sum_{i=1}^m a_{i,m}ix^{i-1}\right) + \frac{1}{2}\left(\sum_{i=2}^m a_{i,m}i(i-1)x^{i-2}\right) + \tilde{\lambda}\left(x^m - \sum_{i=0}^m a_{i,m}x^i\right) \tag{F.39}$$

Notice that the solution consist in the following system of equations

$$a_{m,m} = 1\ ;\ a_{m-1,m} = \frac{\tilde{\nu}ma_{m,m}}{\tilde{\lambda}}\ ;\ a_{i,m} = \frac{\tilde{\nu}(i+1)a_{i+1,m} + (i+2)(i+1)a_{i+2,m}/2}{\tilde{\lambda}}. \tag{F.40}$$

The border conditions we have that $A_1$ and $A_2$ satisfy

$$A_{1,m}e^{\xi_1\underline{x}} + A_{2,m}e^{\xi_2\underline{x}} = -\sum_{i=0}^{m-1}a_{i,m}\underline{x}^i, \tag{F.41}$$

$$A_{1,m}e^{\xi_1\overline{x}} + A_{2,m}e^{\xi_2\overline{x}} = -\sum_{i=0}^{m-1}a_{i,m}\overline{x}^i. \tag{F.42}$$

and the solution to this system of equations is given by

$$A_{1,m} = \frac{\left(-\sum_{i=0}^{m-1}a_{i,m}\underline{x}^i\right)e^{\xi_2\overline{x}} - \left(-\sum_{i=0}^{m-1}a_{i,m}\overline{x}^i\right)e^{\xi_2\underline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}} \tag{F.43}$$

$$A_{2,m} = \frac{\left(-\sum_{i=0}^{m-1}a_{i,m}\overline{x}^i\right)e^{\xi_1\underline{x}} - \left(-\sum_{i=0}^{m-1}a_{i,m}\underline{x}^i\right)e^{\xi_1\overline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}}. \tag{F.44}$$

For the next verifications it is useful to simply these expressions in several ways. Note that the coefficient are an equation in difference

$$a_{i,m} = \left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)(i+1)a_{i+1,m} + \frac{(i+2)(i+1)}{2\tilde{\lambda}}a_{i+2,m}. \tag{F.45}$$

Using the guess and verify method with $a_{i,m} = \frac{c}{i!}x^i$, we have that

$$a_{i,m} = \frac{m!}{i!}\left[c_1\xi_1^{i-m} + c_2\xi_2^{i-m}\right], \tag{F.46}$$

and using the border conditions

$$a_{i,m} = \frac{m!}{i!}\left[\frac{\xi_1 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1 - \xi_2}\xi_1^{i-m} - \frac{\xi_2 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2 - \xi_1}\xi_2^{i-m}\right], \tag{F.47}$$

Putting all the results togethers we have (F.35).

The function $h_m(x) = \mathbb{E}^x[(\hat{x} - x_\tau)^m]$ satisfies the HJB

$$0 = \nu h_m'(x) + \frac{\sigma^2}{2}h_m''(x) + \lambda\left((\hat{x} - x)^m - h_m(x)\right) \tag{F.48}$$

with the border conditions $v_m(\underline{x}) = (\hat{x} - \underline{x})^m$ and $v_m(\overline{x}) = (\hat{x} - \overline{x})^m$. The homogenous solution is given by

$$v^h(x) = B_1e^{\xi_1 x} + B_2e^{\xi_2 x}. \tag{F.49}$$

To compute the non-homogenous solution let us use a guess and verify $v^{nh}(x) = \sum_{i=0}^{m} b_{i,m}(\hat{x}-x)^i$. Then

$$0 = -\tilde{\nu}\left(\sum_{i=1}^{m} b_i i(\hat{x}-x)^{i-1}\right) + \frac{1}{2}\left(\sum_{i=2}^{m} b_i i(i-1)(\hat{x}-x)^{i-2}\right) + \tilde{\lambda}\left(x^m - \sum_{i=0}^{m} b_i(\hat{x}-x)^i\right). \tag{F.50}$$

Note that the solution consist in the following system of equations

$$b_{m,m} = 1\ ;\ b_{m-1,m} = -\frac{\tilde{\nu} m b_m}{\tilde{\lambda}}\ ;\ b_{i,m} = \frac{-\tilde{\nu}(i+1)b_{i+1} + (i+2)(i+1)a_{i+2}/2}{\tilde{\lambda}}. \tag{F.51}$$

with the solution given by

$$b_{i,m} = \frac{m!}{i!}\left[\frac{\xi_1 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1-\xi_2}(-\xi_1)^{i-m} + \frac{\xi_2 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2-\xi_1}(-\xi_2)^{i-m}\right], \tag{F.52}$$

with the border conditions we have that $B_1$ and $B_2$ satisfy

$$B_{1,m}e^{\xi_1\underline{x}} + B_{2,m}e^{\xi_2\underline{x}} = -\sum_{i=0}^{m-1} b_{i,m}(\hat{x}-\underline{x})^i, \tag{F.53}$$

$$B_{1,m}e^{\xi_1\overline{x}} + B_{2,m}e^{\xi_2\overline{x}} = -\sum_{i=0}^{m-1} b_{i,m}(\hat{x}-\overline{x})^i. \tag{F.54}$$

and the solution to this system of equations is given by

$$B_{1,m} = \frac{\left(-\sum_{i=0}^{m-1} b_{i,m}(\hat{x}-\underline{x})^i\right)e^{\xi_2\overline{x}} - \left(-\sum_{i=0}^{m-1} b_{i,m}(\hat{x}-\overline{x})^i\right)e^{\xi_2\underline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}} \tag{F.55}$$

$$B_{2,m} = \frac{\left(-\sum_{i=0}^{m-1} b_{i,m}(\hat{x}-\overline{x})^i\right)e^{\xi_1\underline{x}} - \left(-\sum_{i=0}^{m-1} b_{i,m}(\hat{x}-\underline{x})^i\right)e^{\xi_1\overline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}} \tag{F.56}$$

Putting all the results togethers we have (F.36). □

**Proposition F.3.** *The moments of the ergodic distribution are given by*

$$\mathcal{M}_m[x] = \frac{-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_m^m(\underline{x}) - \underline{\alpha}_2\kappa_m^m(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_m^m(\overline{x}) - \overline{\alpha}_1\kappa_m^m(\underline{x})\right] + \kappa_m^m(\hat{x})}{\lambda\mathbb{E}^{\hat{x}}[\tau]} \tag{F.57}$$

*Proof.* Remember that $\mathcal{M}_m[x] = \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^m dt\right]}{\mathbb{E}^{\hat{x}}[\tau]}$. Define $h_m(x) = \mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^m dt\right]$. Then function $h_m(x)$ satisfies the HJB

$$0 = x^m + \nu h_m'(x) + \frac{\sigma^2}{2}h_m''(x) + \lambda\left(-h_m(x)\right) \tag{F.58}$$

with the border conditions $h_m(\underline{x}) = 0$ and $h_m(\overline{x}) = 0$. The homogenous solution is given by

$$h^h(x) = B_{1,m}e^{\xi_1 x} + B_{2,m}e^{\xi_2 x}. \tag{F.59}$$

For the non-homogenous solution we use a guess and verify $v^{nh}(x) = \sum_{i=0}^{m} b_{i,m}x^i$. Then

$$0 = \frac{x^m}{\sigma^2} + \tilde{\nu}\left(\sum_{i=1}^{m} b_{i,m} i x^{i-1}\right) + \frac{1}{2}\left(\sum_{i=2}^{m} b_{i,m} i(i-1)x^{i-2}\right) + \tilde{\lambda}\left(-\sum_{i=0}^{m} b_{i,m}x^i\right). \tag{F.60}$$

The solution consist in the following system of equations

$$b_{m,m} = \frac{1}{\lambda}\ ;\ b_{m-1,m} = \frac{\tilde{\nu} m b_{m,m}}{\tilde{\lambda}}\ ;\ b_{i,m} = \frac{\tilde{\nu}(i+1)b_{i+1,m} + (i+2)(i+1)b_{i+2,m}/2}{\tilde{\lambda}}. \tag{F.61}$$

The solution of the previous differential equation is given by

$$b_{i,m} = \frac{1}{\lambda}\frac{m!}{i!}\left[\frac{\xi_1 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1-\xi_2}\xi_1^{i-m} + \frac{\xi_2 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2-\xi_1}\xi_2^{i-m}\right], \tag{F.62}$$

With the border conditions we have that $A_1$ and $A_2$ satisfy

$$B_{1,m}e^{\xi_1 \underline{x}} + B_{2,m}e^{\xi_1 \underline{x}} = -\sum_{i=0}^{m} b_i \underline{x}^i, \tag{F.63}$$

$$B_{1,m}e^{\xi_1 \overline{x}} + B_{2,m}e^{\xi_1 \overline{x}} = -\sum_{i=0}^{m} b_i \overline{x}^i. \tag{F.64}$$

and the solution to this system of equations is given by

$$B_{1,m} = \frac{\left(-\sum_{i=0}^{m} b_{i,m}\underline{x}^i\right)e^{\xi_2\overline{x}} - \left(-\sum_{i=0}^{m} b_{i,m}\overline{x}^i\right)e^{\xi_2\underline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}} \tag{F.65}$$

$$B_{2,m} = \frac{\left(-\sum_{i=0}^{m} b_{i,m}\overline{x}^i\right)e^{\xi_1\underline{x}} - \left(-\sum_{i=0}^{m} b_{i,m}\underline{x}^i\right)e^{\xi_1\overline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}} \tag{F.66}$$

Putting all the results togethers we have that

$$\mathcal{M}_m(\hat{x}) = \frac{-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_m^m(\underline{x}) - \underline{\alpha}_2\kappa_m^m(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_m^m(\overline{x}) - \overline{\alpha}_1\kappa_m^m(\underline{x})\right] + \kappa_m^m(\hat{x})}{\lambda\mathbb{E}[\tau]} \tag{F.67}$$

□

**Proposition F.4.** *The ergodic distribution is given*

$$f(x) = A\begin{cases} \frac{e^{\beta 1(x-\underline{x})} - e^{\beta 2(x-\underline{x})}}{e^{\beta 1(\hat{x}-\underline{x})} - e^{\beta 2(\hat{x}-\underline{x})}} & \text{if } x \in (\underline{x}, \hat{x}] \\ \frac{e^{\beta 1(x-\overline{x})} - e^{\beta 2(x-\overline{x})}}{e^{\beta 1(\hat{x}-\overline{x})} - e^{\beta 2(\hat{x}-\overline{x})}} & \text{if } x \in [\hat{x}, \overline{x}) \end{cases} \tag{F.68}$$

$$A = \left[\frac{\frac{e^{\beta 1(\hat{x}-\underline{x})}-1}{\beta 1} - \frac{e^{\beta 2(\hat{x}-\underline{x})}-1}{\beta 2}}{e^{\beta 1(\hat{x}-\underline{x})} - e^{\beta 2(\hat{x}-\underline{x})}} + \frac{\frac{1-e^{\beta 1(\hat{x}-\overline{x})}}{\beta 1} - \frac{1-e^{\beta 2(\hat{x}-\overline{x})}}{\beta 2}}{e^{\beta 1(\hat{x}-\overline{x})} - e^{\beta 2(\hat{x}-\overline{x})}}\right]^{-1} \tag{F.69}$$

*Proof.* The distribution of state solves the following KFE given by

$$\lambda f(x) = -\nu f'(x) + \frac{\sigma_a^2}{2}f''(x) \quad \forall x \in (\underline{x}, \overline{x}) \cap \{\hat{x}\}^c \tag{F.70}$$

with the border conditions $f(\underline{x}) = f(\overline{x}) = 0$ and $\int_{\underline{x}}^{\overline{x}} f(x)dx = 1$. The solution for $f(x)$ is given by

$$f(x) = \begin{cases} A_{11}e^{\beta_1 x} + A_{12}e^{\beta_2 x} & \text{if } x \in (\underline{x}, \hat{x}) \\ A_{21}e^{\beta_1 x} + A_{22}e^{\beta_2 x} & \text{if } x \in (\hat{x}, \overline{x}) \end{cases} \tag{F.71}$$

$$\beta_1 = \tilde{\nu} + \sqrt{\tilde{\nu}^2 + 2\tilde{\lambda}} \tag{F.72}$$

$$\beta_2 = \tilde{\nu} - \sqrt{\tilde{\nu}^2 + 2\tilde{\lambda}}. \tag{F.73}$$

The border conditions of zero mass in the boundary of the continuation region imply that

$$f(x) = \begin{cases} A_1\left(e^{\beta_1(x-\underline{x})} - e^{\beta_2(x-\underline{x})}\right) & \text{if } x \in (\underline{x}, \hat{x}) \\ A_2\left(e^{\beta_1(x-\overline{x})} - e^{\beta_2(x-\overline{x})}\right) & \text{if } x \in (\hat{x}, \overline{x}) \end{cases}. \tag{F.74}$$

