Advanced Macroeconomics II

Lecture 2

Investment: Frictionless and Convex Adjustment Costs

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Motivation

• Investment increases productive capacity of the economy
  ⇒ key to determine standards of living in the long-run.

• Investment is highly volatile
  ⇒ key to understand short run (business cycle) fluctuations.

• Investment depends on real interest rates
  ⇒ key to understand impact of monetary policy

• Investment is a channel through which many fiscal instruments act
  ⇒ key to understand impact of fiscal policy

• Some facts...
Facts about investment

- **Fact 1:** Aggregate investment is relatively volatile.

- **Fact 2:** High correlation of aggregate investment with output.


<table>
<thead>
<tr>
<th>Variable</th>
<th>St Dev (%)</th>
<th>Correlation with GNP</th>
</tr>
</thead>
<tbody>
<tr>
<td>GNP</td>
<td>1.72</td>
<td>1</td>
</tr>
<tr>
<td>Consumption (non-durables)</td>
<td>0.86</td>
<td>0.77</td>
</tr>
<tr>
<td>Investment (gross private domestic)</td>
<td>8.24</td>
<td>0.91</td>
</tr>
<tr>
<td>Hours Worked</td>
<td>1.59</td>
<td>0.86</td>
</tr>
</tbody>
</table>

Cooley and Prescott (1996)
Facts about investment

- **Fact 1:** Aggregate investment is relatively volatile.
Facts about investment

- Fact 2: High correlation of aggregate investment with output.
  - Investment gradually increases and decreases along the business cycle.
Facts about investment

- Fact 3: Investment to Capital Ratio is relatively stable

![Graph of Investment / Capital Stock](image)

- Law of motion: \( K_{t+1} = (1 - \delta)K_t + I_t \)
- In steady state, \( K_{t+1} = K_t \implies \frac{I_t}{K_t} = \delta \).
Facts about investment

• **Fact 4: Investment at the plant level is “lumpy”**.

• **Lumpiness:** Spikes (infrequent and large changes) and Inaction

• **Doms and Dunne (1998), Cooper and Haltiwanger (2005)**
  - 18% of plants report investment rates of 20% (relative to capital).
  - 50% of plants experience a 1 year capital adjustment of $\geq 37\%$.
  - 80% of plants in a given year change their net capital stock $<10\%$.

• **Becker et al (2006)**
  - Between 9 and 28% of plants have exactly zero investment in a year.
Facts about investment

- **Fact 4:** Investment at the plant level is “lumpy”.

Roadmap

1 Frictionless investment

- Rental model
- Ownership model and user cost of capital
- Some empirical evidence and a quick fix

2 Convex adjustment costs

- Tobin’s $q$ theory
- Quadratic adjustment cost (microfounds $q$ theory)
- Empirical Evidence
Frictionless investment

- Start from model **without adjustment costs**.
- First, we assume that firms **rent** capital every period from households.
  - Static problem.
  - No cost to change the level of capital they rent.
  - Same as in the Solow and Ramsey models.
- Second, we assume that the representative firm **owns** capital.
  - Dynamic problem: depreciation and future production.
  - No costs to change their investment (capital they purchase).
- Note: Without adjustment costs, who owns the capital does not matter.
- In both models: \[ \text{marginal benefit} = \text{marginal cost} \]
  - \( \text{marginal productivity of capital} \) = \( \text{rental rate or user cost} \)
Frictionless investment: Rental model

- A firm **rents** capital $K_t$ every period to produce.
- Suppose we can write profits, after optimizing over other inputs, as $\Pi(K_t, x_t)$, where $x_t$ are other inputs’ costs.
- Let $r_K$ the rental cost of a unit of capital, then the firm solves:

$$\max_{K_t} \Pi(K_t, x_t) - r_K K_t$$

- The first order condition (FOC) for the demand of capital is:

$$\Pi_K(K_t, x_t) = r_K$$  \tag{1}$$

- If profit function exhibits decreasing returns to capital and the usual Inada conditions, then LHS is decreasing in $K$ and RHS is constant $\Rightarrow$ unique $K^*$ that solves (1).
Frictionless investment: Ownership model (1)

- A (risk neutral) representative firm owns capital $K_t$, rents labour $L_t$, produces output $Y_t$ and maximizes profits $\Pi_t$.

