Advanced Macroeconomics II

Lecture 5

Consumption: Permanent Income Hypothesis

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Motivation

• Consumption is a large fraction (70%) of aggregate output.

• Households derive direct utility from consumption
  ⇒ key determinant of welfare, both at short and long run.

• Savings vary a lot over the life cycle.
  ⇒ key to understand investment in the long run (e.g. ageing populations)

• Relatively stable in the business cycle.
  ⇒ still key for business cycle fluctuations

• Consumption depends on real interest rates.
  ⇒ key to understand impact of monetary policy

• Consumption is affected by fiscal instruments (consumption tax, income tax through its effects on labor supply, etc)
  ⇒ key to understand impact of fiscal policy
A little bit of history (1)

- **Keynesian consumption function (1936)**
  - Consumption is a constant fraction of disposable income.
    \[ C_t = \alpha Y_t, \quad \alpha \in (0, 1) \]
  - Assumption of Solow’s growth model.
  - Implication: Big role for government stimulus.

- **Friedman’s Permanent Income Hypothesis [PIH] (1957)**
  - Individual consumption tracks permanent income, which is the “normal” level of income.
    \[
    \begin{align*}
    c_{i,t}^P &= \alpha_t Y_{i,t}^P \\
    Y_{i,t} &= Y_{i,t}^P + \epsilon_{i,t}^Y \\
    c_{i,t} &= c_{i,t}^P + \epsilon_{i,t}^C
    \end{align*}
    \]
  - Implication: Smaller role for government stimulus.
A little bit of history (2)

- Ando and Modigliani’s Life Cycle Theory (1963)
  - Savings smooth consumption over the life cycle while income varies.
  
  \[ c_{i,t} = \alpha_t, a Y_{i,t}, \quad a = \text{age} \quad i = \text{individual} \]

- Afterwards, microfoundation of consumption optimization problem.
  - PIH is a special case of a general framework.
  - Adding uncertainty, heterogenous agents, borrowing constraints, GE ...
Roadmap

1. Facts about aggregate and household level consumption

2. Consumption-savings problem with certainty
   - Euler Equation and Consumption Smoothing
   - Full Solution
   - Application: Ricardian Equivalence

3. Consumption-savings problem with uncertainty
   - Special case: Quadratic utility
   - Propensity to consume and income persistence

4. Empirical evidence
Facts about consumption

- Fact 1: Consumption is a large fraction of aggregate output.
  - Focus on consumption of non-durables + services (≈ 60% of output).
  - Later we will study durables.
Facts about consumption (cont...)

- **Fact 2:** Consumption is less volatile than output.

- **Fact 3:** Relatively low correlation of consumption with output.


<table>
<thead>
<tr>
<th>Variable</th>
<th>St Dev (%)</th>
<th>Correlation with GNP</th>
</tr>
</thead>
<tbody>
<tr>
<td>GNP</td>
<td>1.72</td>
<td>1</td>
</tr>
<tr>
<td>Consumption (non-durables)</td>
<td>0.86</td>
<td>0.77</td>
</tr>
<tr>
<td>Investment (gross private domestic)</td>
<td>8.24</td>
<td>0.91</td>
</tr>
<tr>
<td>Hours Worked</td>
<td>1.59</td>
<td>0.86</td>
</tr>
</tbody>
</table>

Cooley and Prescott (1996)

- This facts point towards **aggregate consumption smoothing.**
Aside: Hodrick-Prescott Filter (1997)

- Two sided filter that allows for isolating cyclical component of a time series.
- Let \( y_t = \log Y_t \), which has a trend \( \tau_t \) and a cyclical component \( y_t^c \):
  \[
  y_t = \tau_t + y_t^c + \varepsilon_t
  \]
- Trend solves following problem for a given \( \lambda \) (1600 for quarterly data):
  \[
  \min_{\{\tau_t\}_{t=1}^T} \left( \sum_{t=1}^{T} (y_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1} \left[ (\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1}) \right]^2 \right)
  \]
- Other filters: Baxter and King, BandPass filters.
Facts about consumption (cont...)

- Fact 4: Within household’s life cycle, the correlation of consumption and income is large and positive during early and late years, and negative in the middle.

Source: Gourinchas & Parker (2002), “Consumption over the Life Cycle”. CES data for US.
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4. Empirical evidence
The building blocks

- **Time:** Discrete and infinite $t = 0, 1, 2, \ldots, \infty$.

- **Many identical optimizing households who live forever**
  - Choosing consumption sequence to maximize utility $U(\{c_{t}\}_{t=0}^{\infty})$.
  - Dynasty where current generations care about future ones (bequests).
  - Discrete time version of the consumption problem in Ramsey model.

- **Endowments:**
  - Labor income $\{y_{t}\}_{t=0}^{\infty}$ is a non-stochastic sequence.
  - Initial wealth $a_{0}$.

- **Financial markets:** borrowing and saving through a one period risk-free bond with return $R = 1 + r$ (with certainty, one asset is complete markets).