Continuity at the reset state implies

$$f(x) = \begin{cases} A\frac{e^{\beta 1(x-\underline{x})} - e^{\beta 2(x-\underline{x})}}{e^{\beta 1(\hat{x}-\underline{x})} - e^{\beta 2(\hat{x}-\underline{x})}} & \text{if } x \in (\underline{x}, \hat{x}) \\ A\frac{e^{\beta 1(x-\overline{x})} - e^{\beta 2(x-\overline{x})}}{e^{\beta 1(\hat{x}-\overline{x})} - e^{\beta 2(\hat{x}-\overline{x})}} & \text{if } x \in (\hat{x}, \overline{x}) \end{cases} \tag{F.75}$$

$$\tag{F.76}$$

Using the unitary mass of firms condition we have that

$$A = \left[\frac{\frac{e^{-\beta_1\underline{x}}}{\beta_1}\left(e^{\beta_1\hat{x}} - e^{\beta_1\underline{x}}\right) - \frac{e^{-\beta_2\underline{x}}}{\beta_2}\left(e^{\beta_2\hat{x}} - e^{\beta_2\underline{x}}\right)}{e^{\beta_1(\hat{x}-\underline{x})} - e^{\beta_2(\hat{x}-\underline{x})}} + \frac{\frac{e^{-\beta_1\overline{x}}}{\beta_1}\left(e^{\beta_1\overline{x}} - e^{\beta_1\hat{x}}\right) - \frac{e^{-\beta_2\overline{x}}}{\beta_2}\left(e^{\beta_2\overline{x}} - e^{\beta_2\hat{x}}\right)}{e^{\beta_1(\hat{x}-\overline{x})} - e^{\beta_2(\hat{x}-\overline{x})}}\right]^{-1}$$

$$= \left[\frac{\frac{e^{\beta_1(\hat{x}-\underline{x})}-1}{\beta_1} - \frac{e^{\beta_2(\hat{x}-\underline{x})}-1}{\beta_2}}{e^{\beta_1(\hat{x}-\underline{x})} - e^{\beta_2(\hat{x}-\underline{x})}} + \frac{\frac{1-e^{\beta_1(\hat{x}-\overline{x})}}{\beta_1} - \frac{1-e^{\beta_2(\hat{x}-\overline{x})}}{\beta_2}}{e^{\beta_1(\hat{x}-\overline{x})} - e^{\beta_2(\hat{x}-\overline{x})}}\right]^{-1} \tag{F.77}$$

□

**Proposition F.5.** *The following relations hold*

$$\mathbb{E}^{\hat{x}}[e^{\varphi\tau}x_\tau^m] = \frac{\lambda}{\lambda-\varphi}\Big[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\overline{\kappa}^m_{m-1}(\underline{x},\varphi)-\underline{\alpha}_2(\varphi)\overline{\kappa}^m_{m-1}(\overline{x},\varphi)\right] \tag{F.78}$$

$$\cdots - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\overline{\kappa}^m_{m-1}(\overline{x},\varphi)-\overline{\alpha}_1(\varphi)\overline{\kappa}^m_{m-1}(\underline{x},\varphi)\right]+\overline{\kappa}^m_m(\hat{x})\Big]+\dots \tag{F.79}$$

$$\cdots + \frac{\varphi}{\lambda-\varphi}\left[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\underline{x}^m-\underline{\alpha}_2(\varphi)\overline{x}^m\right]+e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\overline{x}^m-\overline{\alpha}_1(\varphi)\underline{x}^m\right]\right] \tag{F.80}$$

$$\begin{aligned}\mathbb{E}^{\hat{x}}[e^{\varphi\tau}\Delta x^m] &= \frac{\lambda}{\lambda-\varphi}\Big[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\overline{\kappa}^m_{m-1}(\hat{x}-\underline{x},\varphi)-\underline{\alpha}_2(\varphi)\overline{\kappa}^m_{m-1}(\hat{x}-\overline{x},\varphi)\right]\dots\\ &\cdots - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\overline{\kappa}^m_{m-1}(\hat{x}-\overline{x},\varphi)-\overline{\alpha}_1(\varphi)\overline{\kappa}^m_{m-1}(\hat{x}-\underline{x},\varphi)\right]+\overline{\kappa}^m_0(0)\Big]\dots\\ &\cdots+\frac{\varphi}{\lambda-\varphi}\left[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)(\hat{x}-\underline{x})^m-\underline{\alpha}_2(\varphi)(\hat{x}-\overline{x})^m\right]-e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)(\hat{x}-\overline{x})^m-\overline{\alpha}_1(\varphi)(\hat{x}-\underline{x})^m\right]\right]\end{aligned} \tag{F.81}$$

*Proof.* The function $v_m(x) = \mathbb{E}^x[e^{\varphi\tau}(\hat{x}-x_\tau)^m]$ satisfies the HJB

$$-\varphi v_m(x) = \nu v_m'(x) + \frac{\sigma^2}{2}v_m''(x) + \lambda\left((\hat{x}-x)^m - v_m(x)\right) \tag{F.82}$$

with the border conditions $v_m(\underline{x}) = (\hat{x}-\underline{x})^m$ and $v_m(\overline{x}) = (\hat{x}-\overline{x})^m$. The homogenous solution is given by

$$v^h(x) = B_{1,m}e^{\xi_1(\varphi)x} + B_{2,m}e^{\xi_2(\varphi)x}, \tag{F.83}$$

To compute the non-homogenous solution let us use a guess and verify $v^{nh}(x) = \sum_{i=0}^m b_{i,m}(\hat{x}-x)^i$. Then

$$0 = -\tilde{\nu}\left(\sum_{i=1}^m b_{im}i(\hat{x}-x)^{i-1}\right) + \frac{1}{2}\left(\sum_{i=2}^m b_{im}i(i-1)(\hat{x}-x)^{i-2}\right) + \tilde{\lambda}(\varphi)\left(\frac{\tilde{\lambda}}{\tilde{\lambda}(\varphi)}(\hat{x}-x)^m - \sum_{i=0}^m b_i(\hat{x}-x)^i\right) \tag{F.84}$$

Notice that the solution consist in the following system of equations

$$b_{m,m} = \frac{\tilde{\lambda}}{\tilde{\lambda}(\varphi)}\ ;\ b_{m-1,m} = -\frac{\tilde{\nu}mb_{m,m}}{\tilde{\lambda}(\varphi)}\ ;\ b_{i,m} = \frac{-\tilde{\nu}(i+1)b_{i+1,m}+(i+2)(i+1)b_{i+2,m}/2}{\tilde{\lambda}(\varphi)}. \tag{F.85}$$

with the solution given by

$$b_{i,m}(\varphi) = \frac{\tilde{\lambda}}{\tilde{\lambda}(\varphi)}\frac{m!}{i!}\left[\frac{\xi_1(\varphi)-\xi_1(\varphi)\xi_2(\varphi)\frac{\tilde{\nu}}{\tilde{\lambda}(\varphi)}}{\xi_1(\varphi)-\xi_2(\varphi)}(-\xi_1(\varphi))^{i-m} + \frac{\xi_2(\varphi)-\xi_1(\varphi)\xi_2(\varphi)\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2(\varphi)-\xi_1(\varphi)}(-\xi_2(\varphi))^{i-m}\right], \tag{F.86}$$

with the border conditions we have that $B_1$ and $B_2$ satisfy

$$B_{1,m}(\varphi)e^{\xi_1(\varphi)\underline{x}} + B_{2,m}(\varphi)e^{\xi_2(\varphi)\underline{x}} = -\sum_{i=0}^m b_{i,m}(\varphi)(\hat{x}-\underline{x})^i + (\hat{x}-\underline{x})^m, \tag{F.87}$$

$$B_{1,m}(\varphi)e^{\xi_1(\varphi)\overline{x}} + B_{2,m}(\varphi)e^{\xi_2(\varphi)\overline{x}} = -\sum_{i=0}^m b_{i,m}(\varphi)(\hat{x}-\overline{x})^i + (\hat{x}-\underline{x})^m. \tag{F.88}$$

Putting all the results togethers we have that

$$h_m(\hat{x}) = \frac{\lambda}{\lambda-\varphi}\Big[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\overline{\kappa}^m_{m-1}(\hat{x}-\underline{x},\varphi)-\underline{\alpha}_2(\varphi)\overline{\kappa}^m_{m-1}(\hat{x}-\overline{x},\varphi)\right] \tag{F.89}$$

$$\cdots - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\overline{\kappa}^m_{m-1}(\hat{x}-\overline{x},\varphi)-\overline{\alpha}_1(\varphi)\overline{\kappa}^m_{m-1}(\hat{x}-\underline{x},\varphi)\right]+\overline{\kappa}^m_0(0)\Big]+\dots \tag{F.90}$$

$$\cdots + \frac{\varphi}{\lambda-\varphi}\left[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)(\hat{x}-\underline{x})^m-\underline{\alpha}_2(\varphi)(\hat{x}-\overline{x})^m\right]+e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)(\hat{x}-\overline{x})^m-\overline{\alpha}_1(\varphi)(\hat{x}-\underline{x})^m\right]\right] \tag{F.91}$$

In a similar way, the function $k_m(x) = \mathbb{E}^x[e^{\varphi\tau}x_\tau^m]$ satisfies the HJB

$$-\varphi k_m(x) = \nu k_m'(x) + \frac{\sigma^2}{2}k_m''(x) + \lambda\left(x^m - k_m(x)\right) \tag{F.92}$$

with the border conditions $k_m(\underline{x}) = \underline{x}^m$ and $k_m(\overline{x}) = \overline{x}^m$. The homogenous solution is given by

$$v^h(x) = B_{1,m}e^{\xi_1(\varphi)x} + B_{2,m}e^{\xi_2(\varphi)x}. \tag{F.93}$$

To compute the non-homogenous solution let us use a guess and verify $v^{nh}(x) = \sum_{i=0}^{m} b_{i,m} x^i$. Then

$$0 = \tilde{\nu}\left(\sum_{i=1}^{m} b_{im} i x^{i-1}\right) + \frac{1}{2}\left(\sum_{i=2}^{m} b_{i,m} i(i-1)x^{i-2}\right) + \tilde{\lambda}(\varphi)\left(\frac{\tilde{\lambda}}{\tilde{\lambda}(\varphi)}x^m - \sum_{i=0}^{m} b_i x^i\right) \tag{F.94}$$

Notice that the solution consist in the following system of equations

$$b_{m,m} = \frac{\tilde{\lambda}}{\tilde{\lambda}(\varphi)} \; ; \; b_{m-1,m} = \frac{\tilde{\nu} m b_m}{\tilde{\lambda}(\varphi)} \; ; \; b_{i,m} = \frac{\tilde{\nu}(i+1)b_{i+1} + (i+2)(i+1)a_{i+2}/2}{\tilde{\lambda}(\varphi)}. \tag{F.95}$$

with the solution given by

$$b_{i,m}(\varphi) = \frac{\tilde{\lambda}}{\tilde{\lambda}(\varphi)} \frac{m!}{i!}\left[\frac{\xi_1(\varphi) - \xi_1(\varphi)\xi_2(\varphi)\frac{\tilde{\nu}}{\tilde{\lambda}(\varphi)}}{\xi_1(\varphi) - \xi_2(\varphi)}(\xi_1(\varphi))^{i-m} + \frac{\xi_2(\varphi) - \xi_1(\varphi)\xi_2(\varphi)\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2(\varphi) - \xi_1(\varphi)}(\xi_2(\varphi))^{i-m}\right], \tag{F.96}$$

with the border conditions we have that $B_1$ and $B_2$ satisfy

$$B_{1,m}(\varphi)e^{\xi_1(\varphi)\underline{x}} + B_{2,m}(\varphi)e^{\xi_2(\varphi)\underline{x}} = -\sum_{i=0}^{m} b_{i,m}(\varphi)\underline{x}^i + \underline{x}^m, \tag{F.97}$$

$$B_{1,m}(\varphi)e^{\xi_1(\varphi)\overline{x}} + B_{2,m}(\varphi)e^{\xi_2(\varphi)\overline{x}} = -\sum_{i=0}^{m} b_{i,m}(\varphi)\overline{x}^i + \underline{x}^m. \tag{F.98}$$

Putting all the results togethers we have that

$$k_m(\hat{x}) = \frac{\lambda}{\lambda - \varphi}\Big[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\overline{\kappa}_{m-1}^m(\underline{x},\varphi) - \underline{\alpha}_2(\varphi)\overline{\kappa}_{m-1}^m(\overline{x},\varphi)\right] \tag{F.99}$$

$$\cdots - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\overline{\kappa}_{m-1}^m(\overline{x},\varphi) - \overline{\alpha}_1(\varphi)\overline{\kappa}_{m-1}^m(\underline{x},\varphi)\right] + \overline{\kappa}_m^m(\hat{x})\Big] + \dots \tag{F.100}$$

$$\cdots + \frac{\varphi}{\lambda - \varphi}\left[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\underline{x}^m - \underline{\alpha}_2(\varphi)\overline{x}^m\right] + e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\overline{x}^m - \overline{\alpha}_1(\varphi)\underline{x}^m\right]\right] \tag{F.101}$$

□

**Proposition F.6.** *The following relation holds*

$$\mathcal{M}_{m,1}[x,a] = \frac{h_m'(0)}{\mathbb{E}^{\hat{x}}[\tau]} \tag{F.102}$$

$$h_m(\varphi) = \frac{1}{\lambda - \varphi}\Big[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\kappa_m^m(\underline{x},\varphi) - \underline{\alpha}_2(\varphi)\kappa_m^m(\overline{x},\varphi)\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\kappa_m^m(\overline{x},\varphi) - \overline{\alpha}_1(\varphi)\kappa_m^m(\underline{x},\varphi)\right] + \overline{\kappa}_m^m(\hat{x})\Big] \tag{F.103}$$

*Proof.* The function $v_m(x) = \mathbb{E}^x[\int_0^\tau e^{\varphi t}x_t^m dt]$ satisfies the HJB

$$-\varphi v_m(x,\varphi) = x^m + \nu\frac{\partial_m^v(x,\varphi)}{\partial x^2} + \frac{\sigma^2}{2}\frac{\partial_m^v(x,\varphi)}{\partial x^2} + \lambda\left(-v_m(x,\varphi)\right) \tag{F.104}$$

with the border conditions $v_m(\underline{x},\varphi) = 0$ and $v_m(\overline{x},\varphi) = 0$. The homogenous solution is given by

$$v^h(x,\varphi) = B_{1,m}e^{\xi_1(\varphi)x} + B_{2,m}e^{\xi_2(\varphi)x}, \tag{F.105}$$

and the non-homogenous solution is given by $v^{nh}(x,\varphi) = \sum_{i=0}^{m} b_{i,m}x^i$ with

$$b_{i,m}(\varphi) = \frac{1}{\lambda - \varphi}\frac{m!}{i!}\left[\frac{\xi_1(\varphi) - \xi_1(\varphi)\xi_2(\varphi)\frac{\tilde{\nu}}{\tilde{\lambda}(\varphi)}}{\xi_1(\varphi) - \xi_2(\varphi)}(\xi_1(\varphi))^{i-m} + \frac{\xi_2(\varphi) - \xi_1(\varphi)\xi_2(\varphi)\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2(\varphi) - \xi_1(\varphi)}(\xi_2(\varphi))^{i-m}\right], \tag{F.106}$$