- Let $V_0$ be the expected discounted value of profits for the firm.

$$V_0 = \max_{\{K_t, L_t\}_{t=0}^\infty} \mathbb{E}_0 \left[ \sum_{t=0}^{\infty} \left( \frac{1}{R} \right)^t \Pi_t \right]$$

where period profits and investment are given by

$$\Pi_t = Y_t - p_t l_t - w L_t$$

$$l_t = K_t - (1 - \delta) K_{t-1}$$

- $p_t = \text{price of the capital goods.}$
- $w = \text{wage (constant, partial eq.)}$
- $R = 1 + r = \text{gross risk free rate, assumed constant (partial eq.)}$
- $\delta = \text{depreciation rate.}$

- Think: Why does the firm discount with $R$? How would the discount change if firm was owned by a household?
Frictionless investment: Ownership model (2)

- The **production function** $F$ transforms inputs $(K_t, L_t)$ into output $Y_t$:

  $$Y_t = F(K_t, L_t)$$

  *where: $F_K > 0$, $F_L > 0$, $F_{KK} < 0$, $F_{LL} < 0$, $F_{LK} > 0$*

  *Note 1: Capital is productive in the same period it is purchased. It can be modelled also with a time to build, where $Y_t = F(K_{t-1}, L_t)$.

  *Note 2: In partial equilibrium, no need to impose constant returns to scale.*

- We restate the problem net of the flexible factors (only labor in this case):

  $$L_t^* (K_t, w, p_t) = \arg \max F(K_t, L_t^*) - p_t I_t - wL_t^*$$

  and substitute into the value function:

  $$V_0 = \max_{\{K_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left( \frac{1}{R} \right)^t \mathbb{E}_0 [ F(K_t, L_t^*) - p_t I_t - wL_t^* ]$$ (3)
Frictionless investment: Ownership model (3)

- Substituting the expression for investment $I_t$, the problem becomes:

$$V_0 = \max_{\{K_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left( \frac{1}{R} \right)^t \mathbb{E}_0 \left\{ F(K_t, L^*_t) - p_t [K_t - (1 - \delta) K_{t-1}] - wL^*_t \right\}$$

$$= \max_{\{K_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left( \frac{1}{R} \right)^t \mathbb{E}_0 \left\{ F(K_t, L^*_t) - p_t [K_t - (1 - \delta) K_{t-1}] - wL^*_t \right\}$$ (4)

- **Optimality**: First order condition for capital at a generic time $t$ is:

$$F_k(K_t, L^*_t) - p_t + \frac{1}{R} (1 - \delta) \mathbb{E}_t [p_{t+1}] = 0$$

- Rearrange to express as:

$$p_t = F_k(K_t, L^*_t) + \frac{1}{R} (1 - \delta) \mathbb{E}_t [p_{t+1}]$$ (5)

- Pay $p_t$ today (units of output)
- Produce today $F_k$ (units of output)
- Future resale value of undepreciated capital $(1 - \delta) \mathbb{E}_t [p_{t+1}]$
Frictionless investment: Ownership model (4)

- Iterate on (5), use the law of iterated expectations and a transversality condition to express the price of capital as discounted marginal product:

\[ p_t = \sum_{j=0}^{\infty} \left( \frac{1 - \delta}{R} \right)^j \mathbb{E} \left[ F_k (K_{t+j}, L_{t+j}^*) \right] \]

- Comparing FOC to that of rental model, we define the user cost of capital:

\[ UK_t \equiv p_t - \frac{1}{R} (1 - \delta) \mathbb{E}_t [p_{t+1}] \]

- \( UK_t \) is an estimate of the rental rate of capital \( r_K \).