- **Partial equilibrium approach:**
  - No production sector and no population growth.
  - Interest rate $R$ fixed, and labor income $y$ is exogenous.
The building blocks

- Preferences:
  \[ U = \max \sum_{t=0}^{\infty} \beta^t u(c_t) \]

1. Stationarity: \( u_s(\cdot) = u(\cdot) \) and \( \beta_s = \beta \) for all periods \( s \).
2. Homogeneity: \( u^i = u \) and \( \beta^i = \beta \) for all households \( i \).
3. No habit formation: \( u \) at \( t \) only depends on consumption at \( t \).
4. Monotonicity and Risk Aversion: \( u' > 0, u'' < 0 \)

- Later on, we change assumptions:
  - \( u''' > 0 \iff \) Prudence (precautionary saving)
  - Habits
  - Hyperbolic discounting
Net assets and budget constraint

- Define net assets $a_{t+1}$ at the beginning of period $t + 1$:
  
  $$a_{t+1} = Ra_t - c_t + y_t \quad \text{with initial condition } a_0 \text{ given}$$

- Implicit timing assumption:
  1. Begin period with net assets $a_t$
  2. Receive interest payments $r a_t$ and income $y_t$
  3. Decide consumption $c_t$ (or equivalently net assets for next period $a_{t+1}$).

- We can explore different timings, but no big changes in results.

- The maximization problem is then:
  
  $$V_0 = \max_{\{c_t\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u(c_t)$$

  subject to: 
  
  $$a_{t+1} = Ra_t - c_t + y_t, \quad a_0 \text{ given}$$
What about transversality conditions?

- Substituting the budget constraint recursively forward, starting from \( t = 0 \):
  \[
  a_0 = \frac{1}{R} (a_1 + c_0 - y_0) = \frac{1}{R} \left( \frac{1}{R} (a_2 + c_1 - y_1) + c_0 - y_0 \right) = \ldots
  \]

- Moving incomes to LHS and consumptions to RHS:
  \[
  a_0 + \frac{1}{R} \left( y_0 + \frac{y_1}{R} + \ldots + \frac{y_t}{R^t} \right) - \frac{a_{t+1}}{R^{t+1}} = \frac{1}{R} \left( c_0 + \frac{c_1}{R} + \ldots + \frac{c_t}{R^t} \right)
  \]

- Keep substituting and take limits:
  \[
  Ra_0 + \sum_{j=0}^{\infty} \left( \frac{1}{R} \right)^j y_j - \lim_{t \to \infty} \frac{a_{t+1}}{R^{t+1}} = \sum_{j=0}^{\infty} \left( \frac{1}{R} \right)^j c_j
  \]

- If allowed to increase its borrowing over time, the household will choose \( a_{t+1} \) equal to negative infinity, and consumption equal to infinity.
No Ponzi schemes

- To eliminate this possibility a “No Ponzi game” condition is imposed:

\[
\lim_{t \to \infty} \frac{a_{t+1}}{R_{t+1}} \geq 0
\]

- This condition is an exogenous constraint, not an optimality condition necessary for the maximization.

- Since it is then not optimal for \( a_t \) to grow too large, the condition above must hold with exact equality for the household to maximise utility.

\[
\lim_{t \to \infty} \frac{a_{t+1}}{R_{t+1}} = 0
\]

- For now we abstract from borrowing constraints and only use “No Ponzi”.
Different borrowing constraint

- Ad-hoc borrowing constraints (usually binding):
  - Borrowing only up to some value $B \geq 0$:
    \[ a_{t+1} \geq -B \]
  - Extreme case (no borrowing, only saving)
    \[ a_{t+1} \geq 0 \]

- Natural borrowing constraint (NBC):
  \[ a_{t+1} \geq \tilde{a}_t \quad \text{where} \quad \tilde{a}_t \equiv -\sum_{j=1}^{\infty} \frac{y_{t+j}}{R_j} \]

Why is it “natural” not to borrow more than the stream future income? More on this later.
Back to the problem

- Reformulate the sequential problem in terms of $a_{t+1}$ by substituting the budget constraint:

$$V_0(a_0) = \max_{\{a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(Ra_t + y_t - a_{t+1})$$

- FOC: $-u'(c_t) + \beta Ru'(c_{t+1}) = 0$.

- Alternatively, use the recursive problem

$$V(a, y) = \max_{a'} u(Ra + y - a') + \beta V(a', y')$$

- FOC: $-u'(c) + \beta \frac{\partial V(a', y')}{\partial a'} = 0$

- Envelope: $\frac{\partial V(a, y)}{\partial a} = Ru'(c)$

- FOC + Forward Envelope: $-u'(c) + \beta Ru'(c') = 0$. 
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4. Empirical evidence
The Euler equation for this problem is given by:

\[ u'(c_t) = \beta R u'(c_{t+1}) \]

Interpretation:
- Let \( \rho = \) subjective intertemporal discount rate, such that \( \beta = \frac{1}{1+\rho} \).
- \( \rho = 0.02 \) means consuming tomorrow rather than today reduces \( u \) by 2%.

