# Advanced Macroeconomics II

## Lecture 6

## Consumption: Precautionary Savings

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February 15, 2016

# Saving motives

1. **Intertemporal motive:** patience vs. returns to savings ($\beta R > 1$)
2. **Smoothing motive:** equalize $u'(c)$ through time ($c_t$ is a normal good).
3. **Life-cycle motive:** smoothing between working life and retirement.
4. **Bequest motive:** altruism towards offspring, leaves behind assets.
5. **Precautionary motive or self-insurance:** associated with future income uncertainty.
   - Desire to save, accumulate wealth, and face less fluctuations in $c$ and $u'$ in the future.
   - Arises through preferences (prudence) or borrowing constraints.

# Roadmap: Precautionary Savings

1. **Prudence**
   - Preliminaries: Jensen's Inequality and Convex Marginal Utility
   - Intuition in two-period model
   - General results with Taylor approximation
   - Closed form: CARA + Random walk income
2. Borrowing constraints
   - Natural
   - Ad-hoc
3. Empirical evidence

# Preliminaries: Jensen's Inequality for Convex Functions

- Let $X$ be a convex set in a real vector space and let $f : X \to R$ be a function.
- $f$ is called **convex** if $\forall x_1, x_2 \in X, \forall \alpha \in [0, 1]$

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$$

- Consider a random variable $x$ that takes value $x_1$ with prob $\alpha$ and value $x_2$ with probability $1 - \alpha$.
  - Let $\bar{x} = \mathbb{E}[x] = \alpha x_1 + (1-\alpha)x_2$.
  - Using definition of convexity, we get: $f(\mathbb{E}[x]) \leq \mathbb{E}[f(x)]$.
- Jensen generalized the inequality using measure theory.
  - Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, $x$ an integrable real-valued random variable and $f$ a convex function. Then:

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)] \quad \textbf{Jensen's Inequality for Convex Functions}$$

# Convex marginal utility (1)

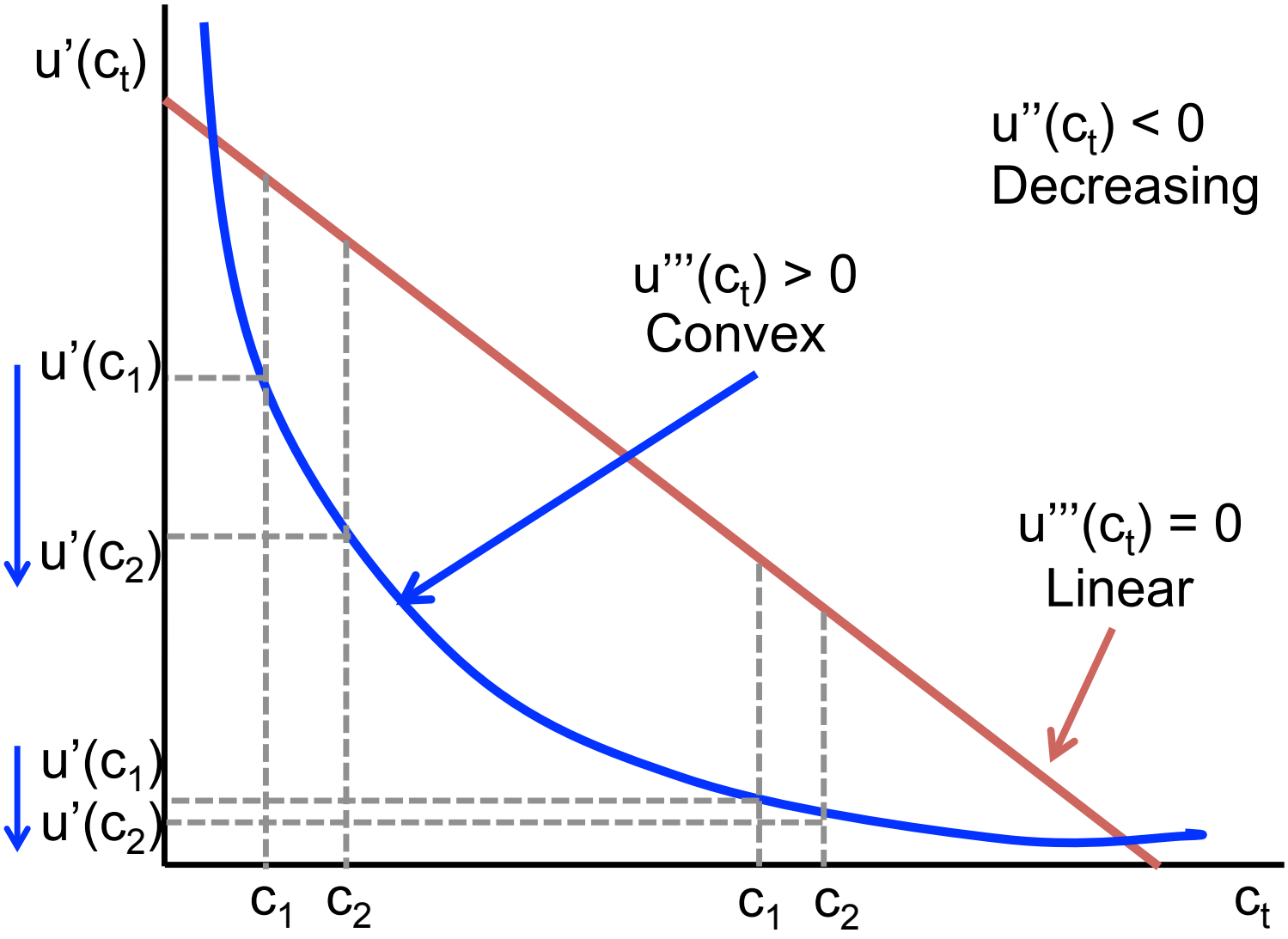
- We will abandon quadratic utility framework ($u''' = 0$).
- Now we adopt a **utility function with positive third derivative**.

$$u'''(c) > 0$$

  - $u''(c) < 0$ means that marginal utility is *decreasing.*
  - $u'''(c) > 0$ means that marginal utility is *convex.*

- Convex $u'$ means that an additional unit of consumption is very valuable when consumption is low, compared to when consumption is already high.
- With linear $u'$, in contrast, an additional unit of consumption provides always the same marginal utility.

# Convex marginal utility (2)



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# Prudence: Intuition in two-period model (1)

- An agent will exhibit **prudent** behavior if $u''' > 0$.
- Consider the simple two-period consumption-saving problem