- \( UK_t \) increases with riskless rate \( R \), depreciation \( \delta \), and current price \( p_t \), and decreases with future price of capital goods \( \mathbb{E}[p_{t+1}] \).
Frictionless investment: Ownership model (5)

- Particular case: Cobb-Douglas production with productivity $\theta_t$:

$$Y_t = \theta_t K_t^\alpha L_t^\beta$$  \hspace{1cm} (6)

Then the FOC reads:

$$\alpha \theta_t K_t^{\alpha-1} L_t^\beta = UK_t$$

- Solve for $K_t$, we get the “desired” level of capital without adjustment costs:

$$K_t^* = (L_t)^{\frac{\beta}{1-\alpha}} \left( \frac{\alpha \theta_t}{UK_t} \right)^{\frac{1}{1-\alpha}}$$  \hspace{1cm} (7)

- If labour $\bar{L}$ is fixed, $K_t^*$ follows closely $\theta_t$ and $UK_t$.

$$K_t^* = (\bar{L})^{\frac{\beta}{1-\alpha}} \left( \frac{\alpha \theta_t}{UK_t} \right)^{\frac{1}{1-\alpha}}$$
Frictionless investment: Consequences

• The UK model determines the stock of capital:

\[ K_t^* = (L)^{\frac{\beta}{1-\alpha}} \left( \frac{\alpha \theta_t}{UK_t} \right)^{\frac{1}{1-\alpha}} \]

• Fluctuation in “desired” capital \( K_t^* \) are matched by equal fluctuations in observed capital \( K_t \).

• Empirical studies: Not rejected as a long run relation.

• But it does not explain short-run fluctuations:
  
  ▶ \( K_t^* \) does not depend on past or expected future levels of capital.
  
  ▶ \( K_t^* \) is a jump variable ⇒ Investment adjusts immediately.
  
  ▶ Excessive volatility of \( I_t \) against data (Cooper & Haltinwanger, 2000).

• Need for something that slows down adjustment of capital stock.
Frictionless investment: A quick fix (1)

• Distinction between net investment $I^n_t$ (after depreciation) and replacement investment $I^r_t = \delta K_{t-1}$

• The flexible accelerator model:

$$I^n_t = \beta (K^*_t - K_t), \quad \beta < 1$$

Net investment closes the gap between desired and current capital stock.

• Delayed adjustment of $K_t$ to $K^*_t$ generates a negative correlation over time between $UK_t$ and net investment rate.

• But such correlation is not strong in the data!
Frictionless investment: A quick fix (2)

- Low empirical correlation between user cost $UK_t$ and investment rate (equipment spending).

- Better strategy: micro-founded model with convex adjustment costs.
Roadmap

1. Frictionless investment
   - Rental model
   - Ownership model and user cost of capital
   - Some empirical evidence and a quick fix

2. Convex adjustment costs
   - Tobin’s $q$ theory
   - Quadratic adjustment cost (microfound $q$ theory)
   - Empirical Evidence
Tobin’s q

- Define the discounted return of **installed capital** at $t$, which assumes no further investments, as:

$$W(K_t) \equiv \sum_{j=0}^{\infty} \left( \frac{1}{R} \right)^j F((1 - \delta)^j K_t)$$

where the argument of the production function $F(\cdot)$ assumes capital evolves as $K_{t+j} = (1 - \delta)^j K_t$ since $I_{t+j} = 0 \ \forall j > 0$.

- For simplicity we do not write explicitly labor and productivity but they do affect firms production.

- Define **average return on installed capital** $Q$ as follows:

$$Q_t \equiv \frac{W(K_t)/K_t}{p_t}$$

  - Note: we must divide by $p_t$ to convert units, since the numerator is measured in terms of output and the denominator in terms of capital.
Tobin’s q

• Define **marginal return on capital** $q$ as follows:

$$q_t \equiv \frac{\mathbb{E}_t [\partial W(K_t)/\partial K_t]}{p_t} = \frac{\mathbb{E}_t \left[ \sum_{j=0}^{\infty} \left(\frac{1-\delta}{R}\right)^j F_K((1 - \delta)^j K_t) \right]}{p_t}$$

(8)

• Note the derivative $F_K$ in the previous expression.