Euler Eq can be written comparing subjective and objective discounts:

\[ \frac{u'(c_t)}{u'(c_{t+1})} = \frac{1 + r}{1 + \rho} \]

Since \( u''(c) < 0 \), then
- If \( r = \rho \), then perfect smoothing \( c_0 = c_1 = \ldots = c_t = c_{t+1} \).
- If \( r > \rho \), intertemporal saving motive: \( c_t < c_{t+1} \).
- If \( r < \rho \), intertemporal borrowing motive: \( c_t > c_{t+1} \).
Euler Equation and Consumption Smoothing (2)

- Example: Constant Relative Risk Aversion (CRRA) Utility

\[
  u(c_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma}
\]

\[
  u'(c_t) = c_t^{-\gamma}
\]

- Euler equation:

\[
  \frac{c_t^{-\gamma}}{1 + \rho} = 1 + r
  \implies
  \frac{c_{t+1}}{c_t} = \left(\frac{1 + r}{1 + \rho}\right)^{\frac{1}{\gamma}}
\]

- Denote consumption growth with \( g^c = \log \left( \frac{c_{t+1}}{c_t} \right) \).

- Then Euler implies a constant trend:

\[
  g^c = \frac{1}{\gamma} \log \left( \frac{1 + r}{1 + \rho} \right) \approx \frac{r - \rho}{\gamma}
\]
More in general, we can linearize the FOC.

\[ u'(c_t) = \frac{1 + r}{1 + \rho} u'(c_{t+1}) \]

A first order linear approximation of the RHS around \( c_t \):

\[ u'(c_t) \approx \frac{1 + r}{1 + \rho} \left[ u'(c_t) + u''(c_t)(c_{t+1} - c_t) \right] \]

Dividing both sides by \( u'(c_t) \) and multiply and divide by \( c_t \) rearranging:

\[
1 \approx \frac{1 + r}{1 + \rho} \left[ 1 + \frac{u''(c_t)}{u'(c_t)} (c_{t+1} - c_t) \right] \approx \frac{1 + r}{1 + \rho} \left[ 1 + \frac{u''(c_t) c_t}{u'(c_t) c_{t+1} - c_t} \right]
\]

where the coefficient of relative risk aversion is defined as:

\[ \gamma(c_t) \equiv -\frac{u''(c_t) c_t}{u'(c_t)} \]
Euler Equation and Consumption Smoothing (4)

- Under the linear approximation, consumption growth is given by:

\[-\gamma(c_t) \frac{c_{t+1} - c_t}{c_t} \approx \frac{1 + \rho}{1 + r} - 1\]

\[\frac{c_{t+1} - c_t}{c_t} \approx \frac{r - \rho}{R\gamma(c_t)}\]

- Consumption trend depends on \(\rho\), \(r\) and \(\gamma(c_t)\).

- Under the example of CRRA utility: \(\gamma(c_t) = \gamma\). Proof.

\[u(c_t) = c_t^{\frac{1-\gamma}{1-\gamma}}; ~ u'(c_t) = c_t^{-\gamma}; ~ u''(c_t) = -\gamma c_t^{-(1+\gamma)}\]

\[-\frac{u''(c_t) c_t}{u'(c_t)} = \frac{\gamma c_t^{-(1+\gamma)} c_t}{c_t^{-\gamma}} = \gamma\]

- Trend becomes constant: \(\frac{r - \rho}{R\gamma}\)

- Consumption is smoothed over time, even without uncertainty.
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4. Empirical evidence
• Solution of the problem: consumption as a function of $a_0, \{y_t\}_{t=0}^{\infty}$. Close form solution in the deterministic case.

• Solution: Euler equation + budget constraint

• So we have two equations:

$$\sum_{t=0}^{\infty} \left( \frac{1}{R} \right)^t c_t = Ra_0 + \sum_{t=0}^{\infty} \left( \frac{1}{R} \right)^t y_t$$

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta R$$
Solution with constant consumption (perfect smoothing)

• Let us impose \( r = \rho \) or equivalently \( \beta R = 1 \).
  
  ▶ This is a sensible assumption: in GE, with homogeneous agents, the steady state implies constant consumption.

• From EE it yields immediately \( c_t = \bar{c} \ \forall t \).

• It implies that the LHS of the budget reads:

\[
\sum_{t=0}^{\infty} \left( \frac{1}{R} \right)^t c_t = \bar{c} \sum_{t=0}^{\infty} \left( \frac{1}{R} \right)^t = \frac{\bar{c}}{1 - \frac{1}{R}} = \bar{c} \frac{R}{r}
\]

• Therefore:

\[
\bar{c} = \frac{r}{1 + r} \left[ Ra_0 + \sum_{t=0}^{\infty} \left( \frac{1}{R} \right)^t y_t \right] \equiv y_p
\]

• Consumption is constant since the value of \( w_0^p \) is perfectly known at \( t = 0 \).
Permanent Income Hypothesis

• This is the *permanent income hypothesis (PIH)*.

\[ c_t = \bar{c} = y^p = \frac{r}{1 + r} w_0^p \text{ for any } t \]

• Consumption equals permanent income, or the annuity value of wealth.