The border conditions for $B_1$ and $B_2$ are given by

$$B_{1,m}(\varphi)e^{\xi_1(\varphi)\underline{x}} + B_{2,m}(\varphi)e^{\xi_2(\varphi)\underline{x}} = -\sum_{i=0}^{m} b_{i,m}(\varphi)\underline{x}^i, \tag{F.107}$$

$$B_{1,m}(\varphi)e^{\xi_1(\varphi)\overline{x}} + B_{2,m}(\varphi)e^{\xi_2(\varphi)\overline{x}} = -\sum_{i=0}^{m} b_{i,m}(\varphi)\overline{x}^i. \tag{F.108}$$

Putting all the results togethers we have that

$$h_m(\varphi) = v_m(\hat{x}, \varphi) \tag{F.109}$$

$$= \frac{1}{\lambda - \varphi}\Big[ - e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\kappa_m^m(\underline{x}, \varphi) - \underline{\alpha}_2(\varphi)\kappa_m^m(\overline{x}, \varphi)\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\kappa_m^m(\overline{x}, \varphi) - \overline{\alpha}_1(\varphi)\kappa_m^m(\underline{x}, \varphi)\right] + \overline{\kappa}_m^m(\hat{x})\Big] \tag{F.110}$$

□

**Proposition F.7.** *The following relations hold*

$$\mathbb{E}^{\hat{x}}\left[\int_0^\tau e^{\xi_1 x_t} dt\right] = -\frac{-e^{\xi_1\hat{x}}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\lambda_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_1\hat{x}}\hat{x}}{\sigma^2(\tilde{\nu} + \xi_1)} \tag{F.111}$$

$$\mathbb{E}^{\hat{x}}\left[\int_0^\tau e^{\xi_2 x_t} dt\right] = -\frac{-e^{\xi_1\hat{x}}(e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\lambda_2\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_2\hat{x}}\hat{x}}{\sigma^2(\tilde{\nu} + \xi_2)} \tag{F.112}$$

*Proof.* The computation of the solution is equivalent for $\xi_1$ and $\xi_2$, so we only do it for $\xi_1$. Define $p(x) = \mathbb{E}^x\left[\int_0^\tau e^{\xi_1 x_t} dt\right]$. Then $p(x)$ satisfies the HJB

$$0 = e^{\xi_1 x} + \nu p'(x) + \frac{\sigma^2}{2} p''(x) + \lambda\left(-v_m(x)\right) \tag{F.113}$$

with the border conditions $p(\underline{x}) = 0$ and $p(\overline{x}) = 0$. The homogenous solution is given by

$$v^h(x) = A_1 e^{\xi_1 x} + A_2 e^{\xi_2 x}. \tag{F.114}$$

To compute the non-homogenous solution let us use a guess and verify $v^{nh}(x) = e^{\xi_1 x} Cx$. Then

$$0 = e^{\xi_1 x} + \nu\left[e^{\xi_1 x}C + \xi_1 e^{\xi_1 x} Cx\right] + \frac{\sigma^2}{2}\left[2\xi_1 e^{\xi_1 x}C + \xi_1^2 e^{\xi_1 x} Cx\right] - \lambda e^{\xi_1 x} Cx, \tag{F.115}$$

or equivalently

$$0 = \frac{1}{\sigma^2} + \tilde{\nu}\left[C + \xi_1 Cx\right] + \frac{1}{2}\left[2\xi_1 C + \xi_1^2 Cx\right] - \tilde{\lambda}Cx, \tag{F.116}$$

$$= x(\tilde{\nu}\xi_1 C + \frac{\xi_1^2 C}{2} - \tilde{\lambda}C) + \frac{1}{\sigma^2} + \tilde{\nu}C + \xi_1 C \tag{F.117}$$

$$= x\underbrace{(\tilde{\nu}\lambda_1 + \frac{\lambda_1^2}{2} - \tilde{\lambda})}_{=0} C + \frac{1}{\sigma^2} + \tilde{\nu}C + \xi_1 C \tag{F.118}$$

Since the previous equation must be satisfied for all $x$ we have that

$$C = -\frac{1}{\sigma^2(\tilde{\nu} + \xi_1)}. \tag{F.119}$$

Therefore the solution is given by

$$v(x) = A_{1,1}e^{\xi_1 x} + A_{2,1}e^{\xi_2 x} + \frac{e^{\xi_1 x}x}{\sigma^2(\tilde{\nu} + \xi_1)}, \tag{F.120}$$

Using the boundaries conditions we have that

$$A_{1,1} = \left(\frac{e^{\xi_1\underline{x}}\underline{x}}{\sigma^2(\tilde{\nu} + \xi_1)}\right)\overline{\alpha}_2 - \left(\frac{e^{\lambda_1\overline{x}}\overline{x}}{\sigma^2(\tilde{\nu} + \xi_1)}\right)\underline{\alpha}_2, \tag{F.121}$$

$$A_{2,1} = \left(\frac{e^{\xi_1\overline{x}}\overline{x}}{\sigma^2(\tilde{\nu} - \xi_1)}\right)\underline{\alpha}_1 - \left(\frac{e^{\xi_1\underline{x}}\underline{x}}{\sigma^2(\tilde{\nu} + \xi_1)}\right)\overline{\alpha}_1. \tag{F.122}$$

Putting all the results together

$$v(x) = -\frac{-e^{\xi_1 x}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\lambda_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2 x}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_1 x}x}{\sigma^2(\tilde{\nu} + \xi_1)} \tag{F.123}$$

□

**Verification of $\frac{\mathbb{E}[\Delta x]}{\mathbb{E}[\tau]} = -\nu$ .** From proposition F.2, we have that

$$\mathbb{E}^{\hat{x}}[\Delta x] = -e^{\xi_1 \hat{x}} \left[\overline{\alpha}_2 \overline{\kappa}_0^1(\hat{x} - \underline{x}) - \underline{\alpha}_2 \overline{\kappa}_0^1(\hat{x} - \overline{x})\right] - e^{\xi_2 \hat{x}} \left[\underline{\alpha}_1 \overline{\kappa}_0^1(\hat{x} - \overline{x}) - \overline{\alpha}_1 \overline{\kappa}_0^1(\hat{x} - \underline{x})\right] + \overline{\kappa}_0^1(0)$$
$$= -\frac{\tilde{\nu}}{\tilde{\lambda}} \left[-e^{\xi_1 \hat{x}} \left[\overline{\alpha}_2 - \underline{\alpha}_2\right] - e^{\xi_2 \hat{x}} \left[\underline{\alpha}_1 - \overline{\alpha}_1\right] + 1\right], \tag{F.124}$$

where we have used that

$$= \overline{\kappa}_0^1(\hat{x} - \underline{x}) = \overline{\kappa}_0^1(\hat{x} - \overline{x}) = \overline{\kappa}_0^1(0)$$
$$= \frac{\xi_1 - \xi_1\xi_2 \frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1 - \xi_2} (-\xi_1)^{-1} + \frac{\xi_2 - \xi_1\xi_2 \frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2 - \xi_1} (-\xi_2)^{-1}$$
$$= \frac{-1 + \xi_2 \frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1 - \xi_2} + \frac{1 - \xi_1 \frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1 - \xi_2}$$
$$= -\frac{\tilde{\nu}}{\tilde{\lambda}} \tag{F.125}$$

From proposition F.5, we have that

$$\mathbb{E}^{\hat{x}}[e^{\varphi\tau} \Delta x^0] = \frac{\lambda}{\lambda - \varphi} \overline{\kappa}_0^0(0) + \frac{\varphi}{\lambda - \varphi} \left[-e^{\xi_1(\varphi)\hat{x}} \left[\overline{\alpha}_2(\varphi) - \underline{\alpha}_2(\varphi)\right] - e^{\xi_2 \hat{x}} \left[\underline{\alpha}_1(\varphi) - \overline{\alpha}_1(\varphi)\right]\right] \tag{F.126}$$
$$= \frac{\lambda}{\lambda - \varphi} + \frac{\varphi}{\lambda - \varphi} \underbrace{\left[-e^{\xi_1(\varphi)\hat{x}} \left[\overline{\alpha}_2(\varphi) - \underline{\alpha}_2(\varphi)\right] - e^{\xi_2 \hat{x}} \left[\underline{\alpha}_1(\varphi) - \overline{\alpha}_1(\varphi)\right]\right]}_{= \mathcal{H}(\tilde{\nu}, \tilde{\lambda} - \frac{\varphi}{\sigma^2}, \hat{x}, \underline{x}, \overline{x})}. \tag{F.127}$$

Taking derivative with respect to $\varphi$ and evaluating in zero

$$= \left.\frac{d\mathbb{E}^{\hat{x}}[e^{\varphi\tau} \Delta x^0]}{d\varphi}\right|_{\varphi=0}$$
$$= \left.\left[\frac{\lambda}{(\lambda - \varphi)^2} + \frac{\lambda}{(\lambda - \varphi)^2} \mathcal{H}(\tilde{\nu}, \tilde{\lambda} - \frac{\varphi}{\sigma^2}, \hat{x}, \underline{x}, \overline{x}) - \frac{\varphi}{\sigma^2(\lambda - \varphi)} \mathcal{H}_2(\tilde{\nu}, \tilde{\lambda} - \frac{\varphi}{\sigma^2}, \hat{x}, \underline{x}, \overline{x})\right]\right|_{\varphi=0}$$
$$= \frac{1}{\lambda} \left[-e^{\xi_1 \hat{x}} \left[\overline{\alpha}_2 - \underline{\alpha}_2\right] - e^{\xi_2 \hat{x}} \left[\underline{\alpha}_1 - \overline{\alpha}_1\right] + 1\right], \tag{F.128}$$

where we used $\mathcal{H}_2(\cdot)$ to denote the derivative with respect to the second argument.

From equations (F.124) and (F.128) we have

$$\frac{\mathbb{E}[\Delta x]}{\mathbb{E}[\tau]} = -\frac{\frac{\tilde{\nu}}{\tilde{\lambda}} \left[\mathcal{H}(\tilde{\nu}, \tilde{\lambda}, \hat{x}, \underline{x}, \overline{x}) + 1\right]}{\frac{1}{\lambda} \left[\mathcal{H}(\tilde{\nu}, \tilde{\lambda}, \hat{x}, \underline{x}, \overline{x}) + 1\right]} \tag{F.129}$$
$$= -\nu \tag{F.130}$$

**Verification of $\sigma^2 = \frac{\mathbb{E}[\Delta x^2]}{\mathbb{E}[\tau]} + \frac{2\nu \mathbb{E}[\tau \Delta x]}{\mathbb{E}[\tau]} + \nu^2 \frac{\mathbb{E}[\tau^2]}{\mathbb{E}[\tau]}$ .** From proposition F.2, we have that

$$\mathbb{E}^{\hat{x}}[\Delta x^2] = -e^{\xi_1 \hat{x}} \left[\overline{\alpha}_2 \overline{\kappa}_1^2(\hat{x} - \underline{x}) - \underline{\alpha}_2 \overline{\kappa}_1^2(\hat{x} - \overline{x})\right] - e^{\xi_2 \hat{x}} \left[\underline{\alpha}_1 \overline{\kappa}_1^2(\hat{x} - \overline{x}) - \overline{\alpha}_1 \overline{\kappa}_1^2(\hat{x} - \underline{x})\right] + \overline{\kappa}_0^2(0)$$
$$= 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + \frac{1}{2\tilde{\lambda}}\right] \left[\mathcal{H}(\tilde{\nu}, \tilde{\lambda}, \hat{x}, \underline{x}, \overline{x}) + 1\right] - 2\frac{\tilde{\nu}}{\tilde{\lambda}} \underbrace{\left[-e^{\xi_1 \hat{x}} \left[\overline{\alpha}_2(\hat{x} - \underline{x}) - \underline{\alpha}_2(\hat{x} - \overline{x})\right] - e^{\xi_2 \hat{x}} \left[\underline{\alpha}_1(\hat{x} - \overline{x}) - \overline{\alpha}_1(\hat{x} - \underline{x})\right]\right]}_{= \mathcal{B}(\tilde{\nu}, \tilde{\lambda}, \hat{x}, \underline{x}, \overline{x})}$$
$$= 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + \frac{1}{2\tilde{\lambda}}\right] \left[\mathcal{H}(\tilde{\nu}, \tilde{\lambda}, \hat{x}, \underline{x}, \overline{x}) + 1\right] - 2\frac{\tilde{\nu}}{\tilde{\lambda}} \mathcal{B}(\tilde{\nu}, \tilde{\lambda}, \hat{x}, \underline{x}, \overline{x}), \tag{F.131}$$

where we have used that

$$= \overline{\kappa}_1^2(x)$$
$$= 2\left[\frac{\xi_1 - \xi_1\xi_2 \frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1 - \xi_2} (-\xi_1)^{-2} + \frac{\xi_2 - \xi_1\xi_2 \frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2 - \xi_1} (-\xi_2)^{-2}\right] + 2\left[\frac{\xi_1 - \xi_1\xi_2 \frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1 - \xi_2} (-\xi_1)^{-1} + \frac{\xi_2 - \xi_1\xi_2 \frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2 - \xi_1} (-\xi_2)^{-1}\right] x$$
$$= 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + \frac{1}{2\tilde{\lambda}}\right] - 2\frac{\tilde{\nu}}{\tilde{\lambda}} x \tag{F.132}$$