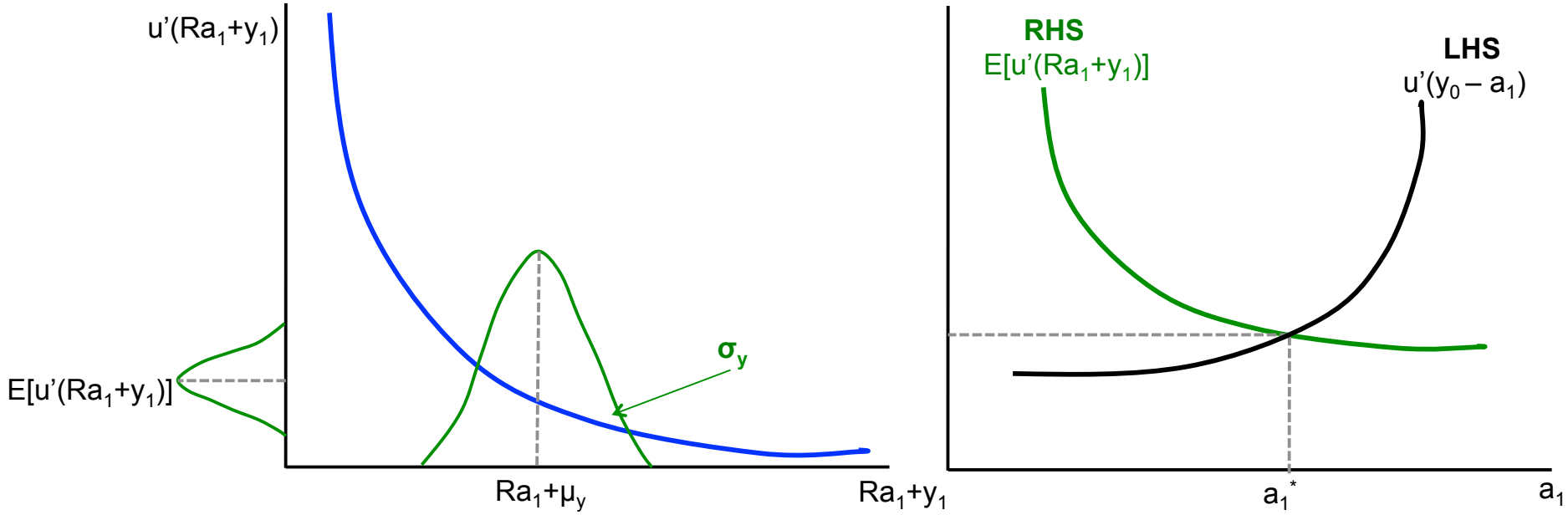
$$\begin{aligned} \max_{\{c_0, c_1, a_1\}} \quad & u(c_0) + \beta \mathbb{E}[u(c_1)] \\ c_0 &= y_0 - a_1 \\ c_1 &= R a_1 + y_1 \end{aligned}$$

  where $y_0$ is given, and $y_1$ is stochastic: $\mathbb{E}[y_1] = \mu_y$ and $\mathbb{V}[y_1] = \sigma_y$.
- Assume $\beta R = 1$ to simplify algebra, Euler equation gives:

$$u'(y_0 - a_1) = \mathbb{E}_{y_1}[u'(R a_1 + y_1)]$$

  - Since $u'' < 0$, LHS is increasing in $a_1$ and RHS is decreasing in $a_1$
  - This implies $a_1^*$ is uniquely determined.
- Policy: $c_0 = y_0 - a_1^*$
  - A rise in savings $a_1^*$ leads to a fall in current consumption.

# Prudence: Intuition in two-period model (2)



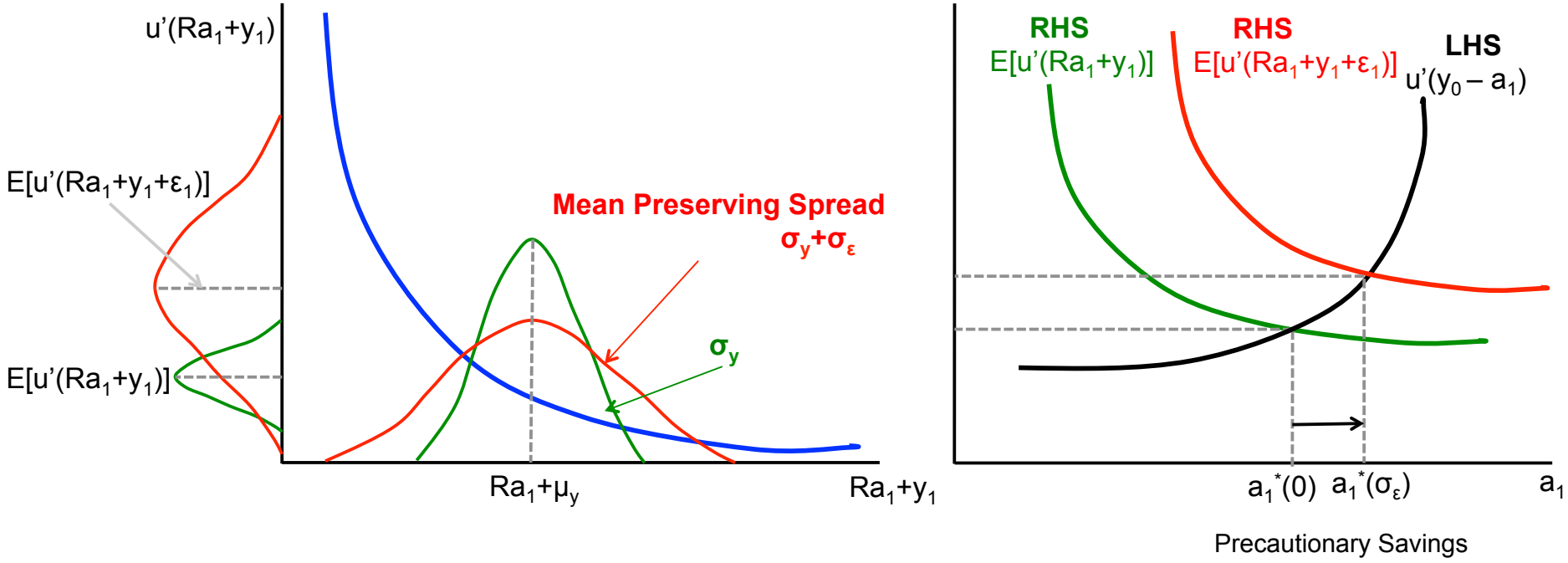
# Prudence: Intuition in two-period model (3)

- Let us increase uncertainty over next period income $y_1$ through a **mean-preserving spread:**
  - Let $\varepsilon_1$ be a random variable with $\mathbb{E}[\varepsilon_1] = 0$ and $\mathbb{V}[\varepsilon_1] = \sigma_\varepsilon$.
  - Define $\tilde{y}_1 = y_1 + \varepsilon_1$, a mean-preserving spread of $y_1$.
    - $\mathbb{E}[\tilde{y}_1] = \mu_y$ and $\mathbb{V}[\tilde{y}_1] = \sigma_y + \sigma_\varepsilon$.
- Under the new more uncertainty world:
  - The LHS of Euler remains the same.
  - The RHS of Euler increases by convexity of $u'$ (Jensen is applied to $\mathbb{E}_{\varepsilon_1}$)

$$\underbrace{\mathbb{E}_{y_1,\varepsilon_1}[u'(Ra_1 + y_1 + \varepsilon_1)]}_{new \quad RHS} > \mathbb{E}_{y_1}[u'(Ra_1 + y_1 + \mathbb{E}[\varepsilon_1])] = \underbrace{\mathbb{E}_{y_1}[u'(Ra_1 + y_1)]}_{old \quad RHS}$$

- RHS shifts upwards and induces a rise in $a_1^*$ and a fall in $c_0$.
- Famous result by Rothschild and Stiglitz (1970).

# Prudence: Intuition in two-period model (4)



# Prudence vs. Certainty Equivalent

- Consider the consumption-savings problem with uncertainty.
- Recall the FOC with uncertainty and $R\beta = 1$

$$u'(c_t) = \mathbb{E}_t\left[u'(c_{t+1})\right]$$

- Under quadratic utility we have **certainty equivalence:**

$$\mathbb{E}_t\left[u'(c_{t+1})\right] = u'\left(\mathbb{E}_t(c_{t+1})\right)$$

which together with the FOC implies that consumption is a random walk:

$$u'(c_t) = \mathbb{E}_t\left[u'(c_{t+1})\right] \Longleftrightarrow u'(c_t) = u'\left[\mathbb{E}_t(c_{t+1})\right] \Longleftrightarrow c_t = \mathbb{E}_t(c_{t+1})$$

- But $\mathbb{E}_t\left[u'(c_{t+1})\right] = u'\left[\mathbb{E}_t(c_{t+1})\right]$ is true only if $u'$ is a linear function.
  - Linearity $\Rightarrow$ $u''$ is a constant and $u''' = 0$.