• An increase in current income has a very minor effect in current consumption, because the increase is spent evenly during all lifetime.

\[
\frac{\partial c_t}{\partial y_t} = \frac{\partial c_t}{\partial y_t} \frac{\partial y_t^p}{\partial y_t} = 1 \frac{\partial y_t^p}{\partial y_t} = \frac{r}{1 + r}
\]

where the last equality can be obtained from:

\[
y_t^p = \frac{r}{1 + r} w_t^p = \frac{r}{1 + r} \left[ Ra_t + y_t + \sum_{j=1}^{\infty} \left( \frac{1}{1 + r} \right)^j y_{t+j} \right]
\]

• Hence: \( \frac{\partial c_t}{\partial y_t} = \frac{r}{1 + r} < 1 \)
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The Ricardian Equivalence (1)

- Imagine a government has to finance a given amount of spending.

- Does the timing of taxation matter?

- Does it matter whether

  a) it sets taxes today or

  b) it issues debt and collects taxes in the future to pay the debt back?

- Ricardian Equivalence: No, it does not matter.
The Ricardian Equivalence (2)

- In the Keynesian model, temporary tax cuts can have a large stimulating effect on demand.

- PIH suggests that a consumer will spread out the gains from a temporary tax cut over a long horizon, and so the stimulus effect will be much smaller.

- Furthermore, PIH implies that an increase in income today that is accompanied by a decrease of income in the future with the same present value (⇒ the permanent income does not change) will not change consumption decisions.

- Ricardian Equivalence is consistent with PIH.

- Let’s add a government to the previous model.
The Ricardian Equivalence (3)

- The government collects lump sum taxes $\tau_t$ and issues debt $B_t$ to finance a given stream of public spending $\{g_t\}_{t=0}^{\infty}$.
- The government has its own budget constraint:

$$RB_t + g_t = B_{t+1} + \tau_t$$

- Substituting forward recursively:

$$B_t = \frac{1}{R} \left( \frac{1}{R} \left( B_{t+2} + \tau_{t-1} - g_{t-1} \right) + \tau_t - g_t \right)$$

- After imposing a transversality condition, obtain the intertemporal budget:

$$\sum_{j=0}^{\infty} \frac{g_{t+j}}{R^j} = \sum_{j=0}^{\infty} \frac{\tau_{t+j}}{R^j} - RB_t$$

The present value of government spending has to be equal to its *paying capacity*, which is equal to the present value of taxes minus debts.
The Ricardian Equivalence (3)

• The household budget constraint becomes:

\[ c_t + a_{t+1} = (y_t - \tau_t) + Ra_t \]

The corresponding intertemporal budget constraint will be:

\[ \sum_{j=0}^{\infty} \frac{c_{t+j}}{R^j} = Ra_t + \sum_{j=0}^{\infty} \frac{y_{t+j} - \tau_{t+j}}{R^j} \]

• If we impose constant consumption and we solve, we get:

\[ c_t = r \left\{ a_t + \frac{1}{R} \left[ \sum_{j=0}^{\infty} \frac{y_{t+j}}{R^j} - \sum_{j=0}^{\infty} \frac{\tau_{t+j}}{R^j} \right] \right\} \]

Timing of the taxes does not matter, only the NPV of taxes matters.
Ricardian Equivalence: Failure

1 Capital Market imperfections
   ▶ If there are borrowing constraints.

2 Finite lifetimes
   ▶ What if a cut of taxes today comes with higher taxes in 30 years time, once a big fraction of today’s population is already dead?

3 Distortionary taxation
   ▶ If taxes are distortionary, then there exist an optimal intertemporal profile of taxes (the solution to the Ramsey problem).

4 Income uncertainty
   ▶ If changes in government taxation alter the degree of uncertainty that households face in their income.
Ricardian Equivalence: Why bother?

- The Ricardian equivalence has value as a reference point.
  - It defines the conditions we need for the equivalence to hold and we can easily work out cases when it does not.
  - Having in mind the Ricardian equivalence (or PIH) when we look at the real world is a good framework to understand reality.

- From the policy point of view, it is a quantitative issue.

- Are the departures quantitatively important?

- Two papers to read for next week in Box.
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Adding uncertainty

- Basic problem same as before.

- Two potential sources of exogenous uncertainty:
  - Labor income: \( \{y_t\}_{t=0}^{\infty} \) is stochastic (we focus on this now).
  - Capital income \( \{R_t\}_{t=0}^{\infty} \) is stochastic (later with asset pricing).

- First we focus on stochastic labor income and assume a stationary process.

- Only asset that can be traded is a one period bond with return \( R \).
  - Incomplete financial markets: With a continuum of states, a one-period bond is not enough to move wealth across future states.