From proposition F.5 evaluating in $m = 0$

$$
\begin{aligned}
&= \left.\frac{d^2\mathbb{E}^{\hat{x}}[e^{\varphi\tau}\Delta x^0]}{d\varphi^2}\right|_{\varphi=0} \\
&= \left.\left[\frac{2\lambda}{(\lambda-\varphi)^3} + \frac{2\lambda\mathcal{H}(\tilde{\nu},\tilde{\lambda}-\frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x})}{(\lambda-\varphi)^3} - 2\frac{\lambda\mathcal{H}_2(\tilde{\nu},\tilde{\lambda}-\frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x})}{\sigma^2(\lambda-\varphi)^2} + \frac{\varphi}{\sigma^4(\lambda-\varphi)}\mathcal{H}_{2^2}(\tilde{\nu},\tilde{\lambda}-\frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x})\right]\right|_{\varphi=0} \\
&= \frac{2}{\lambda^2}\left[1+\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right] - \frac{2}{\lambda\sigma^2}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}).
\end{aligned} \tag{F.133}
$$

From proposition F.5 evaluating in $m = 1$

$$
\begin{aligned}
\mathbb{E}^{\hat{x}}[e^{\varphi\tau}\Delta x] &= \frac{\lambda}{\lambda-\varphi}\Big[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\overline{\kappa}_0^1(\hat{x}-\underline{x},\varphi) - \underline{\alpha}_2(\varphi)\overline{\kappa}_0^1(\hat{x}-\overline{x},\varphi)\right]\dots \\
&\quad \dots - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\overline{\kappa}_0^1(\hat{x}-\overline{x},\varphi) - \overline{\alpha}_1(\varphi)\overline{\kappa}_0^1(\hat{x}-\underline{x},\varphi)\right] + \overline{\kappa}_0^1(0)\Big]\dots \\
&\quad \dots + \frac{\varphi}{\lambda-\varphi}\left[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)(\hat{x}-\underline{x}) - \underline{\alpha}_2(\varphi)(\hat{x}-\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)(\hat{x}-\overline{x}) - \overline{\alpha}_1(\varphi)(\hat{x}-\underline{x})\right]\right] \\
&= -\frac{\lambda}{\lambda-\varphi}\frac{\tilde{\nu}}{\tilde{\lambda}}\left[1+\mathcal{H}(\tilde{\nu},\tilde{\lambda}-\frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x})\right] + \frac{\varphi}{\lambda-\varphi}\mathcal{B}(\tilde{\nu},\tilde{\lambda}-\frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x}) \\
&= -\frac{\lambda\nu}{(\lambda-\varphi)^2}\left[1+\mathcal{H}(\tilde{\nu},\tilde{\lambda}-\frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x})\right] + \frac{\varphi}{\lambda-\varphi}\mathcal{B}(\tilde{\nu},\tilde{\lambda}-\frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x})
\end{aligned} \tag{F.134}
$$

and taking derivatives and evaluating at zero, we have

$$
\mathbb{E}^{\hat{x}}[\tau\Delta x] = -2\frac{\nu}{\lambda^2}\left[1+\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right] - \frac{\nu}{\lambda\sigma^2}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + \frac{1}{\lambda}\mathcal{B}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}). \tag{F.135}
$$

Using equations (F.124) to (F.135)

$$
= \frac{\mathbb{E}[\Delta x^2]}{\mathbb{E}[\tau]} + \frac{2\nu\mathbb{E}[\tau\Delta x]}{\mathbb{E}[\tau]} + \nu^2\frac{\mathbb{E}[\tau^2]}{\mathbb{E}[\tau]} \tag{F.136}
$$

$$
\begin{aligned}
&= \frac{2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + \frac{1}{2\tilde{\lambda}}\right]\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})+1\right] - 2\frac{\tilde{\nu}}{\tilde{\lambda}}\mathcal{B}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})}{\frac{1}{\lambda}\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})+1\right]}\dots \\
&\quad \dots + 2\nu\frac{-2\frac{\nu}{\lambda^2}\left[1+\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right] - \frac{\nu}{\lambda\sigma^2}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + \frac{1}{\lambda}\mathcal{B}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})}{\frac{1}{\lambda}\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})+1\right]}\dots \\
&\quad + \nu^2\frac{\frac{2}{\lambda^2}\left[1+\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right] - \frac{2}{\lambda\sigma^2}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})}{\frac{1}{\lambda}\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})+1\right]} \\
&= 2\lambda\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + \frac{1}{2\tilde{\lambda}}\right] - 4\lambda\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + 2\lambda\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 \\
&= \sigma^2
\end{aligned} \tag{F.137}
$$

**Verification that if $\hat{x} = \frac{\mathbb{E}[\tau\Delta x]}{\mathbb{E}[\tau]} + \nu\frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]}$, then $\mathcal{M}_1[x] = 0$.** From equations (F.133) and (F.135)

$$
\begin{aligned}
\hat{x} &= \frac{\mathbb{E}[\tau\Delta x]}{\mathbb{E}[\tau]} + \nu\frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]} \\
&= \frac{-2\frac{\nu}{\lambda^2}\left[1+\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right] - \frac{\nu}{\lambda\sigma^2}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + \frac{1}{\lambda}\mathcal{B}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})}{\frac{1}{\lambda}\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})+1\right]}\dots \\
&\quad + \nu\frac{\frac{2}{\lambda^2}\left[1+\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right] - \frac{2}{\lambda\sigma^2}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})}{\frac{2}{\lambda}\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})+1\right]} \\
&= -\frac{\tilde{\nu}}{\tilde{\lambda}} + \frac{\mathcal{B}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})}{\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})+1\right]}
\end{aligned} \tag{F.138}
$$

From proposition F.3, we have that

$$\mathcal{M}_1[x] = \frac{-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(\underline{x})\right] + \kappa_1^1(\hat{x})}{\lambda\mathbb{E}[\tau]}. \tag{F.139}$$

Since the $\kappa_1^1(x)$ satisfies

$$\kappa_1^1(x) = \frac{1!}{0!}\left[\frac{\xi_1 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1 - \xi_2}(\xi_1)^{-1} + \frac{\xi_2 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2 - \xi_1}(\xi_2)^{-1}\right] + \frac{1!}{1!}\left[\frac{\xi_1 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1 - \xi_2} + \frac{\xi_2 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2 - \xi_1}\right]x \tag{F.140}$$

$$= \frac{\tilde{\nu}}{\tilde{\lambda}} + x \tag{F.141}$$

Using this result

$$\begin{aligned}
\mathcal{M}_1[x] &= \frac{-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(\underline{x})\right] + \kappa_1^1(\hat{x})}{\lambda\mathbb{E}[\tau]} \\
&= \frac{-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2 - \underline{\alpha}_2\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1 - \overline{\alpha}_1\right] + 1}{\lambda\mathbb{E}[\tau]} + \frac{-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x}\right] + \hat{x}}{\lambda\mathbb{E}[\tau]} \\
&= \frac{\frac{\tilde{\nu}}{\tilde{\lambda}}\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + 1\right]}{\frac{\lambda}{\tilde{\lambda}}\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + 1\right]} + \frac{-\left[\mathcal{B}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right] + \hat{x}\left(-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x}\right]\right) + \hat{x}}{\frac{\lambda}{\tilde{\lambda}}\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + 1\right]} \\
&= \frac{\tilde{\nu}}{\tilde{\lambda}} - \frac{\mathcal{B}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})}{\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + 1\right]} + \hat{x} \\
&= 0
\end{aligned} \tag{F.142}$$

**Observability with respect to $\mathcal{M}_1[a]$.** We need to show that $\mathcal{M}_1[a] = \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]}$. Equations (F.128) and (F.133) shows the functional form for $\mathbb{E}[\tau]$ and $\mathbb{E}[\tau^2]$. Using these two equations we have that

$$\frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]} = \frac{\frac{2}{\lambda^2}\left[1 + \mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right] - \frac{2}{\lambda\sigma^2}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})}{2\frac{1}{\lambda}\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + 1\right]} = \frac{\frac{1}{\lambda}\left[1 + \mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right] - \frac{1}{\sigma^2}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})}{\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + 1\right]}. \tag{F.143}$$

From proposition F.6 we have that

$$\begin{aligned}
\mathcal{M}_{0,1}[x,a] &= \frac{h_0'(0)}{\mathbb{E}^{\hat{x}}[\tau]} \\
h_m(\varphi) &= \frac{1}{\lambda - \varphi}\Big[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\kappa_0^0(\underline{x},\varphi) - \underline{\alpha}_2(\varphi)\kappa_0^0(\overline{x},\varphi)\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\kappa_0^0(\overline{x},\varphi) - \overline{\alpha}_1(\varphi)\kappa_0^0(\underline{x},\varphi)\right] + \overline{\kappa}_0^0(\hat{x})\Big] \\
&= 1 + \mathcal{H}(\tilde{\nu},\tilde{\lambda} - \frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x}),
\end{aligned} \tag{F.144}$$

Where we have that

$$h_0'(0) = \frac{1}{\lambda - \varphi}\left(1 + \mathcal{H}(\tilde{\nu},\tilde{\lambda} - \frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x})\right)\Bigg|_{\varphi=0} \tag{F.145}$$

$$= \frac{1}{(\lambda - \varphi)^2}\left(1 + \mathcal{H}(\tilde{\nu},\tilde{\lambda} - \frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x})\right) + \frac{1}{(\lambda - \varphi)\sigma^2}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda} - \frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x})\Bigg|_{\varphi=0} \tag{F.146}$$

$$= \frac{1}{\lambda^2}\left(1 + \mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right) - \frac{1}{\lambda\sigma^2}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}). \tag{F.147}$$

Therefore, using (F.143) we have that

$$\mathcal{M}_{0,1}[x,a] = \mathcal{M}_1[a] = \frac{\frac{1}{\lambda^2}\left(1 + \mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right) - \frac{1}{\lambda\sigma^2}\mathcal{H}_2(\tilde{\mu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})}{\frac{1}{\lambda}\left[\mathcal{H}(\tilde{\mu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + 1\right]} = \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]} \tag{F.148}$$

# G Lumpy Investment in the Calvo-Taylor Model

This section describe briefly the environment with the change in the investment adjustment policy.

**Environment.** The main change in the environment is the decision problem of the firms, all other problems remain the same. Let $\{\tau_i\}$ be a sequence of stopping time describing the adjusment dates. We assume that $\tau_{i+1} = \inf_t\{t \geq t \geq \tau_i : \tilde{N}_{zs} - \tilde{N}_{zt} \geq 1 \text{ } or \text{ } t - \tau_i = T\}$, where $\{\tilde{N}\}$ is a Poisson counter with unit increase and arrival rate $\lambda$ and T is fixed parameter. The firm's problem is to choose investment rates $\Delta K_{\tau_{zi}} = K_{\tau_{zi}} - K_{\tau_{zi}^-}$ that solve the following problem

$$\max_{\{\Delta K_{\tau_{\omega,i}}\}_{i=1}^{\infty}} \mathbb{E}\left[\int_0^{\infty} Q_s Y_{\omega,s}\, ds - \sum_{i=1}^{\infty} Q_{\tau_{\omega,i}}\left(\Delta K_{\tau_{\omega,i}}\right)\middle| \mathcal{F}_t\right], \tag{G.149}$$

where output is given by (??) and $E_{zt}$ follows (??).

**Controlled States.** Let $S_t = (k_t, N_t, a_t)$ be the controlled state of the firm. $k_t$ follows a Brownian motion with drift given by

$$dk_t = \nu dt + \sigma dW_t, \tag{G.150}$$

where $W_t$ is a Brownian motion. $N_t$ is a Poisson counter with arrival rate $\lambda$ and $a_t$ a satisfies the law of motion $da_t = dt$. The reset state is given by $\hat{S} = (\hat{k}, 0, 0)$ and the stopping time is given by

$$\tau = \inf\{t \geq 0 : N_t \geq 1 \text{ } or \text{ } a \geq T\} \tag{G.151}$$

Notice that this model has only 4 parameters: $(\nu, \sigma, \lambda, T)$ and the distribution of stopping times is independent of $k$ state and it is given by

$$g(\tau) = \lambda e^{-\lambda\tau} I(\tau < T), \quad with \quad Pr(\tau = T) = e^{-\lambda T} \tag{G.152}$$

**Policy.** Using the assumption of constant real interest rate and writing (G.149) recursively with the capital-gaps as a state, we have that

$$\tilde{v}(k) = \mathbb{E}_0\left[\int_0^{\tau} e^{-\left(\rho-\mu-\sigma^2/2\right)} e^{\alpha k_s} ds + e^{-(\rho-\mu)\tau}\left[\max_{\hat{y}}\left\{-\left[e^{\hat{k}} - e^{k_\tau}\right] + \tilde{v}(\hat{k})\right]\right\}\right]. \tag{G.153}$$

Using the result above, the reset capital gap for teh adjusting firm $\hat{k}$, can be written as

$$\hat{k} = \arg\max_{\hat{k}}\left\{\mathbb{E}\left[\int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/2)t}\left(e^{\alpha\left(\hat{k}+\nu t+\sigma B_t\right)} + \lambda e^{\hat{k}+\nu t+\sigma_a B_t} + (v(\hat{k}) - \hat{k})\right) dt\right] - e^{\hat{k}}\right\} \tag{G.154}$$

The next proposition characterizes the reset capital gap in the Calvo Taylor model.