# Prudence vs. Certainty Equivalent

- Assuming $u''' > 0$, implies that $u'$ is a strictly convex function and by Jensen's inequality:

$$u'\left[\mathbb{E}_t\left(c_{t+1}\right)\right] < \mathbb{E}_t\left[u'\left(c_{t+1}\right)\right]$$

- Therefore if we set $c_t = \mathbb{E}_t\left(c_{t+1}\right)$ the marginal utility of consumption in period $t$ would be too low (too much $c_t$), and it violates Euler:

$$u'\left(c_t\right) \underbrace{=}_{c_t = \mathbb{E}_t\left(c_{t+1}\right)} u'\left(\mathbb{E}_t\left(c_{t+1}\right)\right) \underbrace{\leq}_{Jensen} \mathbb{E}_t\left[u'\left(c_{t+1}\right)\right]$$

- For the FOC to hold with equality, we need to reduce $c_t$ below $\mathbb{E}_t\left(c_{t+1}\right)$.

$$\Longrightarrow \textbf{Precautionary Savings}$$

# Recap of results

- Summing up the case with $\beta R = 1$ :
  - Certainty: $c_t = c_{t+1}$
    - Consumption is constant.
  - Uncertainty and $u''' = 0$ : $\quad c_t = \mathbb{E}_t\left(c_{t+1}\right)$
    - Consumption is a martingale.
  - Uncertainty and $u''' > 0$ : $\quad c_t < \mathbb{E}_t\left(c_{t+1}\right)$.
    - Consumption is a supermartingale.

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# Prudence: Taylor Approx (1)

- A more general result.
- A second order Taylor approximation to $u'(\cdot)$ around the point $c_t = c_{t+1}$

$$\begin{aligned} u'(c_t) &= \beta R \mathbb{E}_t \left[ u'(c_{t+1}) \right] \\ u'(c_t) &\approx \beta R \mathbb{E}_t \left[ u'(c_t) + u''(c_t)(c_{t+1} - c_t) + \tfrac{1}{2} u'''(c_t)(c_{t+1} - c_t)^2 \right] \end{aligned}$$

$$1 \approx \beta R \mathbb{E}_t \left[ 1 + \underbrace{\frac{c_t u''(c_t)}{u'(c_t)}}_{-\gamma(c_t)} \frac{c_{t+1} - c_t}{c_t} + \frac{1}{2} \underbrace{\frac{c_t u''(c_t)}{u'(c_t)}}_{-\gamma(c_t)} \underbrace{\frac{c_t u'''(c_t)}{u''(c_t)}}_{-\psi(c_t)} \left( \frac{c_{t+1} - c_t}{c_t} \right)^2 \right]$$

$$1 \approx \beta R \mathbb{E}_t \left[ 1 - \gamma(c_t) \left( \frac{c_{t+1} - c_t}{c_t} \right) + \frac{1}{2} \gamma(c_t) \psi(c_t) \left( \frac{c_{t+1} - c_t}{c_t} \right)^2 \right]$$

- with $\gamma(c_t) \equiv -c_t u''(c_t) / u'(c_t)$ = coefficient of relative risk aversion (RAA).
- with $\psi(c_t) \equiv -c_t u'''(c_t) / u''(c_t)$ = coefficient of relative prudence (RP).

# Prudence: Taylor Approx (2)

- Note that $\gamma(c_t) = \gamma_t$ and $\psi(c_t) = \psi_t$ are known at time $t$.
- Since $c_{t+1} \approx c_t$, then $\mathbb{E}_t[c_{t+1}] \approx c_t$. Opening the expectation:

$$\mathbb{E}_t\left[\frac{c_{t+1} - c_t}{c_t}\right] \approx \frac{1}{\gamma_t}\frac{\beta R - 1}{\beta R} + \tfrac{1}{2}\psi_t \mathbb{V}_t\left[\frac{c_{t+1} - c_t}{c_t}\right]$$

- In the approximation, $c_{t+1} \approx c_t \Rightarrow \frac{c_{t+1}-c_t}{c_t} \approx \Delta \log c_{t+1}$
- If $\beta R \approx 1$, $\frac{\beta R - 1}{\beta R} = -\log \beta R$.

$$\mathbb{E}_t\left[\Delta \log c_{t+1}\right] \approx \underbrace{-\frac{1}{\gamma_t}\log \beta R}_{\text{intertemporal motive}} + \tfrac{1}{2}\underbrace{\psi_t \mathbb{V}_t\left[\Delta \log c_{t+1}\right]}_{\text{precautionary motive}}$$

- Higher variance of consumption growth $\Rightarrow$ Higher expected consumption growth $\Rightarrow$ Lower current consumption $\Rightarrow$ Precautionary savings.
- The previous equation nests all our previous models.

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# Prudence: CARA Utility

- A closed form solution of the optimal consumption problem can be obtained even without certainty equivalence.

- Closed-form can be obtained with two elements:

  1. (Constant Absolute Risk Aversion) CARA utility with coefficient $\gamma > 0$

$$u(c) = -\exp(-\gamma c)$$

     - Absolute risk aversion: $-\frac{u''}{u'} = -\frac{-\gamma^2 \exp(-\gamma c)}{\gamma \exp(-\gamma c)} = \gamma > 0$
     - Absolute prudence: $-\frac{u'''}{u''} = -\frac{\gamma^3 \exp(-\gamma c)}{-\gamma^2 \exp(-\gamma c)} = \gamma > 0$

  2. Income follows a **random walk.**

# Prudence: CARA Utility (1)

- The Euler equation becomes:

$$\exp\left(-\gamma c_t\right) = \beta R \mathbb{E}_t \left[\exp\left(-\gamma c_{t+1}\right)\right]$$

- Consider a slightly modified timing for the evolution of wealth.
  - Savings at the end of period $t$ are $a_t + y_t - c_t$, thus:

$$a_{t+1} = R\left(a_t + y_t - c_t\right)$$

- Income is a random walk:

$$y_t = y_{t-1} + \varepsilon_t, \quad \varepsilon_{t|t-1} \sim \mathcal{N}\left(0, \sigma^2\right)$$

# Prudence: CARA Utility (2)

- Guess: linear policy rule

$$c(a_t, y_t) = Aa_t + By_t + k$$

- In the next period, consumption is also given by rule:

$$c_{t+1}(a_{t+1}, y_{t+1}) = Aa_{t+1} + By_{t+1} + k$$

- Substitute evolution of assets $a_{t+1}$, income $y_t$ and consumption $c_t$

$$\begin{aligned} c_{t+1} &= AR\left(a_t + y_t - c_t\right) + B\left(y_t + \varepsilon_{t+1}\right) + k \\ &= AR\left(a_t + y_t - Aa_t - By_t - k\right) + B\left(y_t + \varepsilon_{t+1}\right) + k \\ &= \left[AR\left(1 - A\right)\right] a_t + \left[AR\left(1 - B\right) + B\right] y_t + \left(1 - AR\right) k + B\varepsilon_{t+1} \end{aligned}$$

- **Note:** have to carry $\varepsilon_{t+1}$ even though $\mathbb{E}_t(\varepsilon_{t+1}) = 0$, because its variance will have effects in policy (no more certainty equivalence).