- Interest rate \( R_t = R \) is still exogenous and constant over time.
  - GE with aggregate uncertainty: \( R_t \) is stochastic and correlated to \( y_t \).
Stationary Income Process

- A process \( \{y_t\}_{t=0}^{\infty} \) is strictly stationary if the joint probability distribution does not change when shifted in time:
  \[
  F(y_{t_1}, y_{t_2}, \ldots, y_{t_k}) = F(y_{t+\tau}, y_{t_1+\tau}, \ldots, y_{t_k+\tau}), \quad \forall k, \tau, t_1, t_2, \ldots
  \]

- It implies 2nd order stationarity: unconditional mean \( \mathbb{E}[y_t] = \mu_y \), variance \( \mathbb{V}[y_t] = \sigma_y^2 \) and autocovariance \( \text{Cov}[y_t, y_{t+\tau}] = \gamma(\tau) \) are finite and invariant.

- For example, a stationary process for income could be an AR(1) structure:
  \[
  y_t = (1 - \rho)\bar{y} + \rho y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim iid(0, \sigma_{\varepsilon}^2)
  \]

  Homework: \( \mu_y = \bar{y} \), \( \gamma(\tau) = \rho \gamma(\tau - 1), \gamma(0) = \sigma_y^2 = \frac{\sigma_{\varepsilon}^2}{1 - \rho^2} \)

- Cases:
  - If \( \rho = 1 \), \( y_t \) is a random walk
  - if \( \rho = 0 \), \( y_t \) is iid
  - if \( \rho \in (0, 1) \), \( y_t \) is persistent
Consumption-saving problem with uncertainty

• The problem:

$$\max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

subject to

$$a_{t+1} = Ra_t + y_t - c_t, \quad a_0 \text{ given}$$

$$y_t = (1 - \rho)\bar{y} + \rho y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim iid(0, \sigma_\varepsilon^2)$$

$$\lim_{t \to \infty} \mathbb{E}_0 \left[ \frac{a_{t+1}}{R^{t+1}} \right] \geq 0$$

• Timing:

1. Start period $t$ with net assets $a_t$.
2. Receive interest payments $ra_t$.
3. $\varepsilon_t$ is realized, and household learns realization of $y_t$
4. Consumption $c_t(a_t, y_t)$ is decided.
5. Residual wealth $a_{t+1}$ is invested (if positive) or borrowed (if negative).

• For now we assume there are no borrowing constraints or they don’t bind.
Recursive problem with uncertainty

- Problem in recursive form (substituting budget constraint):

\[ V(a_t, y_t) = \max_{a_{t+1}} u(Ra_t + y_t - a_{t+1}) + \beta \mathbb{E}_t [V(a_{t+1}, y_{t+1})|y_t] \]

- Note: \( y_t \) is a state variable for two reasons:
  - it determines disposable income \( Ra_t + y_t \)
  - it provides information on future income when \( \rho \neq 0 \)

- FOC & Envelope & Forward Envelope:

\[
\begin{align*}
  u'(c_t) &= \beta \mathbb{E}_t \left[ \frac{\partial V(a_{t+1}, y_{t+1})}{\partial a_{t+1}} \right] \\
  \frac{\partial V(a_t, y_t)}{\partial a_t} &= Ru'(c_t) \implies \frac{\partial V(a_{t+1}, y_{t+1})}{\partial a_{t+1}} = Ru'(c_{t+1})
\end{align*}
\]

- Euler equation:

\[ u'(c_t) = \beta R \mathbb{E}_t [u'(c_{t+1})] \]
Policy

- The solution is a policy function $c_t = c(a_t, y_t)$ that satisfies:
  
  - Bellman equation:
    
    $$c(a_t, y_t) = \arg\max u[c(a_t, y_t)] + \beta E_t[V(a_{t+1}, y_{t+1})]$$
    
  - Feasibility:
    
    $$a_{t+1} = Ra_t + y_t - c(a_t, y_t)$$
    
  - Euler equation: $\forall(a_t, y_t)$
    
    $$u'[c(a_t, y_t)] = \beta R E_t u' \left( c \left( Ra_t + y_t - c(a_t, y_t) \right) \right)$$
    
- We discuss the Euler equation first, then full solution.
Stochastic Euler Equation

- Euler Equation:
  \[ u'(c_t) = \beta R E_t \left[ u'(c_{t+1}) \right] \]

- Implies that \( u'(c_t) \) is a sufficient statistic to predict \( u'(c_{t+1}) \):
  \[ E_t \left[ u'(c_{t+1}) \right] = \frac{u'(c_t)}{R \beta} \]

- Let the expectation error be given by \( \varepsilon_{t+1}^u \equiv u'(c_{t+1}) - E_t \left[ u'(c_{t+1}) \right] \), then
  \[ u'(c_{t+1}) = \frac{u'(c_t)}{R \beta} + \varepsilon_{t+1}^u, \quad E_t \left[ \varepsilon_{t+1}^u \right] = 0 \]

- From rational expectations, \( \varepsilon_{t+1}^u \) is independent of all the information available at time \( t \), and of \( c_t \) in particular.
  