**Proposition G.8.** *The reset state is given by*

$$\hat{k} = \log\left[\left(\frac{\alpha\frac{1-e^{-(\rho+\lambda-\mu-\sigma^2/2-\alpha\nu-\frac{\sigma^2\alpha^2}{2})T}}{\rho+\lambda-\mu-\sigma^2/2-\alpha\nu-\frac{\sigma^2\alpha^2}{2}}}{1-\lambda\frac{1-e^{-(\rho+\lambda-\mu-\sigma^2/2-\nu-\frac{\sigma^2}{2})T}}{\rho+\lambda-\mu-\sigma^2/2-\nu-\frac{\sigma^2}{2}}}\right)^{\frac{1}{1-\alpha}}\right] \tag{G.155}$$

*Proof.* Operating over the expectation

$$\begin{aligned}\mathbb{E}\left[\int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/2)t+\alpha\left(\hat{k}+\nu t+\sigma W_t\right)} dt\right] &= \int_0^T \mathbb{E}\left[e^{-(\rho+\lambda-\mu-\sigma^2/2)t+\alpha\left(\hat{k}+\nu t+\sigma W_t\right)}\right] dt \\ &= \int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/2-\alpha\nu-\frac{\alpha^2\sigma^2}{2})t+\alpha\hat{k}} dt \end{aligned} \tag{G.156}$$

$$\mathbb{E}\left[\int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/2)t+\left(\hat{k}+\nu t+\sigma B_t\right)} dt\right] = \int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/2-\nu-\frac{\sigma^2}{2})t+\hat{k}} dt \tag{G.157}$$

Therefore the optimality conditions for the reset state in the Calvo-Taylor model is given by

$$\hat{k} = \arg\max_{\hat{k}}\left[\int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/-\alpha\nu-\frac{\alpha^2\sigma^2}{2})t+\alpha\hat{k}} dt + \lambda\int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/2-\nu-\frac{\sigma^2}{2})t+\hat{k}} dt - e^{\hat{k}}\right]. \tag{G.158}$$

Taking FOC we have that

$$\alpha e^{\alpha\hat{k}}\int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/2-\alpha\nu-\frac{\alpha^2\sigma^2}{2})t} dt + \lambda e^{\hat{k}}\int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/2-\nu-\frac{\sigma^2}{2})t} dt - e^{\hat{k}} = 0, \tag{G.159}$$

and operating we have that

$$e^{(\alpha-1)\hat{k}} = \frac{1-\lambda\int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/2-\nu-\frac{\sigma^2}{2})t}dt}{\alpha\int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/2-\alpha\nu-\frac{\alpha^2\sigma_a^2}{2})t}dt}, \tag{G.160}$$

or equivalently, (G.155). □

**Re-normalization** We normalize the policy function in the model to work with centralized moments. We redefine the reset state $\hat{x}$ in the following way

$$\hat{x} = \hat{k} - \mathcal{M}_1[k], \tag{G.161}$$

where $\mathcal{M}_1[k]$ is the mean capital-gap under the the firms' policies $\hat{k}$. It is easy to check that under the re-normalized policy $\mathcal{M}_1[x] = 0$.

**Moments to verify observation property in the structural parameters and reset state** The next proposition computes all the inputs to verify observability in the structural parameters: $\mathbb{E}[\tau]$, $\mathbb{E}[\tau^2]$,$\mathbb{E}[\tau\Delta x]$, $\mathbb{E}[\Delta x]$,$\mathbb{E}[\Delta x^2]$.

**Proposition G.9.** *In the CalvoTaylor model the following relationships hold*

$$\mathbb{E}\left[\tau^n\right] = \frac{n!\left[1-e^{-\lambda T}\sum_{i=0}^{n-1}\frac{(\lambda T)^i}{i!}\right]}{\lambda^n} \tag{G.162}$$

$$\mathbb{E}\left[\Delta x^n\right] = \sum_{k=0,k\ even}^{n}\binom{n}{k}(-\nu)^{n-k}\sigma^k(k-1)!\mathbb{E}[\tau^{n-k/2}] \tag{G.163}$$

$$\mathbb{E}[\tau\Delta x] = -\nu\mathbb{E}[\tau^2] \tag{G.164}$$

*Proof.* In this proof, we will use the incomplete gamma function $\Gamma(\cdot)$, and recall that $\Gamma(n+1,x) = n!e^{-x}\sum_{k=0}^{n}\frac{x^k}{k!}$. By definition we have that

$$\mathbb{E}\left[\tau^n\right] = Pr(\tau=T)T^n + \lambda\int_0^T s^n e^{-\lambda s}ds = e^{-\lambda T}T^n + \left[-\lambda^{-n}\Gamma(n+1,\lambda x)\right]_0^T \tag{G.165}$$

$$= e^{-\lambda T}T^n + \lambda^{-n}\left[\Gamma(n+1,0) - \Gamma(n+1,\lambda T)\right] \tag{G.166}$$

$$= e^{-\lambda T}T^n + \lambda^{-n}\left[n! - n!e^{-\lambda T}\sum_{k=0}^{n}\frac{(\lambda T)^k}{k!}\right] \tag{G.167}$$

$$= e^{-\lambda T}T^n + \lambda^{-n}\left[n! - n!e^{-\lambda T}\sum_{k=0}^{n-1}\frac{(\lambda T)^k}{k!}\right] - n!e^{-\lambda T}\lambda^{-n}\frac{(\lambda T)^n}{n!} \tag{G.168}$$

$$= \frac{n!\left[1-e^{-\lambda T}\sum_{i=0}^{n-1}\frac{(\lambda T)^i}{i!}\right]}{\lambda^n} \tag{G.169}$$

To compute the moment generating function of the changes, note that by definition

$$\mathbb{E}\left[\Delta x^n\right] = \mathbb{E}\left[(-\nu\tau+\sigma B_\tau)^n\right] \tag{G.170}$$

$$= \mathbb{E}\left[\sum_{k=0}^{n}\binom{n}{k}(-\nu\tau)^{n-k}(\sigma B_\tau)^k\right] \tag{G.171}$$

$$= \mathbb{E}\left[\sum_{k=0,k\ even}^{n}\binom{n}{k}(-n\tau)^{n-k}(\sigma B_\tau)^k\right], \tag{G.172}$$

where in the last step we use that $\mathbb{E}\left[\tau^{n-k}B_\tau^k\right] = 0$ for $k$ odd, since $t^{n-k}B_t^k$ is a martingale for $k$ odd with zero initial condition and therefore $\tau^{n-k}B_\tau^k$ is a martingale. Thus

$$\mathbb{E}\left[\Delta x^n\right] = \mathbb{E}\left[\sum_{k=0,k\ even}^{n}\binom{n}{k}(-\nu\tau)^{n-k}(\sigma B_\tau)^k\right] = \sum_{k=0,k\ even}^{n}\binom{n}{k}(-\nu)^{n-k}\sigma^k\mathbb{E}\left[\tau^{n-k}B_\tau^k\right], \tag{G.173}$$

where $B_\tau$ is a normal random variable with mean 0 and variance $\tau$. Using this property it is easy to check that $\mathbb{E}[\tau^{n-k}B_\tau^k] = \mathbb{E}[\tau^{n-k}\mathbb{E}[B_t^k|t=\tau]] = \mathbb{E}[\tau^{n-k}\tau^{k/2}(k-1)!]$, and therefore

$$\mathbb{E}\left[\Delta x^n\right] = \sum_{k=0,k\ even}^{n}\binom{n}{k}(-\nu)^{n-k}\sigma^k(k-1)!\mathbb{E}[\tau^{n-k/2}]. \tag{G.174}$$

Finally with similar arguments as before

$$\mathbb{E}[\tau\Delta x] = (\psi + \mu)\mathbb{E}[\tau^2]. \tag{G.175}$$

□

**Distribution of the State.** To verify observability and representability, we first compute the ergodic distribution of the state. Let $h(x, a)$ denote the joint distribution of the capita gap and the time since the last adjustment and $f(x)$ the marginal distribution over capital gaps. The following proposition characterizes both objects.

**Proposition G.10.** *The distribution of the state and the marginal distribution of the capital-gap are given by*

$$h(x,a) = \frac{\lambda \exp\left(-\lambda a - \frac{(x-\nu a-\hat{x})^2}{2\sigma^2 a}\right)}{(1-e^{-\lambda T})\sqrt{2\pi\sigma^2 a}} \tag{G.176}$$

$$f(x) = \frac{\lambda}{(1-e^{-\lambda T})\sqrt{2\sigma^2}}\mathcal{H}_{\sqrt{\lambda+\frac{\nu^2}{2\sigma^2}},\frac{-\nu}{\sqrt{2\sigma^2}},T}\left(\frac{x-\hat{x}}{\sqrt{2\sigma^2}}\right) \tag{G.177}$$

$$H_{z,p,T}(x) = \frac{e^{-2(|x|z+px)}}{2|z|}\left[1 - erf\left(\frac{|x|-Tz}{\sqrt{T}}\right) + e^{4|x|z}\left(erf\left(\frac{|x|+Tz}{\sqrt{T}}\right)-1\right)\right], \tag{G.178}$$

*here $erf(x)$ is the error function.*

*Proof.* To compute the distribution of relative price we need to extend the firm's state space. Let $(a, x_t)$ be the state of the firm where $a$ is the time since the last adjustment and follows the process $da = dt$ (notice that $N$ has a probability atom in $N = 0$ of 1). Let $h(x, a)$ be the ergodic distribution. Then $h(x, a)$ satisfies

$$\lambda h(\hat{x}, a) = -\frac{\partial h(x,a)}{\partial a} - \nu\frac{\partial h(x,s)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 h(x,a)}{\partial x^2} \quad (a,x) \in (0,T)\times\mathbb{R} \tag{G.179}$$

with the border conditions $\lim_{t\to\infty} h(x,a) = \lim_{t\to-\infty} h(x,a) = 0$, $\int\int h(x,a)dxda = 1$ and the border condition $\lim_{a\to 0} u(x,a) = \delta_d(x-\hat{x})$ where $\delta_d(\hat{x})$ is the Direc delta function. Let $h(x,a) = \alpha e^{-\lambda a}\tilde{h}(x,a)$. It is easy to see that $h$ is a solution for some $\alpha$ iff $\tilde{h}(x,a)$ satisfies the standard heat equation

$$\frac{\partial\tilde{h}(x,a)}{\partial a} = -\nu\frac{\partial\tilde{h}(x,a)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2\tilde{h}(x,a)}{\partial x^2} \quad (a,\hat{x}) \in (0,T)\times\mathbb{R}. \tag{G.180}$$

The solution is given by

$$\tilde{h}(x,a) = A\frac{\int_{-\infty}^{\infty}\delta_d(z-\hat{x})\exp\left(-\frac{(x-\nu a-z)^2}{2\sigma^2 a}\right)dz}{\sqrt{a}} = A\frac{\exp\left(-\frac{(x-\nu a-\hat{x})^2}{2\sigma^2 a}\right)}{\sqrt{a}}. \tag{G.181}$$

Therefore

$$h(x,a) = A\frac{\exp\left(-\lambda a - \frac{(x-\nu a-\hat{x})^2}{2\sigma^2 a}\right)}{\sqrt{a}}, \tag{G.182}$$

and using the border condition

$$h(x,a) = \frac{\lambda\exp\left(-\lambda a - \frac{(x-\nu a-\hat{x})^2}{2\sigma^2 a}\right)}{(1-e^{-\lambda T})\sqrt{2\pi\sigma^2 a}}. \tag{G.183}$$

The marginal distribution of the estate is given by

$$\begin{aligned} f(x) &= \int_0^T h(x,a)da \\ &= \frac{\lambda}{(1-e^{-\lambda T})\sqrt{2\sigma^2}}\left[\int_0^T \frac{\exp\left(-\lambda a - \frac{(x-\nu a-\hat{x})^2}{2\sigma^2 a}\right)}{\sqrt{\pi a}}da\right] \\ &= \frac{\lambda}{(1-e^{-\lambda T})\sqrt{2\sigma^2}}\mathcal{H}_{\sqrt{\lambda+\frac{\nu^2}{2\sigma^2}},\frac{-\nu}{\sqrt{2\sigma^2}},T}\left(\frac{x-\hat{x}}{\sqrt{2\sigma^2}}\right) \end{aligned} \tag{G.184}$$

where $\mathcal{H}_{z,p,y}(x)$ satisfies (when $x \neq 0$)

$$
\begin{aligned}
&= H_{z,p,y}(x) &\text{(G.185)}\\
&= \int_0^y \frac{\exp\left(-(z^2-p^2)s - \frac{(x+ps)^2}{s}\right)}{\sqrt{\pi s}} ds \\
&= \frac{e^{-2(|x||z|+px)}}{2|z|}\left[1 - erf\left(\frac{|x|-sz}{\sqrt{s}}\right) + e^{4|x||z|}\left(erf\left(\frac{|x|+sz}{\sqrt{s}}\right)-1\right)\right]\Bigg|_{s=0}^{s=y} \\
&= \frac{e^{-2(|x||z|+px)}}{2|z|}\left[1 - erf\left(\frac{|x|-yz}{\sqrt{y}}\right) + e^{4|x|z}\left(erf\left(\frac{|x|+yz}{\sqrt{y}}\right)-1\right)\right] - \dots \\
&\dots - \frac{e^{-2(|x||z|+px)}}{2|z|}\left[1 - \lim_{K\to\infty} erf(K) + e^{4|x|z}\left(\lim_{K\to\infty} erf(K) - 1\right)\right] \\
&= \frac{e^{-2(|x|z+px)}}{2|z|}\left[1 - erf\left(\frac{|x|-yz}{\sqrt{y}}\right) + e^{4|x|z}\left(erf\left(\frac{|x|+yz}{\sqrt{y}}\right)-1\right)\right], &\text{(G.186)}
\end{aligned}
$$

where in the last equality we used the result that $\lim_{K\to\infty} erf(K) = 1$. In the case $x = 0$, we have that

$$1 - erf(0) + e^0(erf(0) - 1) = 0 \tag{G.187}$$

Thus, we have the result. □

**Observability of structural parameters.** With the results in Proposition G.9, we can check observability for the structural parameters.