# Prudence: CARA Utility (3)

- Substitute back $c_{t+1}$ in the RHS of Euler equation:

$$\begin{aligned}
u'(c_t) &= \beta R \mathbb{E}_t \left[ u'(c_{t+1}) \right] \\
e^{-\gamma c_t} &= \beta R \mathbb{E}_t \left[ e^{-\gamma c_{t+1}} \right] \\
e^{-\gamma(Aa_t + By_t + k)} &= \beta R \mathbb{E}_t \left[ e^{-\gamma([AR(1-A)]a_t + [AR(1-B)+B]y_t + (1-AR)k + B\varepsilon_{t+1})} \right] \\
e^{-\gamma A a_t - \gamma B y_t - \gamma k} &= e^{-\gamma[AR(1-A)]a_t - \gamma[AR(1-B)+B]y_t - \gamma(1-AR)k} \beta R \mathbb{E}_t \left[ e^{-\gamma B \varepsilon_{t+1}} \right]
\end{aligned}$$

- Matching coefficients for states:

$$\begin{aligned}
A &= AR(1-A) \implies A = 1 - \frac{1}{R} = \frac{r}{R} \\
B &= AR(1-B) + B \implies B = 1
\end{aligned}$$

- Matching constants:

$$\begin{aligned}
e^{-\gamma k} &= e^{-\gamma(1-AR)k} \beta R \mathbb{E}_t \left[ e^{-\gamma B \varepsilon_{t+1}} \right] \\
e^{-\gamma k} &= e^{-\gamma \left[ (1-AR)k - \frac{\log(\beta R)}{\gamma} - \frac{\log \mathbb{E}_t \left[ e^{-\gamma B \varepsilon_{t+1}} \right]}{\gamma} \right]}
\end{aligned}$$

# Prudence: CARA Utility (4)

- Substitute $B = 1$ and $A = \frac{r}{R}$ and solve for $k$:

$$k = -\frac{1}{r}\left[\frac{\log(\beta R)}{\gamma} + \frac{\log \mathbb{E}_t\left[e^{-\gamma\varepsilon_{t+1}}\right]}{\gamma}\right]$$

- Recall $\varepsilon_{t|t-1} \sim \mathcal{N}\left(0, \sigma^2\right)$, then $\mathbb{E}_t\left[e^{-\gamma\varepsilon_{t+1}}\right] = e^{0\gamma + \frac{\gamma^2\sigma^2}{2}} = e^{\frac{\gamma^2\sigma^2}{2}}$

$$k = -\frac{1}{r}\left[\frac{\log(\beta R)}{\gamma} + \frac{\log e^{\frac{\gamma^2\sigma^2}{2}}}{\gamma}\right] = -\frac{1}{r}\left[\frac{\log(\beta R)}{\gamma} + \tfrac{1}{2}\gamma\sigma^2\right]$$

- Finally, we obtain the policy function:

$$c_t(a_t, y_t) = \frac{r}{R}a_t + y_t - \frac{1}{r}\left[\frac{\log(\beta R)}{\gamma} + \tfrac{1}{2}\gamma\sigma^2\right]$$

# Prudence: CARA Utility (5)

- With CARA utility and random walk income, consumption policy is given by:

$$c(a_t, y_t) = \frac{r}{R} a_t + y_t - \frac{1}{r}\left[\frac{\log(\beta R)}{\gamma} + \frac{1}{2}\gamma\sigma^2\right]$$

- Interpretation
  - Annuity value of $a_t$ (adjustment for $R$ because we changed the timing)
  - Consume all $y_t$ because it is permanent income (random walk).
  - Relative patience ($\frac{\log(\beta R)}{\gamma}$) decreases consumption, as usual.
  - Only difference with certainty equivalence: additive variance term $\frac{1}{2}\gamma\sigma^2$

# Prudence: Summing Up

- From closed-form solution (CARA utility) or second order Taylor approximation, we find that:

  (a) Precautionary savings increase with risk $\sigma^2$.

  (b) Precautionary saving increases with prudence ($\psi_t$).

- For commonly used utilities, risk aversion and prudence are closely related.

  - CARA: $\gamma$ is both absolute risk aversion and absolute prudence.
  - CRRA: $\gamma$ is relative risk aversion and $1 + \gamma$ is relative prudence.
  - Benefits: simple; Drawbacks: one parameter for different functions.

# Prudence: Further comments

- Quadratic utility is unrealistic: marginal utility equal to zero at a finite point.
- Positive third derivative is realistic.
  - The more we expect future consumption to fluctuate in response to shock, the higher is $u'(c_t) = \mathbb{E}_t(u'(c_{t+1}))$ in equilibrium.
  - We fear disaster in the future and we are willing to save in order to shield from its effect.
- Decreasing absolute risk aversion (DARA) implies $u''' > 0$. (Homework)
  - Precautionary motive becomes smaller and smaller as we become richer.
  - Consequence: $c_t$ is positively related to $a_t$, even conditional on the same expected future income.

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# Borrowing constraints: Idea (1)

- Borrowing constraints (BC) take the following form: $a_{t+1} \geq \underline{a}$
- **Why do BC they induce precautionary savings?**
- Let us abstract from prudence and focus on the quadratic case.
- Suppose there is a *no-borrowing constraint*: $a_{t+1} \geq 0$.
- Then the policy is given by:

$$c_t = \begin{cases} \mathbb{E}_t[c_{t+1}] & \text{if } a_{t+1} > 0 \quad \text{not binding, FOC holds} \\ y_t + a_t & \text{if } a_{t+1} = 0 \quad \text{binding, budget holds} \end{cases}$$

- We can write **optimal consumption plan:**

$$c_t = \min\{y_t + a_t, \mathbb{E}_t[c_{t+1}]\} = \min\{y_t + a_t, \mathbb{E}_t[\min\{y_{t+1} + a_{t+1}, \mathbb{E}_t[c_{t+2}]\}]$$

- The constraint household would like to borrow to finance $c_t$, but she is not allowed and thus consumes all her resources.

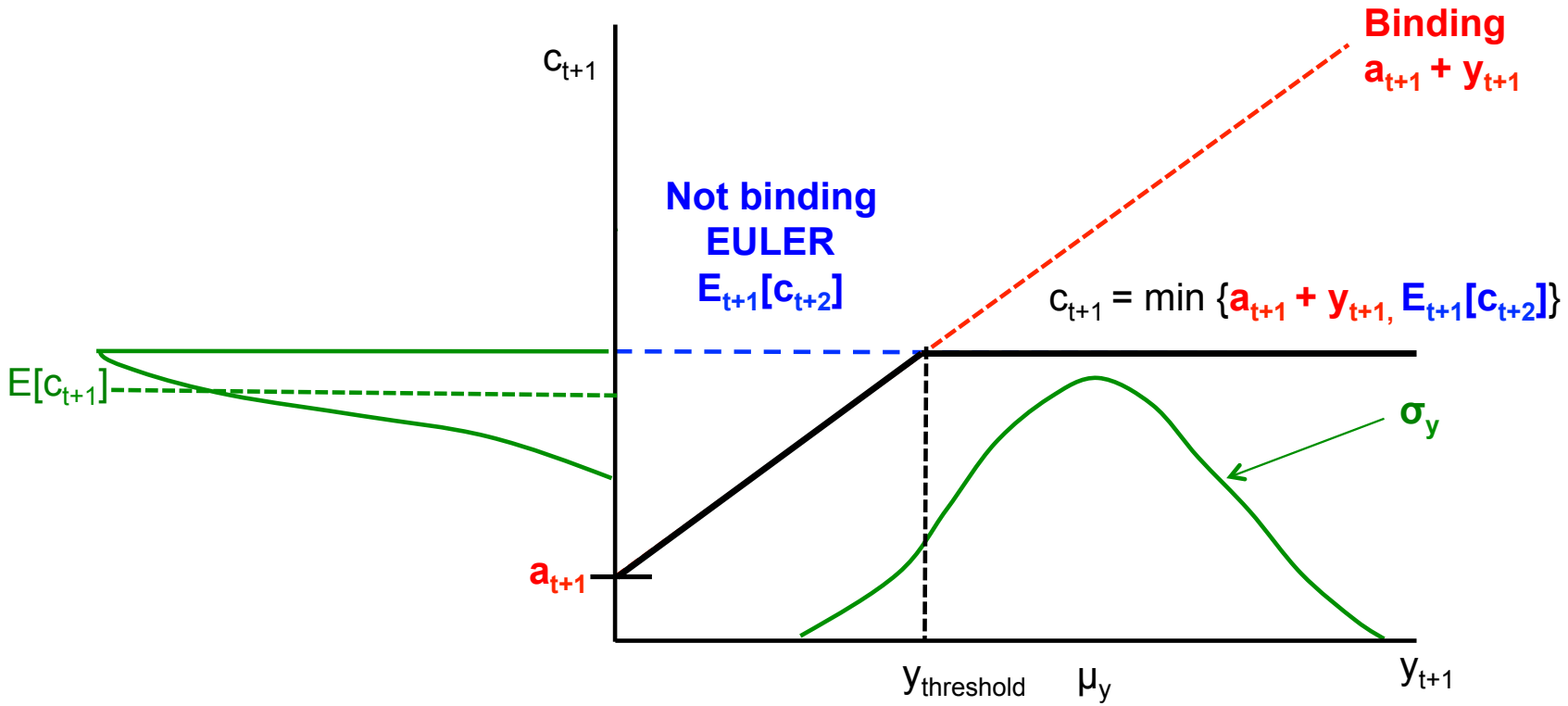
# Borrowing constraints: Idea (2)