  - Orthogonality conditions for GMM estimation:
    \[ E_t \left[ \varepsilon_{t+1}^u x_t \right] = 0, \quad \text{where} \ x_t \in \{ c_t, y_t, \ldots \} \]
Special case $R^\beta = 1$

- Assume $R^\beta = 1$
  $$u'(c_{t+1}) = u'(c_t) + \varepsilon_{t+1}$$

- Marginal utility is a random walk (a thus a martingale).
  - Martingale: $E_t[x_{t+1}] = x_t$
  - Submartingale: $E_t[x_{t+1}] \geq x_t$
  - Supermartingale: $E_t[x_{t+1}] \leq x_t$

- Recall that with certainty, $R^\beta = 1$ meant perfect smoothing:
  $$u'(c_{t+1}) = u'(c_t) \iff c_{t+1} = c_t = \bar{c}$$

- Which preferences would imply that consumption itself is a martingale?
  $$E_t[c_{t+1}] = c_t$$
Roadmap

1. Facts about aggregate and household level consumption

2. Consumption-savings problem with certainty
   - Application: The Ricardian equivalence

3. Consumption-savings problem with uncertainty
   - Special case: Quadratic utility
   - Propensity to consume and income persistence

4. Empirical evidence
Special case: Quadratic utility

- Consider a linear quadratic utility function

\[ u(c_t) = \alpha_0 c_t - \frac{\alpha_1}{2} c_t^2, \quad \alpha_i > 0 \]

- Provided that \( c_t < \frac{\alpha_0}{\alpha_1} \), the utility satisfies required conditions:

\[ u'(c_t) = \alpha_0 - \alpha_1 c_t > 0 \quad u''(c_t) = -\alpha_1 < 0 \]

- Substituting in the Euler equation:

\[ \mathbb{E}_t [u'(c_{t+1})] = \frac{u'(c_t)}{\beta R} \quad \Rightarrow \quad \mathbb{E}_t [\alpha_0 - \alpha_1 c_{t+1}] = \frac{\alpha_0 - \alpha_1 c_t}{R \beta} \]

- Solving for expected consumption at \( t + 1 \):

\[ \mathbb{E}_t [c_{t+1}] = \frac{1}{\beta R} c_t + \frac{\alpha_0}{\alpha_1} \left( 1 - \frac{1}{\beta R} \right) \]
Special case: Quadratic utility and $\beta R = 1$

- By further assuming that $\beta R = 1$:
  \[ \mathbb{E}_t (c_{t+1}) = c_t \]
  or alternatively
  \[ c_{t+1} = c_t + \varepsilon_{t+1}, \quad \mathbb{E}_t [\varepsilon_{t+1}] = 0 \]

- **Consumption is a random walk (martingale).**

- By the Law of Iterated Expectations and the martingale property:
  \[ \mathbb{E}_t [c_{t+2}] = \mathbb{E}_t [\mathbb{E}_{t+1} [c_{t+2}]] = \mathbb{E}_t [c_{t+1}] = c_t \]
  therefore in general for any future period:
  \[ \mathbb{E}_t [c_{t+j}] = c_t, \quad \forall j \geq 0 \]

  i) The best predictor of future consumption is current consumption.
  
  ii) No other information available at the current period can help predict next periods’ consumption.
Special case: Quadratic utility and $\beta R = 1$

- To solve the model, we use both the information in the Euler equation $(\mathbb{E}_t[c_{t+j}] = c_t, \forall j \geq 0)$ and the budget constraint.

- Let us iterate forward the budget constraint from time $t$, use the No Ponzi condition, and obtain:

$$\sum_{j=0}^{\infty} \frac{1}{R^j} \mathbb{E}_t[c_{t+j}] = R a_t + \sum_{j=0}^{\infty} \frac{1}{R^j} \mathbb{E}_t[y_{t+j}]$$

  - Financial Wealth
  - Human Wealth

- Note: We used conditional expectations to deal with uncertain future realizations of income and consumption as seen from time $t$.

- Note: Wealth is now stochastic (vs. certainty case where was a constant).
Special case: Quadratic utility and $\beta R = 1$

- Use the martingale property on the LHS:

$$
\sum_{j=0}^{\infty} \frac{1}{R^j} \mathbb{E}_t[c_{t+j}] = c_t \sum_{j=0}^{\infty} \frac{1}{R^j} = \left( \frac{R}{R - 1} \right) c_t = \left( \frac{1 + r}{r} \right) c_t
$$

and substitute back to get the optimal consumption:

$$
c_t = \frac{r}{1 + r} \left( Ra_t + \sum_{j=0}^{\infty} \frac{1}{R^j} \mathbb{E}_t[y_{t+j}] \right) \equiv y_t^p
$$

where we define permanent income $y_t^p$ as the annuity value of total wealth.

- Note that permanent income is a random variable (vs. deterministic case).