- For the drift $\nu$ note that

$$
\begin{aligned}
\mathbb{E}[\tau] &= \frac{1-e^{-\lambda T}}{\lambda} &\text{(G.188)}\\
\mathbb{E}[\Delta x] &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}(-\nu)^1(0-1)!\mathbb{E}[\tau^1] = -\nu\frac{1-e^{-\lambda T}}{\lambda} &\text{(G.189)}
\end{aligned}
$$

  and therefore

$$\frac{\mathbb{E}[\Delta x]}{\mathbb{E}[\tau]} = -\nu\frac{\frac{1-e^{-\lambda T}}{\lambda}}{\frac{1-e^{-\lambda T}}{\lambda}} = (\psi + \mu). \tag{G.190}$$

- For the volatility $\sigma^2$ note that

$$
\begin{aligned}
\mathbb{E}[\tau^2] &= \frac{2\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda^2} &\text{(G.191)}\\
\mathbb{E}[\Delta x\tau] &= -\nu\mathbb{E}[\tau^2] = -\nu\frac{2\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda^2} &\text{(G.192)}\\
\mathbb{E}[\Delta x^2] &= \begin{pmatrix} 2 \\ 0 \end{pmatrix}(-\nu)^{2-0}\sigma^0(0-1)!\mathbb{E}[\tau^2] + \begin{pmatrix} 2 \\ 2 \end{pmatrix}(-\nu)^{2-2}\sigma^2(2-1)!\mathbb{E}[\tau^1] &\text{(G.193)}\\
&= \nu^2\mathbb{E}[\tau^2] + \sigma^2\mathbb{E}[\tau] &\text{(G.194)}\\
&= \nu^2\frac{2\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda^2} + \sigma^2\frac{1-e^{-\lambda T}}{\lambda} &\text{(G.195)}
\end{aligned}
$$

  and therefore

$$
\begin{aligned}
\sigma^2 &= \frac{\mathbb{E}[\Delta x^2]}{\mathbb{E}[\tau]} - 2\frac{\mathbb{E}[\Delta x]\mathbb{E}[\Delta x\tau]}{\mathbb{E}[\tau]^2} + \frac{\mathbb{E}[\Delta x]^2\mathbb{E}[\tau^2]}{\mathbb{E}[\tau]^3} \\
&= \nu^2\frac{2\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda\left(1-e^{-\lambda T}\right)} + \sigma^2 - \nu^2\frac{\frac{\left(1-e^{-\lambda T}\right)4\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda^3}}{\frac{\left(1-e^{-\lambda T}\right)^2}{\lambda^2}} + \nu^2\frac{\frac{\left(1-e^{-\lambda T}\right)^2 2\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda^4}}{\frac{\left(1-e^{-\lambda T}\right)^3}{\lambda^3}} \\
&= \sigma^2 + 2\nu^2\frac{2\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda\left(1-e^{-\lambda T}\right)} - 2\nu^2\frac{2\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda\left(1-e^{-\lambda T}\right)} &\text{(G.196)}\\
&= \sigma^2 &\text{(G.197)}
\end{aligned}
$$

- For the reset state $\hat{x}$, the verification is a little more involved. Notice that if we apply the formula in the main text we have that

$$\hat{x} = \frac{\mathbb{E}[\Delta x\tau]}{\mathbb{E}[\tau]} - \frac{\mathbb{E}[\Delta x]\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]^2} = \nu\frac{(1-e^{-\lambda T}(1+\lambda T))}{\lambda(1-e^{-\lambda T})} \tag{G.198}$$

As we show $\hat{x}$ is given by the number s.t. $\int xh(x,a)dxda = 0$. As we show below $h(x,a) = \frac{\lambda\exp\left(-\lambda a - \frac{(x\nu a-\hat{x})^2}{2\sigma^2 a}\right)}{(1-e^{-\lambda T})\sqrt{2\pi\sigma^2 a}}$, therefore

$$\begin{aligned}
0 = \int xh(x,a)dxda &= \int_a \frac{\lambda\exp(-\lambda a)}{1-e^{-\lambda T}}\int_x x\frac{\exp\left(-\frac{(x-\nu a-\hat{x})^2}{2\sigma^2 a}\right)}{\sqrt{2\pi\sigma^2 a}}dxda \\
&= -\int_0^T \frac{\lambda\exp(-\lambda a)}{1-e^{-\lambda T}}(-\nu a-\hat{x})\,da \\
&= -\left[-\nu\int_0^T \frac{\lambda a\exp(-\lambda a)}{1-e^{-\lambda T}}da - \hat{x}\int_0^T \frac{\lambda\exp(-\lambda a)}{1-e^{-\lambda T}}da\right] \\
&= -\left[-\nu\int_0^T \frac{\lambda\left.\frac{(-e^{-\lambda a})(1+\lambda a)}{\lambda^2}\right|_0^T}{1-e^{-\lambda T}}da - \hat{x}\left.\frac{(-e^{-\lambda a})}{1-e^{-\lambda T}}\right|_0^T\right] \\
&= -\left[-\nu\frac{1-e^{-\lambda T}(1+\lambda T)}{\lambda(1-e^{-\lambda T})} - \hat{x}\right]
\end{aligned} \tag{G.199}$$

From equations (G.198) and (G.199) we get the result between micro-price statistics and the zero ergodic mean of capital gaps.

**Observability with respect to $\mathcal{M}_1[a]$** We need to show that $\mathcal{M}_1[a] = \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]}$. Using the joint distribution $h(x,a)$ from (G.176)

$$\int_z a_z dz = \int_0^T\int_{-\infty}^{\infty} a\frac{\lambda\exp\left(-\lambda a - \frac{(x-\nu a-\hat{x})^2}{2\sigma^2 a}\right)}{(1-e^{-\lambda T})\sqrt{2\pi\sigma_a^2 a}}dxda = \int_0^T a\frac{\lambda\exp(-\lambda a)}{(1-e^{-\lambda T})}da \tag{G.200}$$

$$= \frac{\int_0^T a\exp(-\lambda a)\,da}{\mathbb{E}[\tau]} = \frac{\left.-\frac{e^{-\lambda a}(\lambda x+1)}{\lambda^2}\right|_{a=0}^{a=T}}{\mathbb{E}[\tau]} = \frac{1-e^{-\lambda T}(1+\lambda T)}{\lambda^2\mathbb{E}[\tau]} = \frac{\mathbb{E}\left[\tau^2\right]}{2\mathbb{E}[\tau]} \tag{G.201}$$

**Observability with respect to $\mathbb{E}[x^2]$** We need to verify that

$$\mathcal{M}_2[x] = \frac{\hat{x}^3 - \mathbb{E}[(\hat{x}-\Delta x)^3]}{\mathbb{E}[\Delta x]3} \tag{G.202}$$

Computing the left hand size

$$\begin{aligned}
\mathcal{M}_2[x] &= \int x^2 h(x,a)dxda \\
&= \int_a \frac{\lambda\exp(-\lambda a)}{1-e^{-\lambda T}}\int_x x^2\frac{\exp\left(-\frac{(x-\nu a-\hat{x})^2}{2\sigma^2 a}\right)}{\sqrt{2\pi\sigma^2 a}}dxda = \int_0^T \frac{\lambda\exp(-\lambda a)}{1-e^{-\lambda T}}\left((\hat{x}+\nu a)^2+\sigma^2 a\right)da \\
&= \hat{x}^2 + \frac{\lambda(\sigma^2+2\nu\hat{x})}{1-e^{-\lambda T}}\int_0^T a\exp(-\lambda a)\,da - \frac{\lambda\nu^2}{1-e^{-\lambda T}}\int_0^T a^2\exp(-\lambda a)\,da \\
&= \hat{x}^2 + \frac{\lambda(\sigma^2+2\nu\hat{x})}{1-e^{-\lambda T}}\frac{1-e^{-\lambda T}(1+\lambda T)}{\lambda^2} - 2\frac{\lambda\nu^2}{1-e^{-\lambda T}}\left.\frac{e^{-\lambda a}(1+\lambda a+\frac{(\lambda a)^2}{2})}{\lambda^3}\right|_0^T \\
&= \hat{x}^2 + \frac{(\sigma^2+2\nu\hat{x})\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda(1-e^{-\lambda T})} + 2\frac{\nu^2}{1-e^{-\lambda T}}\frac{1-e^{-\lambda T}(1+\lambda T+\frac{(\lambda T)^2}{2})}{\lambda^2} \\
&= \hat{x}^2 + \hat{x}\frac{2\nu\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda(1-e^{-\lambda T})} + \frac{2\nu^2(1-e^{-\lambda T}(1+\lambda T+\frac{(\lambda T)^2}{2}))}{\lambda^2(1-e^{-\lambda T})} + \frac{\sigma^2\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda(1-e^{-\lambda T})}
\end{aligned} \tag{G.203}$$

Computing the right hand size

$$= \frac{\hat{x}^3 - \mathbb{E}[(\hat{x}-\Delta x)^3]}{\mathbb{E}[\Delta x]3} \tag{G.204}$$

$$= \frac{\hat{x}^3 - \left(\hat{x}^3 - 3\hat{x}^2\mathbb{E}[\Delta x] + 3\hat{x}\mathbb{E}[\Delta x^2] - \mathbb{E}[\Delta x^3]\right)}{\mathbb{E}[\Delta x]3}$$

$$= \hat{x}^2 - \hat{x}\frac{\mathbb{E}[\Delta x^2]}{\mathbb{E}[\Delta x]} + \frac{\mathbb{E}[\Delta x^3]}{3\mathbb{E}[\Delta x]}$$

$$= \hat{x}^2 - \hat{x}\left[\frac{\nu^2\frac{2\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda^2} + \sigma^2\frac{1-e^{-\lambda T}}{\lambda}}{-\nu\frac{1-e^{-\lambda T}}{\lambda}}\right] + \frac{-\nu^3\mathbb{E}[\tau^3] - 3\nu\sigma^2\mathbb{E}[\tau^2]}{-3\nu\frac{1-e^{-\lambda T}}{\lambda}}$$

$$= \hat{x}^2 + \hat{x}\frac{2\nu\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda\left(1-e^{-\lambda T}\right)} + \hat{x}\frac{\sigma^2}{\nu} + \frac{2\sigma^2(1-e^{-\lambda T}(1+\lambda T))}{\lambda(1-e^{-\lambda T})} + \frac{32\nu^2(1-e^{-\lambda T}(1+\lambda T+\frac{(\lambda T)^2}{2}))}{3\lambda^2(1-e^{-\lambda T})}$$

$$= \hat{x}^2 + \hat{x}\frac{2\nu\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda\left(1-e^{-\lambda T}\right)} + \frac{2\nu^2(1-e^{-\lambda T}(1+\lambda T+\frac{(\lambda T)^2}{2}))}{\lambda^2(1-e^{-\lambda T})}\dots$$

$$\dots - \nu\frac{(1-e^{-\lambda T}(1+\lambda T))}{\lambda(1-e^{-\lambda T})}\frac{\sigma^2}{\nu} + \frac{2\sigma^2(1-e^{-\lambda T}(1+\lambda T))}{\lambda(1-e^{-\lambda T})}$$

$$= \hat{x}^2 + \hat{x}\frac{2\nu\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda\left(1-e^{-\lambda T}\right)} + \frac{2\nu^2(1-e^{-\lambda T}(1+\lambda T+\frac{(\lambda T)^2}{2}))}{\lambda^2(1-e^{-\lambda T})} + \frac{\sigma^2(1-e^{-\lambda T}(1+\lambda T))}{\lambda(1-e^{-\lambda T})} \tag{G.205}$$

**Representation with respect to $\Gamma_1$** By definition

$$\Gamma_1 = \frac{\mathbb{E}\left[\int_0^\tau\left(\mathbb{E}^{S_t}[x_\tau] - x_t\right)\right]}{\nu\mathbb{E}[\tau]}. \tag{G.206}$$

Operating, we have that

$$\Gamma_1 = \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau(\mathbb{E}^{S_t}[x_\tau] - x_t)dt\right]}{\nu\mathbb{E}^{\hat{S}}[\tau]} = \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau\mathbb{E}[\tau|a_t]dt\right]}{\mathbb{E}^{\hat{S}}[\tau]} = \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau\frac{1-e^{-\lambda(T-a_t)}dt}{\lambda}dt\right]}{\mathbb{E}^{\hat{S}}[\tau]} \tag{G.207}$$

Let us define $V(a) = \mathbb{E}^a\left[\int_0^\tau\frac{1-e^{-\lambda(T-a_t)}dt}{\lambda}dt\right]$. Then $v(a)$ satsfies the HJB equation

$$\lambda v(a) = \frac{1-e^{-\lambda(T-a)}}{\lambda} + v'(a) \tag{G.208}$$

with $v(T)=0$. It is easy to check that the solution is given by $v(a) = \frac{1}{\lambda^2} + Ae^{\lambda a} + \frac{ae^{-\lambda(T-a)}}{\lambda}$ and with the border condition $v(a) = \frac{1-e^{-\lambda(T-a)}(1+\lambda T)}{\lambda^2} + \frac{ae^{-\lambda(T-a)}}{\lambda}$ and we have that $v(0) = \frac{\mathbb{E}[\tau^2]}{2}$. Therefore

$$\Gamma_1 = \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]} = \mathcal{M}_1[a], \tag{G.209}$$

where we used observability of $\mathcal{M}_1[a]$ in the last step.