- Let us start with the consumption plan in $t+1$:

$$c_{t+1} = \min\{y_{t+1} + a_{t+1}, \mathbb{E}_t[c_{t+2}]\}$$

- Then using the density of $y_{t+1}$, we find

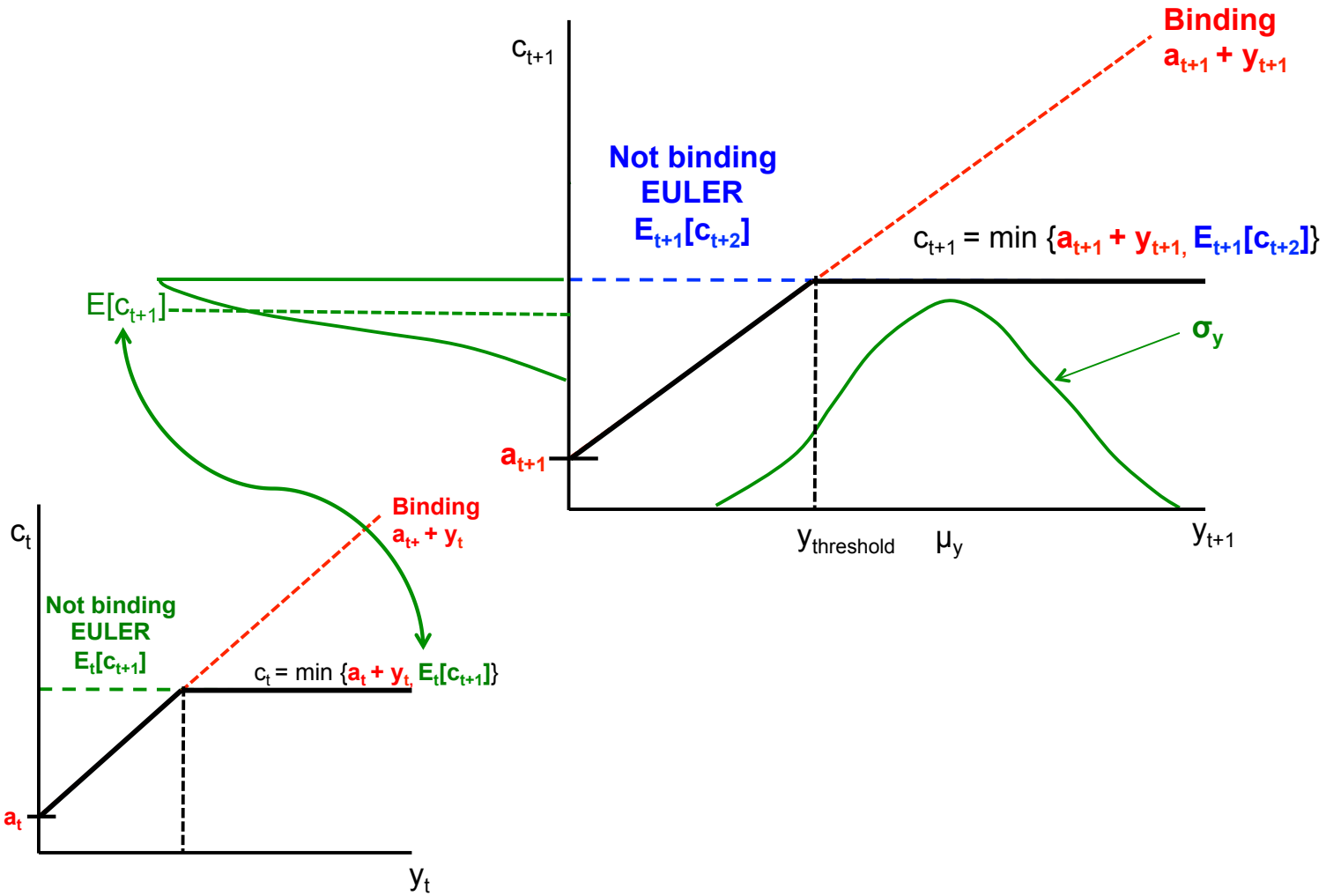
$$\mathbb{E}_t[c_{t+1}] = \mathbb{E}_t[\min\{y_{t+1} + a_{t+1}, \mathbb{E}_t[c_{t+2}]\}]$$