- **Result:** If preferences are quadratic and $\beta R = 1$, then consumption follows a martingale process and equals permanent income.
Certainty equivalence

- Recall that without uncertainty and $\beta R = 1$, the FOC was given by:

$$u'(c_{t+1}) = u'(c_t) \iff c_{t+1} = c_t$$

- And the solution was given by:

$$\bar{c} = c_t = \frac{r}{1+r} \left[ Ra_t + \sum_{j=0}^{\infty} \frac{1}{R^j} y_{t+j} \right]$$

- **Certainty equivalence**: to solve the stochastic problem one can
  1. solve the deterministic problem
  2. substitute conditional expectations of the forcing variables ($y_{t+j}$) in place of the variables themselves.

- **Implication**: Variance and higher moments of the income process DO NOT matter for the determination of consumption, only its expectation.

- However, ex-post consumption is not constant because $y_t^p$ is not constant.
Consumption and news

- Let us compute how consumption changes between periods:

\[ \Delta c_{t+1} = c_{t+1} - c_t = c_{t+1} - E_t[c_{t+1}] = \frac{r}{1 + r} (W_{t+1} - E_t[W_{t+1}]) \]

- Now let's compute the innovation (unexpected change) in wealth:

\[ W_{t+1} - E_t[W_{t+1}] = R(a_{t+1} - E_t[a_{t+1}]) + \sum_{j=0}^{\infty} \frac{1}{R^j} \left( E_{t+1}[y_{t+1+j}] - E_t[E_{t+1}[y_{t+1+j}]] \right) \]

\[ = \sum_{j=0}^{\infty} \frac{1}{R^j} (E_{t+1} - E_t) y_{t+1+j} \]

- Therefore, change in consumption is given by:

\[ \Delta c_{t+1} = \frac{r}{1 + r} \sum_{j=0}^{\infty} \frac{1}{R^j} (E_{t+1} - E_t) y_{t+1+j} \]

- Result: Changes in consumption are proportional to the revision in expected earnings due to the new information (news) arriving in that same period.
Random Walks

- Change in consumption is given by:

\[ \Delta c_{t+1} = \frac{r}{1 + r} \sum_{j=0}^{\infty} \frac{1}{R^j} (E_{t+1} - E_t) y_{t+1+j} \]

- From budget constraint, we find the change in assets:

\[
\begin{align*}
\Delta a_{t+1} &= r a_t + y_t - c_t \\
&= r a_t + y_t - \frac{r}{R} \left[ R a_t + \sum_{j=0}^{\infty} \frac{1}{R^j} E_t [y_{t+j}] \right] \\
&= y_t - \frac{r}{R} \sum_{j=0}^{\infty} \frac{1}{R^j} E_t [y_{t+j}] 
\end{align*}
\]

- Since \( \Delta c_{t+1} \) and \( \Delta a_{t+1} \) are stationary, it means that the original series have unit roots (integrated of order 1).

- Implication: \( c_t \) and \( a_t \) do not converge.
Cointegration

- **Cointegration**: Two non-stationary series of order 1 are said to be cointegrated if there exists a linear combination that is stationary.

- Claim: $a_t$ and $c_t$ are cointegrated series.
  - From the solution of consumption, we obtain a linear combination of $a_t$ and $c_t$ that is stationary:
    \[
    \begin{bmatrix}
    1 & -r
    \end{bmatrix}
    \begin{bmatrix}
    c_t \\
    a_t
    \end{bmatrix}
    = c_t - ra_t = \frac{r}{R} \sum_{j=0}^{\infty} \frac{1}{R^j} \mathbb{E}_t[y_{t+j}]
    \]

- According to Granger and Engle, $[1 - r]'$ is a cointegrating vector that, when applied to the non-stationary vector process $[c_t \ a_t]'$, yields a process that is asymptotically stationary.

- Note: The cointegration vector is not unique. $\left[ \frac{1}{r} - 1 \right]$ (and many others) are also cointegrating vectors.
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4. Empirical evidence
Propensity to consume and income persistence (1)

• Since consumption depends only on permanent income:

\[ c_t = y_t^P \]

• The marginal propensity to consume out of current income is given by:

\[
\frac{\partial c_t}{\partial y_t} = \frac{\partial c_t}{\partial y_t^P} \cdot \frac{\partial y_t^P}{\partial y_t} = \frac{\partial y_t^P}{\partial y_t} = 1
\]

• In the stochastic case, \( \frac{\partial y_t^P}{\partial y_t} \) depends on the process governing income.

1. Permanent shocks: getting a better job
2. Transitory shocks: win lottery, become temporarily unemployed
3. Persistent shocks
Propensity to consume and income persistence (2)

- Let us assume that income follows an AR(1) process:
  \[ y_{t+1} = (1 - \rho)\bar{y} + \rho y_t + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim iid(0, \sigma^2_{\varepsilon}), \quad \rho \in [0, 1] \]

- In the problem set, you will compute the policy using two methods.
  \[ c(a_t, y_t) = ra_t + \frac{R - 1}{R - \rho} y_t \]

- Propensity to consume out of current income: \( \frac{\partial c_t}{\partial y_t} = \frac{R - 1}{R - \rho} \)

  (i) **If shocks are permanent** \((\rho = 1)\), then \( \frac{\partial c_t}{\partial y_t} = 1 \).
  - Consumption responds one to one to changes in income.