**Aggregation** Using the direct approach we have that $\mathcal{A}_1(\delta) = \delta\times\int v'(x,a)h(x,a)dx + o(\delta)$ where $v_1(x,a)$ satisfies

$$v_1(S) = \mathbb{E}^S\left[\int_0^\tau x_t dt\right] = \mathbb{E}^S\left[\int_0^\tau(x+\nu t+\sigma_a B_t)\,dt\right] = x\mathbb{E}^S[\tau|S] + \nu\frac{\mathbb{E}[\tau^2|S]}{2} + \sigma\mathbb{E}[\tau B_\tau]. \tag{G.210}$$

Taking the derivative with respect to $x$ we have that $\frac{\partial v_1(x,a)}{\partial x} = \mathbb{E}^S[\tau|S] = \frac{1-e^{-\lambda(T-a)}}{\lambda}$. To compute the ergodic distribution of $a$ note that

$$f(a) = \int_{-\infty}^{\infty}\frac{\lambda\exp\left(-\lambda a - \frac{(x-\nu a-\hat{x})^2}{2\sigma^2 a}\right)}{(1-e^{-\lambda T})\sqrt{2\pi\sigma_a^2 a}}dx = \frac{\lambda e^{-\lambda a}}{1-e^{-\lambda T}}\int_{-\infty}^{\infty}\frac{\exp\left(-\frac{(x-\nu a-\hat{x})^2}{2\sigma^2 a}\right)}{\sqrt{2\pi\sigma_a^2 a}}dx = \frac{\lambda e^{-\lambda a}}{1-e^{-\lambda T}}$$

Taking the integral

$$\frac{\mathcal{A}_1(\delta)}{\delta} = \int_0^T \frac{1-e^{-\lambda(T-a)}}{\lambda}\frac{\lambda e^{-\lambda a}}{1-e^{-\lambda T}}da + o(\delta) = \frac{1}{1-e^{-\lambda T}}\int_0^T \left(e^{-\lambda a} - e^{-\lambda T}\right) da + o(\delta)$$
$$= \frac{\lambda^{-1}}{1-e^{-\lambda T}}\left[1-e^{-\lambda T}(1+\lambda T)\right] + o(\delta) = \frac{1}{2}\frac{\lambda}{1-e^{-\lambda T}}2\frac{1-e^{-\lambda T}(1+\lambda T)}{\lambda^2} + o(\delta)$$
$$= \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]} + o(\delta) = \Gamma_1 + o(\delta) \tag{G.211}$$

## G.1 Constructing covariance and variance of ergodic moments.

For a given $(\underline{x}, \hat{x}, \overline{x})$, we can define the expected time of investment as

$$\tilde{\nu} = -\frac{\psi+\mu}{\sigma^2}\ ;\ \nu = -\psi+\mu \tag{G.212}$$
$$\tilde{\lambda} = \frac{\lambda}{\sigma^2} \tag{G.213}$$
$$\xi_1 = -\tilde{\nu} - \sqrt{\tilde{\nu}^2 + 2\tilde{\lambda}}\ ;\ \xi_2 = -\tilde{\nu} + \sqrt{\tilde{\nu}^2 + 2\tilde{\lambda}}, \tag{G.214}$$
$$\overline{\alpha}_1 = \frac{e^{\xi_1\overline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}}\ ;\ \overline{\alpha}_2 = \frac{e^{\xi_2\overline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}}, \tag{G.215}$$
$$\underline{\alpha}_1 = \frac{e^{\xi_1\underline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}}\ ;\ \underline{\alpha}_2 = \frac{e^{\xi_2\underline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}}, \tag{G.216}$$
$$\mathbb{E}[\tau] = \frac{1}{\lambda}\left[-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2 - \underline{\alpha}_2\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1 - \overline{\alpha}_1\right] + 1\right], \tag{G.217}$$

Now if we operate, we have that

$$\mathbb{E}[\tau] = \frac{1}{\lambda}\left[-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2 - \underline{\alpha}_2\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1 - \overline{\alpha}_1\right] + 1\right], \tag{G.218}$$
$$= \frac{1}{\lambda}\left(\frac{-e^{\xi_1\hat{x}}\left(e^{\xi_2\overline{x}} - e^{\xi_2\underline{x}}\right) - e^{\xi_2\hat{x}}\left(e^{\xi_1\underline{x}} - e^{\xi_1\overline{x}}\right)}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}} + 1\right] \tag{G.219}$$
$$= \frac{1}{\lambda}\left(\frac{-e^{\xi_1\hat{x}}\left(e^{\xi_2\overline{x}} - e^{\xi_2\underline{x}}\right) - e^{\xi_2\hat{x}}\left(e^{\xi_1\underline{x}} - e^{\xi_1\overline{x}}\right)}{e^{(\xi_1+\xi_2)\hat{x}}\left[e^{\xi_1(\underline{x}-\hat{x})+\xi_2(\overline{x}-\hat{x})} - e^{\xi_2(\underline{x}-\hat{x})+\xi_1((\overline{x}-\hat{x})}\right]} + 1\right] \tag{G.220}$$
$$= \frac{1}{\lambda}\left(\frac{-e^{-\xi_2\hat{x}}\left(e^{\xi_2\overline{x}} - e^{\xi_2\underline{x}}\right) - e^{-\xi_1\hat{x}}\left(e^{\xi_1\underline{x}} - e^{\xi_1\overline{x}}\right)}{\left[e^{\xi_1(\underline{x}-\hat{x})+\xi_2(\overline{x}-\hat{x})} - e^{\xi_2(\underline{x}-\hat{x})+\xi_1((\overline{x}-\hat{x})}\right]} + 1\right] \tag{G.221}$$
$$= \frac{1}{\lambda}\left(\frac{-\left(e^{\xi_2(\overline{x}-\hat{x})} - e^{\xi_2(\underline{x}-\hat{x})}\right) - \left(e^{\xi_1(\underline{x}-\hat{x})} - e^{\xi_1(\overline{x}-\hat{x})}\right)}{e^{\xi_1(\underline{x}-\hat{x})+\xi_2(\overline{x}-\hat{x})} - e^{\xi_2(\underline{x}-\hat{x})+\xi_1((\overline{x}-\hat{x})}} + 1\right] \tag{G.222}$$

# H Quantitative Models of Investment

In this section, we explain the role of each of the three simplifications to the benchmark model in Khan and Thomas (2008), Bachmann, Caballero and Engel (2013), and Winberry (2016)—eliminating free constrained investment, mean reversion and convex adjustment costs—in terms of the cross-sectional implications and transitional dynamics. To facilitate the comparison with the literature, we conduct the analysis in discrete time. As in the model presented in Section **??**, we focus on a partial equilibrium environment.

## H.1 Quantitative Model

**Environment.** Time is discrete and infinite. Consider a steady state economy with a unit mass of production units. Each unit or establishment produces output $y$ using its predetermined capital stock $k$ and labor $n$ via an increasing and concave production function $F$, such that $y = \bar{z}eF(k,n)$, where $\bar{z}$ denotes aggregate productivity and $e$ denotes idiosyncratic productivity. We assume that $F$ is a decreasing returns Cobb-Douglas production function with capital share $\theta$ and labor share $\nu$, such that $\theta + \nu < 1$; $e$ follows an AR(1) process in logs $\log e_t = \rho_e \log e_{t-1} + \sigma_e \eta_t$, where $\eta_t \sim_{iid} \mathcal{N}(0,1)$, and $\bar{z}$ grows at a deterministic rate $\gamma$. All variables are normalized by the aggregate growth rate $\gamma$. Plants discount the future at a rate $\beta$.

At the beginning of each period, a plant hires labor at a wage $w$; then it chooses how much capital to purchase, subject to different types of adjustment costs. A plant can undertake an unconstraint investment $i \in \mathbb{R}$ by paying a fixed adjustment cost $\xi_t \in [0, \bar{\xi}]$ measured in labor units. The fixed cost $\xi$ is uniformly distributed and *iid* across firms and time; its realization is unknown at the moment of making employment decisions. Alternatively, a plant may undertake free but constrained investments, that require that the investment rate is sufficiently small, i.e. $i \in [-ak, ak]$. Finally, plants face quadratic adjustment costs in the form of $\frac{\phi}{2}\left(\frac{i}{k}\right)^2 k$, measured in units of output. Given the investment policy, the plant's capital stock evolves according $(1+\gamma)k' = (1-\delta)k + i$, where $i$ is its current investment, $\delta$ is the depreciation rate, and $\gamma$ denotes aggregate productivity growth.

In each period, a plant is defined by its predetermined stock of capital $k \geq 0$ and its idiosyncratic productivity level $e$. Since the fixed cost realization $\xi$ is *iid*, it is not included in the plant's state. Let $v(k,e)$ be the present discounted value of the firms optimal plan, $v^u(k,e)$ the value when undertaking an unconstrained investment (excluding the adjustment cost) and $v^c(k,e)$ the value of undertaking a constrained investment. Then $v(k,e)$ satisfies the following Bellman equations:

$$v(e,k) = \max_n \left\{ \bar{z}e^{1-\theta-\nu}k^\theta n^\nu - wn \right\} + \mathbb{E}\left[\max\left\{v^u(e,k) - \xi, v^c(e,k)\right\}\right], \tag{H.223}$$

$$v^u(e,k) = \max_i \left\{ -i - \frac{\phi}{2}\left(\frac{i}{k}\right)^2 k + \beta\mathbb{E}\left[v(e',k')\right] \right\}, \tag{H.224}$$

$$v^c(e,k) = \max_{i\in[-ak,ak]} \left\{ -i - \frac{\phi}{2}\left(\frac{i}{k}\right)^2 k + \beta\mathbb{E}\left[v(e',k')\right] \right\}, \tag{H.225}$$

where capital's law of motion is $k' = \frac{(1-\delta)k+i}{1+\gamma}$. The static labor choice yields a labor demand $n(k,e) = \left(e^{1-\theta-\nu}k^\theta \nu / w\right)^{\frac{1}{1-\nu}}$. Substituting it back, and normalizing the wage by a factor $\left[\nu^{\frac{\nu}{1-\nu}} - \nu^{\frac{1}{1-\nu}}\right]^{\frac{1-\nu}{\nu}}$, we can rewrite the plant's problem exclusively in terms of capital as follows:

$$v(e,k) = \bar{z}e^{1-\frac{\theta}{1-\nu}}k^{\frac{\theta}{1-\nu}} + \mathbb{E}\left[\max\left\{v^u(e,k) - \xi, v^c(e,k)\right\}\right] \tag{H.226}$$

where $v^u(e,k)$ and $v^c(e,k)$ remain as before.

**Calibration.** We follow the calibration in Winberry (2016), that targets the average investment rate, the frequency of positive investments, the inaction rate, and positive spikes in the US tax record data as reported by Zwick and Mahon (2017). Column (1) Benchmark in Table I and Table II summarize, respectively, the parametrization and the moments generated by this calibration of the model; and Figure I plots the cross-sectional distribution of capital gaps $\Delta x = 100 \log(1 + i/100)$ in the model and in the data. We observe that while the calibration matches the moments mentioned before, it has trouble in generating very small and very large investment rates and other higher moments of the empirical distribution.

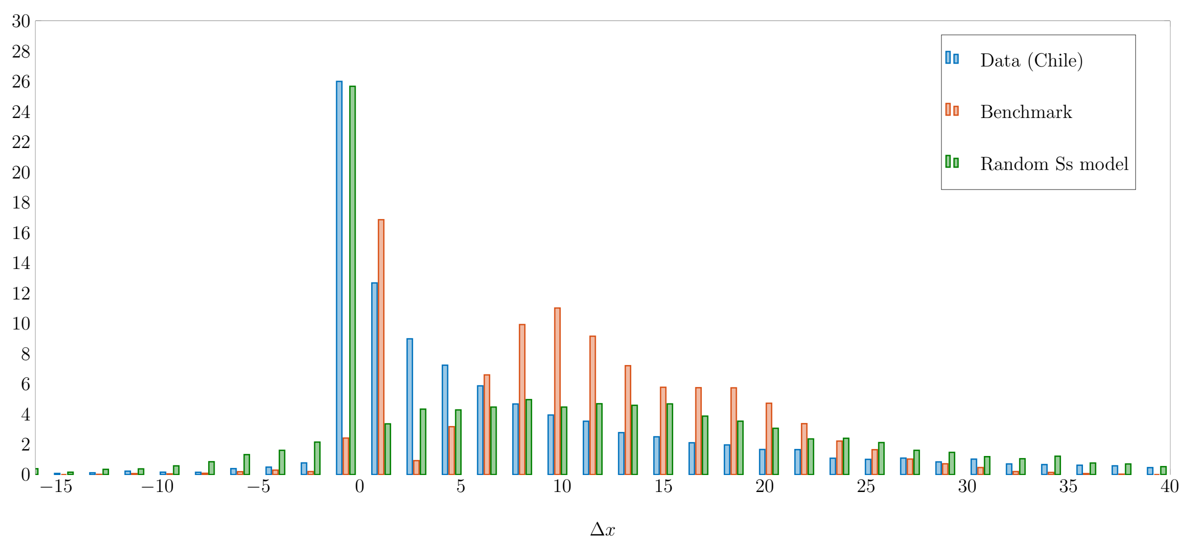
Regarding macro dynamics, Figure III plots the impulse-response of aggregate investment to an unanticipated permanent 1% increase in productivity $\bar{z}$. Capital takes around 10 quarters to converge to its new steady state.

**Table I** – Calibration (Benchmark and simplifications)

| | | (1) Benchmark | (2) No Free Investment ($a = 0$) | (3) High Persistence ($\rho = 0.99$) | (4) No Quadratic Adjustment Cost ($\phi = 0$) | (5) Random Ss Model ($\lambda > 0$) |
|---|---|---|---|---|---|---|
| **Common Parameters** | | | | | | |
| Discount rate | $\beta$ | 0.99 | | | | |
| Capital share | $\theta$ | 0.21 | | | | |
| Labor share | $\nu$ | 0.64 | | | | |
| Aggregate growth | $\gamma$ | 0.004 | | | | |
| Depreciation rate | $\delta$ | 0.025 | | | | |
| Highest fixed adj. cost | $\overline{\xi}$ | 0.0004 | | | | **0.002** |
| Volatility | $\sigma_e$ | 0.053 | | | | |
| **Simplifications** | | | | | | |
| Size of constrained inv. | $a$ | 0.002 | **0** | 0.002 | 0.002 | **0** |
| Convex adj. costs | $\phi$ | 0.025 | 0.025 | 0.025 | **0** | **0** |
| Persistence | $\rho_e$ | 0.94 | 0.94 | **0.99** | 0.94 | **1** |
| Prob. zero adj. cost | $\lambda$ | | | | | **0.3** |

Note: Benchmark calibration from Winberry (2016).