# Borrowing constraints: Idea (3)

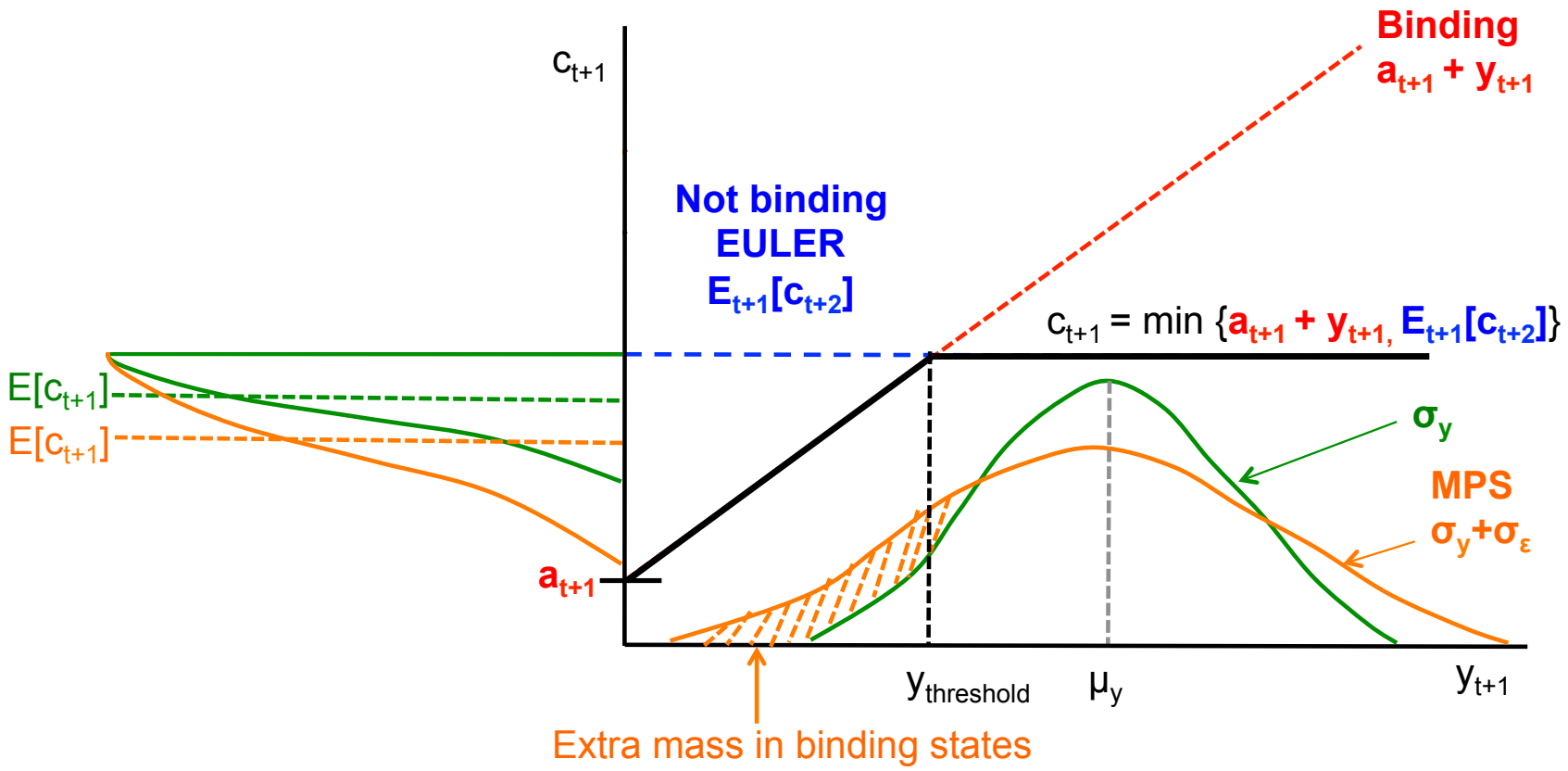
- Given expectation of future consumption, today's consumption is given by

$$c_t = \min\{y_t + a_t, \mathbb{E}_t[c_{t+1}]\}$$



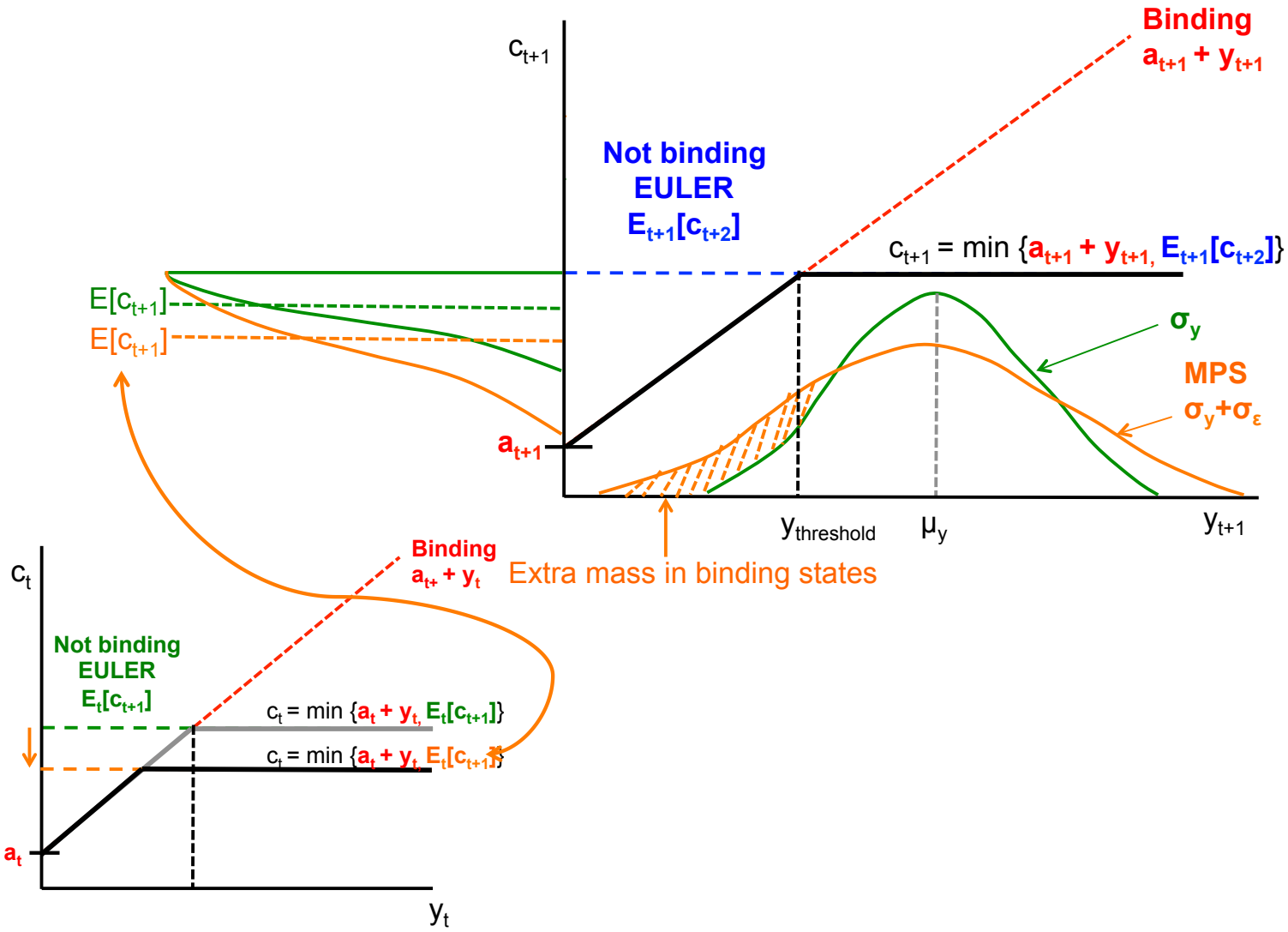
# Borrowing constraints: Idea (4)

- Suppose uncertainty about future $y_{t+1}$ increases (MPS) $\Longrightarrow$ low realizations of $y_{t+1}$ become more likely $\Longrightarrow$ borrowing constraint more likely to bind in the future $\Longrightarrow \mathbb{E}_t[\min\{y_{t+1} + a_{t+1}, \mathbb{E}_t[c_{t+2}]\}]$ falls $\Longrightarrow \mathbb{E}_t[c_{t+1}]$ falls



# Borrowing constraints: Idea (5)

- Today, agents reduce $c_t$ and increase savings for **precautionary motives!**



# Borrowing constraints: Idea (6)

- Agents fear getting consecutive bad income realizations that would push them towards constraint, destroying their ability to smooth consumption.
- Very similar to irreversibility in investment.
- **Result: Even without prudence, in the presence of borrowing constraints a rise in future income uncertainty leads to a rise in current savings.**

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# Natural borrowing constraint

- Is there a borrowing limit that the household naturally will not trespass?
- Let us start with the certainty case where $\{y_t\}_{t=0}^{\infty}$ is a deterministic sequence.
- **Impose** $c_t \geq 0$ **for all** $t$ and iterate forward the budget constraint:

$$c_t = Ra_t + y_t - a_{t+1} \geq 0 \quad \Longrightarrow \quad a_t \geq \frac{a_{t+1} - y_t}{R}$$

- Now substitute forward recursively and use the No Ponzi condition:

$$a_{t+1} \geq \frac{a_{t+2} - y_{t+1}}{R} \geq \frac{\frac{a_{t+3} - y_{t+2}}{R} - y_{t+1}}{R} \geq \frac{a_{t+3}}{R^2} - \frac{y_{t+2}}{R^2} - \frac{y_{t+1}}{R}$$

  which reduces to:

$$a_{t+1} \geq -\sum_{j=1}^{\infty} \frac{y_{t+j}}{R^j} \equiv \widetilde{a}_t \qquad \textbf{Natural Borrowing Constraint}$$

- The household is not allowed to accumulate more debt than what she will ever be able to repay by consuming zero every period thereafter.