  (ii) **If shocks are purely transitory** \((\rho = 0)\), then \( \frac{\partial c_t}{\partial y_t} = \frac{r}{1+r} \).
  - Extra income is spread evenly among all periods.

  (iii) **Propensity to consume increases with persistence** \((\rho)\)
  - Current income carries information about future realizations and the update of permanent income is bigger.
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4. Empirical evidence
Prediction I: $c_t$ summarizes all time $t$ information to forecast $c_{t+1}$.

- If $\beta R = 1$ and certainty equivalence, then consumption is a random walk:
  \[
  E_t (c_{t+1}) = c_t \implies c_{t+1} - c_t = \epsilon_{t+1}
  \]

- Time $t$ variables, besides $c_t$, don’t have additional predictive power on $\Delta c_{t+1}$.

- Hall (1978) tests Prediction I.
  - Confirms random walk hypothesis when adding lagged consumption...
  - ... but rejects it when lagged stock market prices appear significant.

- This evidence stimulated research in the direction of borrowing constraints and precautionary saving.
Prediction II

Prediction II: Predictable changes in $y_t$ have no effect on $c_t$.

- The optimal consumption rule is $c_t = y_t^p$ and $c_{t+1} = y_{t+1}^p$, which implies:
  \[ y_{t+1}^p - y_t^p = y_{t+1}^p - E_t (y_{t+1}^p) = \varepsilon_{t+1} \]

- $\varepsilon_{t+1}$ is the shock (unpredictable component) to permanent income, it is the only thing that should alter consumption.

- Campbell-Mankiw (NBER Macro Annuals, 1989) test Prediction II.
  - Full rationality is too strong: many agents are not rational or constrained.
  - Fraction $\lambda$ of agents consume all income and $1 - \lambda$ follow PIH:
    \[ c_t - c_{t-1} = \lambda (y_t - y_{t-1}) + (1 - \lambda) \varepsilon_t \]
    - Recall that: $\varepsilon_t = y_t^p - E_{t-1} (y_t^p)$
    - Clearly if the PIH is correct $\lambda$ should be zero.
Prediction II

• Implementation: treat $\varepsilon_t$ as an error, and estimate:

$$\Delta c_t = \lambda \Delta y_t + v_t$$

$$v_t = (1 - \lambda) \varepsilon_t$$

▶ What is the problem you face if you estimate $\hat{\lambda}$ with OLS?
▶ Instrument using $\Delta c_{t-j}$ with $j \geq 1$.

• They find a consistent estimate of $\lambda$ around 0.4 - 0.5:

▶Rejects PIH: consumption responds to lagged (or predictable) information.
▶A finding of $\lambda > 0$ is called excess sensitivity of consumption.
▶Around 40-50% are rule of thumb consumers?
Prediction III

Prediction III: $\nabla[\Delta c_t]$ should be equal to $\nabla[\Delta y^p_t]$.

• Consumption changes 1 to 1 with respect to unexpected changes in permanent income, thus variances should be the same.

• If $\Delta c_t = \Delta y^p_t$, then it must be that $\nabla(\Delta c_t) = \nabla(\Delta y^p_t)$.

• Empirical evidence shows that $\hat{\nabla}[\Delta c_t] < \hat{\nabla}[\Delta y^p_t]$.

  ▶ Consumption seems to be excessively smooth with respect to unexpected changes in future income.

  ▶ This is called excess smoothness of consumption.
• Important: $y_t^p$ is not observed, so $\nabla[\Delta y_t^p]$ is computed by estimating a stochastic process for $y_t$.

• Idea behind? If $\rho = 1$ (income is a random walk), then

$$\Delta y_t = \eta_t \quad \implies \quad \nabla[\Delta y^p] = \nabla[\Delta y]$$

• In fact, the process that fits the data best is

$$\Delta y_t = \rho \Delta y_{t-1} + \eta_t \quad \text{with } \rho = 0.44$$

• In this case, $\eta_t$ has more than a one-for-one permanent effect on income: one shock $\eta_t$ and one lagged effect $\rho \eta_t$.

• Hence the Deaton paradox: $\nabla[\Delta c] = \nabla[\Delta y^p] > \nabla[\Delta y]$
  ▶ But data: $\nabla[\Delta c] < \nabla[\Delta y]$
  ▶ Consumption is excessively smooth compared to what we would expect from the model!!
Problems with PIH

- Empirical tests of PIH basically fail.

- Because of certainty equivalence, risk does not matter!

- We will introduce two instances where risk (volatility of income shocks) will matter for consumption and saving decisions:
  1. Prudence
  2. Borrowing constraints (similar to investment irreversibility)

- Both elements will generate precautionary savings that increase with the volatility of income process.