**Figure I** – Investment Rate Distributions




## H.2 Role of each simplification.

Next, we shut down each friction at a time to gauge its effect on the steady state micro-level distribution and on the macro dynamics after an aggregate shock (convergence towards the steady state). First, we shut down the possibility of free constrained investments by setting $a = 0$; we label this exercise "No Free Investment". Second, we increase the persistence of idiosyncratic shocks to $\rho = 0.99$; we label this exercise "High Persistence". Third and last, we eliminate the quadratic adjustment costs $\phi = 0$; we label this exercise "No Quadratic Adjustment Cost". Columns (2)-(4) in Tables Table I and II describe the parametrization and implied moments for each of the experiments, Figure II shows the distributions and Figure III the impulse-responses.

**Simplification I: No free investment.** Motivated by the presence of small investments, Khan and Thomas (2008) introduce the possibility that plants undertake small investments without incurring on the adjustment costs. Two consequences of eliminating free constrained investments are a decrease in the number of investments below than 1% and a shift towards investment rates near the average; both changes push the model further away from the data (and highlight why this feature was introduced). An additional consequence is an increase in the number of plants with exactly zero investment. Interestingly, while the fraction of plants with zero investment increases (there is more inaction), the quantitative response to an aggregate shock is almost identical to the benchmark. This exercise shows that constrained investment reacts mainly to idiosyncratic shocks, and therefore it is harmless to eliminate this assumption for aggregate dynamics.

**Simplification II: Highly persistent productivity.** While there is little agreement on the persistence of the idiosyncratic productivity process, the autocorrelation of investment rates can provide some guidance. In the US data, the serial correlation of investment rates ranges between 0.06 in Cooper and Haltiwanger (2006)'s and 0.4 in Zwick and Mahon (2017). In contrast, the benchmark calibration with $\rho_e = 0.94$ (as well as the experiments with no free investment and no quadratic adjustment costs) generates a small but negative autocorrelation for investment rates. When we increase the persistence of idiosyncratic shocks to $\rho_e = 0.99$, the serial correlation of investment rates remains small but now positive as in the data. To understand this, consider the limit with productivity shocks that follow a random-walk. Then in the absence of adjustment costs, the investment rate becomes *iid*. With some adjustment costs (particularly of the convex type), the investment rate becomes positively serially correlated as investments are spread across several periods. With respect to the response to aggregate shocks, we see that there are no quantitative differences in the IRF with respect to the baseline case. As with the previous simplification, eliminating mean-reversion and opting for a highly persistent productivity process brings the model closer to the data and does not change quantitatively the aggregate response of the economy.

**Simplification III: No quadratic adjustment costs.** The early introduction of quadratic adjustment costs into representative agent investment models was aimed at eliminating the excess volatility of the aggregate investment rate generated by a frictionless model. Cooper and Haltiwanger (2006) show that a model with both convex and non-convex adjustments costs provides a good fit of the micro-data as well. What are the consequences of eliminating this assumption? At the micro level, eliminating convex costs significantly shifts the distribution to the right and increases the fraction of investment rates near zero. Overall, average moments are not affected except for fraction of positive investments (that falls) and positive spikes (that increases). In the aggregate, we see that in the absence of adjustment costs the convergence speed toward the new steady state increases dramatically.

**Table II** – Investment Moments (Data and Models)

| | Data | (1) Benchmark | (2) No Free Investment ($a = 0$) | (3) High Persistence ($\rho = 0.99$) | (4) No Quadratic Adjustment Cost ($\phi = 0$) | (5) Random Ss model |
|---|---|---|---|---|---|---|
| **Investment Moments** | US data | | | | | |
| Average investment | $(0.122; 0.106)$ | 0.114 | 0.114 | 0.115 | 0.113 | 0.101 |
| Inaction rate | $(0.081; 0.237)$ | 0.188 | 0.166 | 0.196 | 0.281 | 0.259 |
| Positive Spike rate | $(0.186; 0.144)$ | 0.161 | 0.161 | 0.179 | 0.191 | 0.214 |
| Negative Spike rate | $(0.018; -)$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.002 |
| Positive rate | $(0.815; 0.602)$ | 0.642 | 0.664 | 0.606 | 0.494 | 0.431 |
| Negative rate | $(0.104; -)$ | 0.009 | 0.010 | 0.019 | 0.033 | 0.095 |
| Serial correlation | $(0.058; 0.400)$ | $-0.058$ | $-0.083$ | 0.027 | $-0.192$ | $-0.171$ |
| Std investment rates | $(0.000; 0.160)$ | 0.081 | 0.082 | 0.088 | 0.102 | 0.128 |
| Zero investment | – | 0.000 | 0.166 | 0.000 | 0.000 | 0.221 |
| **Higher moments** | Chilean data | | | | | |
| $\mathbb{E}[\Delta x]$ | 0.136 | 0.131 | 0.129 | 0.134 | 0.146 | 0.128 |
| $\mathbb{E}[\Delta x^2]$ | 0.051 | 0.021 | 0.021 | 0.023 | 0.028 | 0.031 |
| $\mathbb{E}[\Delta x^3]$ | 0.029 | 0.004 | 0.004 | 0.004 | 0.006 | 0.008 |
| $\mathbb{E}[\Delta x^4]$ | 0.021 | 0.001 | 0.001 | 0.001 | 0.001 | 0.002 |
| $\mathbb{P}erc^5[\Delta x]$ | $-0.012$ | 0.049 | 0.045 | 0.043 | 0.021 | -0.064 |
| $\mathbb{P}erc^{10}[\Delta x]$ | 0.014 | 0.062 | 0.057 | 0.059 | 0.064 | -0.027 |
| $\mathbb{P}erc^{25}[\Delta x]$ | 0.028 | 0.086 | 0.081 | 0.085 | 0.103 | 0.050 |
| $\mathbb{P}erc^{50}[\Delta x]$ | 0.070 | 0.120 | 0.117 | 0.123 | 0.145 | 0.119 |
| $\mathbb{P}erc^{75}[\Delta x]$ | 0.171 | 0.175 | 0.174 | 0.181 | 0.192 | 0.206 |
| $\mathbb{P}erc^{90}[\Delta x]$ | 0.344 | 0.217 | 0.218 | 0.229 | 0.244 | 0.290 |
| $\mathbb{P}erc^{95}[\Delta x]$ | 0.486 | 0.242 | 0.245 | 0.258 | 0.280 | 0.341 |

Notes: Micro investment moments in US reflects the computed by computed moments by Cooper and Haltiwanger (2006) (first column, first entry in parentheses) and Zwick and Mahon (2017) (first column, second entry in parentheses). The moments in US computed in the model are calculated as in their empirical target, $i_t = I_t/K_t$; as the moments in Chile $\Delta x_t = \log(1 + I/0.5 * (K_{t-1} + K_t))$.

**Figure II** – Investment Rate Distributions

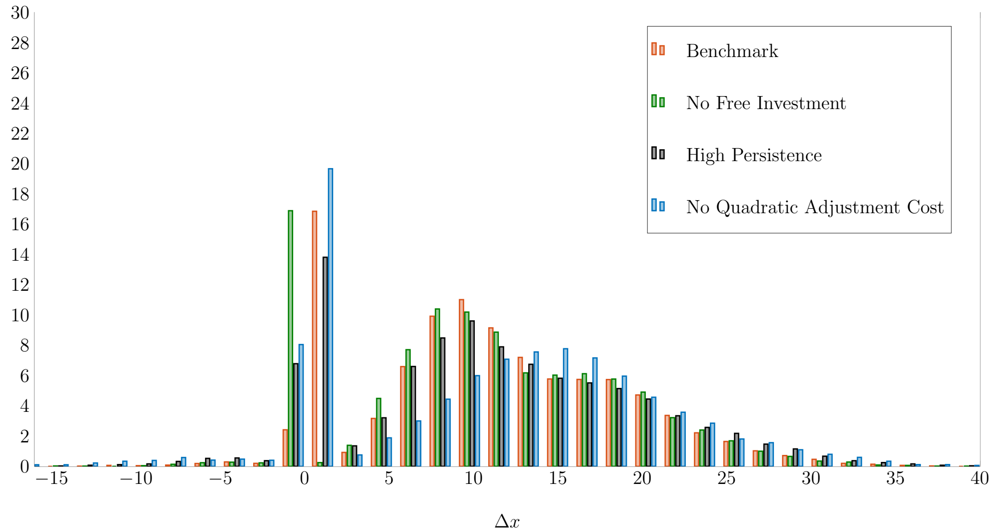
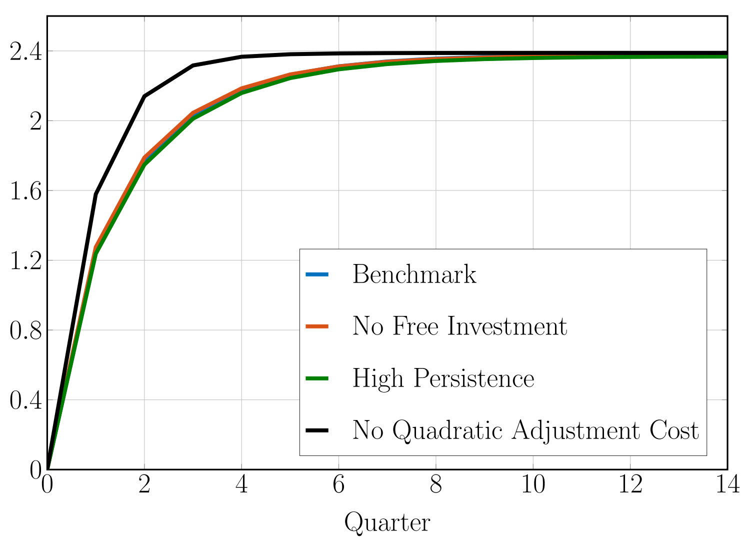



**Figure III** – IRF for Different Calibrations



### H.3 Fit of the random Ss model

Lastly, we present the fit of our random Ss model. The model is the same as in the previous section but with $\rho_e = 1$ and a random fixed cost re-scale by $e$. Additional the distribution of $\xi$ is given by a binomial random variable s.t. with probability $\lambda$, $\xi = 0$; and with $1-\lambda$, $\xi = \bar{\xi}$. In the case it is easy to show that the value function satisfies $v(e,k) = e\tilde{v}\left(\frac{k}{e}\right)$ is given by

$$\tilde{v}(\tilde{k}) = \bar{z}\tilde{k}^{\alpha} + \mathbb{E}\left[\max\left\{\tilde{v}^u(\tilde{k}) - \xi, \tilde{v}^c(\tilde{k})\right\}\right], \tag{H.227}$$

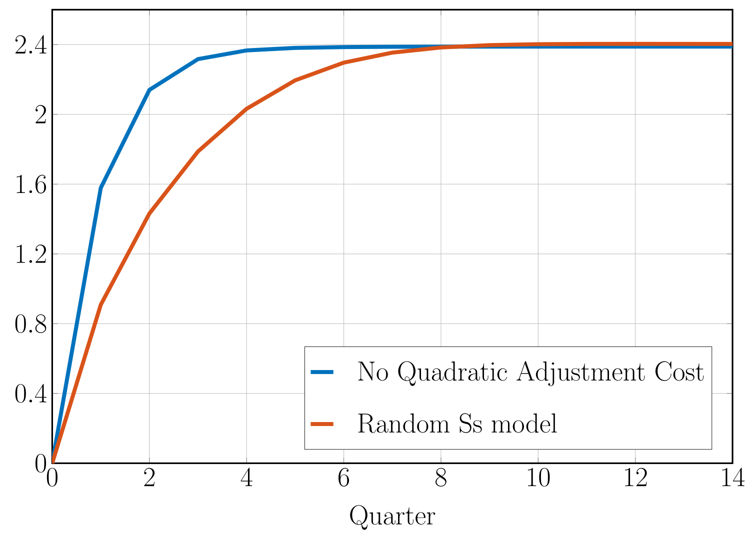
$$\tilde{v}^u(\tilde{k}) = \max_{\tilde{i}}\left\{-\tilde{i} - \frac{\phi}{2}\left(\frac{\tilde{i}}{\tilde{k}}\right)^2 \tilde{k} + \beta\mathbb{E}\left[e^{\sigma_\eta \eta}\tilde{v}(\tilde{k}')\right]\right\}, \tag{H.228}$$

$$\tilde{v}^c(\tilde{k}) = \max_{\tilde{i}\in\{0\}}\left\{-\tilde{i} - \frac{\phi}{2}\left(\frac{\tilde{i}}{k}\right)^2 \tilde{k} + \beta\mathbb{E}\left[e^{\sigma_\eta \eta}\tilde{v}(\tilde{k}')\right]\right\}, \tag{H.229}$$

where we define $\alpha = \frac{\theta}{1-\nu}$ and $k' = e^{-\sigma_\eta \eta}\frac{(1-\delta)k+i}{1+\gamma}$. For simplicity we shut down free investment and the quadratic adjustment cost, thus we have two parameters to calibrate, $\bar{\xi}$ and $\lambda$, that we choose to match the histogram of investment rates—see table I.

As we can see, this calibration of the model does a better match in generating the distribution of investment rates—the target in the calibration. We compare the transition dynamics with the transition dynation generates by the model without adjustment cost.

**Figure IV** – IRF for Random Ss and No Adjustment Cost models

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