# Natural borrowing constraint

- Now consider model with uncertainty.
- How can we be sure that household will repay almost surely (with prob 1)?
  - **Same restriction** $c_t \geq 0$**.**
  - Now $c_t$ has to be positive even if worst income realization happens:
  - Substitute $y_t$ with the lowest possible realization of income $y_{min}$

$$a_{t+1} \geq -\frac{y_{min}}{r} \equiv \widetilde{a} \qquad \textbf{Natural Borrowing Limit}$$

  - This can be very tight. If $y_{\min} = 0$ then $a_{t+1} \geq 0$ No borrowing!
- No exogenous borrowing constraint can ever be looser than the NBC.

# NBC and Inada Conditions

- What do we need for $c_t > 0$ and for the household to avoid the NBC?
  - Inada conditions! Utility satisfies $u(0) = -\infty$.
- Assume she does borrow up to $a_t = -\frac{y_{min}}{r}$ and precisely $y_t = y_{min}$.
- From budget constraint:

$$\begin{aligned} c_t &= Ra_t + y_t - a_{t+1} = -R\frac{y_{min}}{r} + y_{min} - a_{t+1} \\ &= \left(1 - \frac{R}{r}\right) y_{min} - a_{t+1} \leq \frac{y_{min}}{r} - \underbrace{\frac{y_{min}}{r}}_{\text{max that can be borrowed}} = 0 \end{aligned}$$

- Therefore $c_t \leq 0$ and $u(c_t) = -\infty$, never want to reach that state!
- **Message: With Inada conditions, the NBC will never bind (thus is it "natural") and can safely assume interior solutions for Euler Eq.**

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   - Intuition in two-period model
   - General results with Taylor approximation
   - **Closed form: CARA + Random walk income**
2. Borrowing constraints
   - Natural
   - **Ad-hoc**
3. Empirical evidence

# Ad-Hoc Borrowing Constraints (1)

- With Inada conditions, the NBC does not bind.
- Other type of ad-hoc or exogenous limits, where $\underline{a} > \widetilde{a}$ may bind:

$$a_{t+1} \geq \underline{a}$$

- Now we show precautionary behaviour in a general setup.
- The **problem with borrowing constraints** is given by:

$$\max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_t \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

$$\begin{aligned} a_{t+1} &= Ra_t + y_t - c_t && (\lambda_t) \\ a_{t+1} &\geq \underline{a} && (\mu_t) \end{aligned}$$

where $\lambda_t$ and $\mu_t$ are the Lagrange multipliers respectively.

- How do we interpret $\mu_t$?

# Ad-Hoc Borrowing Constraints (2)

- The Lagrangian is given by:

$$\max \mathcal{L} = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[ u\left(c_t\right) + \lambda_t \left[ R a_t - c_t + y_t - a_{t+1} \right] + \mu_t \left( a_{t+1} - \underline{a} \right) \right]$$

- FOC:

$$\begin{aligned} (c_t) \qquad & u'\left(c_t\right) = \lambda_t \\ (a_{t+1}) \qquad & \mu_t + \beta R \mathbb{E}_t \left(\lambda_{t+1}\right) = \lambda_t \end{aligned}$$

and putting them together:

$$u'\left(c_t\right) = \mu_t + \beta R \mathbb{E}_t \left[ u'\left(c_{t+1}\right) \right]$$

# Ad-Hoc Borrowing Constraints (3)

- Let us consider two cases:
- If we have enough wealth to consume optimally, then we are not liquidity constrained
  - $a_{t+1} > \underline{a}$
  - Since the liquidity constraint is not binding, it follows that $\mu_t = 0$ :

$$u'(c_t) = \beta R \mathbb{E}_t [u'(c_{t+1})]$$

- If we do not have enough wealth and we cannot borrow, then we are liquidity constrained:
  - $a_{t+1} = \underline{a}$
  - Since the liquidity constraint is binding, it follows that $\mu_t > 0$ :

$$u'(c_t) > \beta R \mathbb{E}_t [u'(c_{t+1})]$$

# Ad-Hoc Borrowing Constraints (4)

- With binding constraint, $u'(c_t)$ is too high because $c_t$ is constrained by available wealth:

$$c_t = Ra_t + y_t - a_{t+1} \Longrightarrow c_t = Ra_t + y_t - \underline{a}$$

  - Euler equation determines the value of $\mu_t$ (the shadow value of an additional unit of wealth)

$$\mu_t = u'(c_t) - \beta R \mathbb{E}_t \left[ u'(c_{t+1}) \right] > 0$$

  - The lower is wealth, the lower is $c_t$ (relative to $c_{t+1}$), the higher is $\mu_t$.

- **The more liquidity constrained today, the higher is $\mu_t$, the lower is $c_t$.**

$$u'(c_t) = \mu_t + \beta R \mathbb{E}_t \left[ u'(c_{t+1}) \right]$$

# Ad-Hoc Borrowing Constraints (5)

- Hence a positive **liquidity shock** (for example a reduction in $\underline{a}$) reduces $\mu_t$ and increases $c_t$.
  - PIH is violated, liquidity shocks matter for consumption.
- **Future expected financing constraints reduce consumption today (a form of precautionary saving).**
  - Assume $\beta R = 1$ and substitute recursively forward the Euler Eq:

$$u'(c_t) = \mu_t + \mathbb{E}_t \left[\mu_{t+1} + \mu_{t+2} + ... + \mu_{t+j} + ...\right]$$

  - Even if $\mu_t$ is equal to zero (not binding today), $u'(c_t)$ may be higher as a consequence of future expected financing problems.
  - Then consumption falls today.
  - Again, recall firm investment behavior with irriversibilities.

# Roadmap: Precautionary Savings

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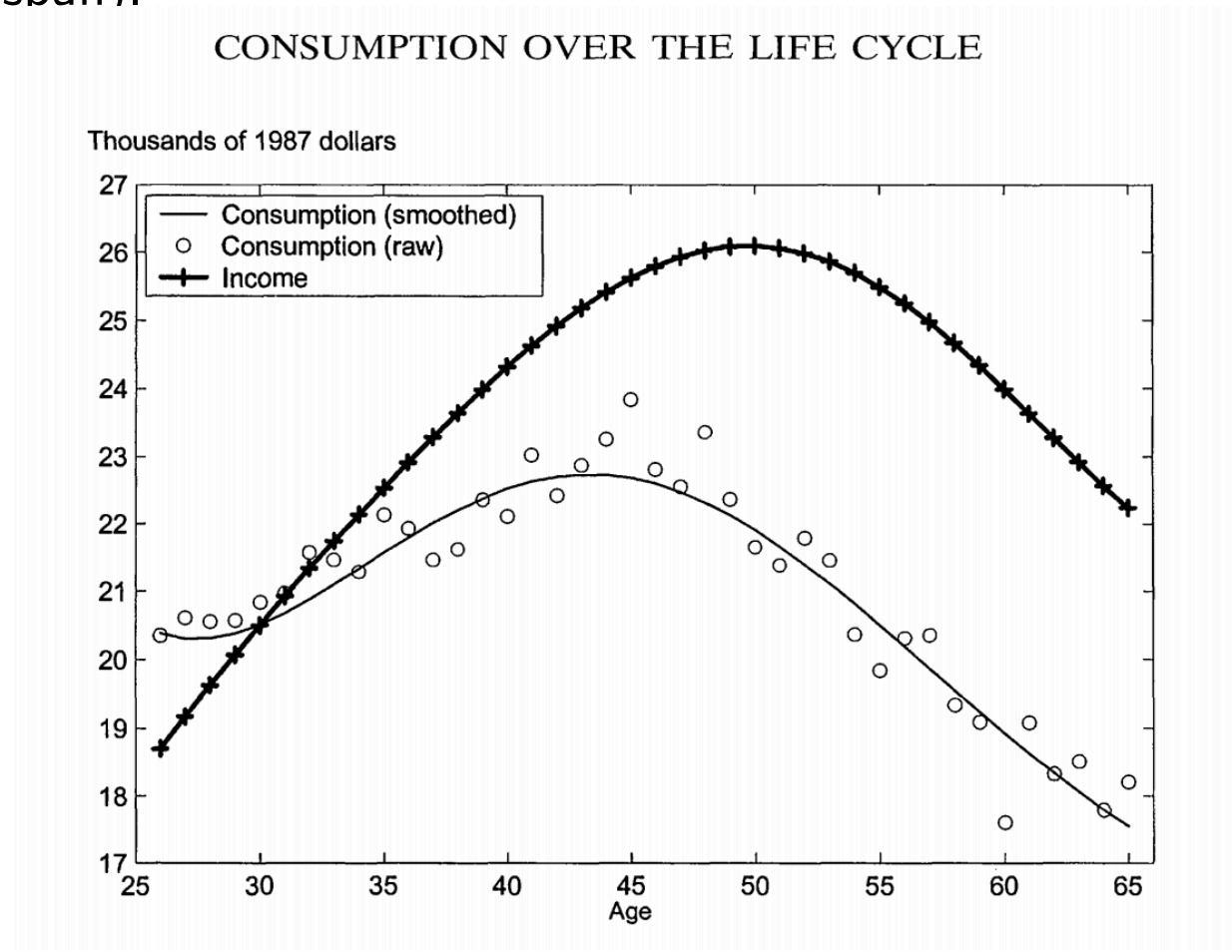
# Precautionary Savings: Empirical Evidence (1)

- Precautionary savings may explain life-cycle behavior:
  - young people do not borrow (want to build up a buffer-stock)
  - old people do not dissave much (want protection against uncertain life-span).



# Precautionary Savings: Empirical Evidence (2)

- Carrol (2001, JEP) "A Theory of the Consumption Function, with and without Liquidity Constraints".
- Test impact of liquidity constraints with a realistically calibrated model:

$$Y_t = Y_t^p \varepsilon_t$$

$$Y_t^p = G Y_{t-1}^p$$

- Transitory shock $\varepsilon_t$ is asymmetric:

$$\log\ \varepsilon_t \sim \begin{cases} \mathcal{N}(0, 0.04) & \text{with prob } p = 0.995 \\ 0 & \text{with prob } p = 0.005 \end{cases}$$

- Benchmark: CRRA with $\gamma = 2$, $\beta = 0.96$, $R = 1.04$ and $G = 1.03$.

# Precautionary Savings: Empirical Evidence (3)

**Steady-State Statistics For Alternative Consumption Models**

| *Income Growth Factor* | *Mean w* | *Median w* | *Aggregate Consumption Growth* | *Mean MPC* | *Frac With w < 0* | *Frac With w = 0* |
|---|---|---|---|---|---|---|
| | | | Panel A: Baseline Model, No Constraints | | | |
| $G = 1.03$ | 0.43 | 0.40 | 1.030 | 0.330 | 0.000 | 0.000 |
| $G = 1.02$ | 0.52 | 0.48 | 1.020 | 0.276 | 0.000 | 0.000 |
| $G = 1.00$ | 2.26 | 2.06 | 1.000 | 0.064 | 0.000 | 0.000 |
| | | | Panel B: Strict Liquidity Constraints | | | |
| $G = 1.03$ | 0.28 | 0.24 | 1.030 | 0.361 | 0.000 | 0.070 |
| $G = 1.02$ | 0.36 | 0.32 | 1.020 | 0.301 | 0.000 | 0.051 |
| $G = 1.00$ | 2.28 | 2.06 | 1.000 | 0.065 | 0.000 | 0.000 |
| | | | Panel C: Borrowing up to 0.3 Allowed | | | |
| $G = 1.03$ | −0.03 | −0.06 | 1.030 | 0.361 | 0.611 | 0.000 |
| $G = 1.02$ | 0.06 | 0.01 | 1.020 | 0.299 | 0.478 | 0.000 |
| $G = 1.00$ | 1.94 | 1.71 | 1.000 | 0.064 | 0.023 | 0.000 |
| | | | Panel D: Borrowing up to 0.3 at $R = 1.15$ Allowed | | | |
| $G = 1.03$ | 0.11 | 0.07 | 1.030 | 0.327 | 0.320 | 0.058 |
| $G = 1.02$ | 0.21 | 0.16 | 1.020 | 0.274 | 0.210 | 0.046 |
| $G = 1.00$ | 2.11 | 1.89 | 1.000 | 0.064 | 0.007 | 0.002 |
| | | | Panel E: Statistics from the 1995 SCF | | | |
| — | 1.02 | 0.29 | — | — | 0.205 | 0.025 |

*Notes:* Results in Panels A through D reflect calculations by the author using simulation programs available at the author's website, ⟨http://www.econ.jhu.edu/people/carroll/ccaroll.html⟩. In Panel A, no constraint is imposed, but income can fall to zero, which prevents consumers from borrowing. In Panels B through D, the worst possible event is for income to fall to half of permanent income. For

# Precautionary Savings: Empirical Evidence (4)

- Early micro studies found that precautionary savings [PS] contribute for as much as 50% of aggregate wealth for individuals below the age of 50.
  - Carrol and Samwick (1997,1998) and Carrol, Dynan and Krane (2003).
- Most of this aggregate wealth is accumulated by entrepreneurial households, who face much more income risk than non-entrepreneurial ones.
- Entrepreneurial households may have motives unrelated to precautionary saving (they do not have a pension; then may have different preferences,...).
- Hurst et al (Restat 2010) estimate PS separately for these two groups, and find that PS contributes to only around 10% of aggregate wealth.

# Empirical tests of PIH (Discussion 1)

**Johnson, Parker and Souleles (AER, 2006) Household Expenditure and the Income Tax Rebates of 2001.**

1. What is the question and its importance? Answer?
2. Why is this approach different from other papers that have asked this question?
3. Data adequate? Reasonable measure of what the author claims to be measuring?
4. Source of the data clear? Manipulations and cleaning?
5. Methodology adequate for answering the question?
6. Clear mapping from the data to the model and vice-versa?
7. Measurement error? If so, is this addressed or considered?
8. Reverse causality, endogeneity problem? Approach to achieve identification?
9. Conclusions tightly linked to the data? Overstatement results?
10. Do you like the paper?

# Empirical tests of PIH (Discussion 2 )

**Hsieh (AER, 2003). Do Consumers React to Anticipated Income Changes? Evidence from the Alaska Permanent Fund.**

1. What is the question and its importance? Answer?
2. Why is this approach different from other papers that have asked this question?
3. Data adequate? Reasonable measure of what the author claims to be measuring?
4. Source of the data clear? Manipulations and cleaning?
5. Methodology adequate for answering the question?
6. Clear mapping from the data to the model and vice-versa?
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