• In other words:

$$q = \frac{\text{Market value of installed capital}}{\text{Replacement cost of installed capital}}$$

• $q$ measures the **perpetual return from one marginal unit of installed capital**.

• Tobin’s $q$ theory states that investment is a function of $q$ and $r$:

$$I = I(q, r) \quad \text{with} \quad \frac{\partial I}{\partial q} > 0 \quad \text{and} \quad \frac{\partial I}{\partial r} < 0$$

• Simple rule: Invest if $q > 1$. 
Tobin’s q and frictionless model

• Is \( q > 1 \) consistent with frictionless model (i.e. \( F_k = U K_t \))? No!!

• From the definition of \( q \) we have that

\[
q_t - 1 = \frac{E_t \left[ \frac{\partial W(K_t)}{\partial K_t} \right] - p_t}{p_t} \tag{9}
\]

where \( E_t \left[ \frac{\partial W(K_t)}{\partial K_t} \right] - p_t \) is the NPV of marginal profits net of the cost of investment \( p_t \).

• From (8) we have that:

\[
E_t \left[ \frac{\partial W(K_t)}{\partial K_t} \right] - p_t = E_t \left[ \sum_{j=0}^{\infty} \left( \frac{1 - \delta}{R} \right)^j F_K((1 - \delta)^j K_t) \right] - p_t = F_K(K_t) - p_t + \frac{1 - \delta}{R} E_t \left[ F_K(K_{t+1}) \right] + \left( \frac{1 - \delta}{R} \right)^2 E_t \left[ F_K(K_{t+2}) \right] + \ldots
\]
Tobin’s q and frictionless model

- Adding and subtracting prices, we recover user cost $UK_t$ at every period:

$$
E_t \left[ \frac{\partial W(K_t)}{\partial K_t} \right] - p_t = F_K(K_t) - p_t + \frac{1 - \delta}{R} E_t[p_{t+1}]
$$

$$
+ \frac{1 - \delta}{R} E_t \left[ F_K(K_{t+1}) - p_{t+1} + \frac{1 - \delta}{R} E_t[p_{t+2}] \right]
$$

$$
+ \left( \frac{1 - \delta}{R} \right)^2 E_t \left[ F_K(K_{t+2}) - p_{t+2} + \frac{1 - \delta}{R} E_t[p_{t+3}] \right] + ... 
$$

- Define the net marginal profits at time $t$, denoted $\pi_t$, as:

$$
\pi_t \equiv F_K(K_t) - p_t + \frac{1}{R} (1 - \delta) E_t[p_{t+1}] = F_K(K_t) - UK_t
$$

and rewrite as:

$$
E_t \left[ \frac{\partial W(K_t)}{\partial K_t} \right] - p_t = \sum_{j=0}^{\infty} \left( \frac{1 - \delta}{R} \right)^j E_t[\pi_{t+j}]
$$
Tobin’s q and frictionless model (cont...)

- Summarizing, the expression for $q$ under the frictionless model reads:

$$q_t - 1 = \frac{E_t \left[ \frac{\partial W(K_t)}{\partial K_t} \right]}{p_t} - p_t = \sum_{j=0}^{\infty} \left( \frac{1-\delta}{R} \right)^j E_t \left[ F_{K_{t+j}} - U_{K_{t+j}} \right] = \sum_{j=0}^{\infty} \left( \frac{1-\delta}{R} \right)^j E_t \left[ \pi_{t+j} \right]$$

(11)

- But the profit maximizing condition of the frictionless model implies that:

$$F_{K_{t+j}} = U_{K_{t+j}} \text{ for any } j \geq 0 \implies \pi_{t+j} = 0 \text{ for any } j \geq 0$$

- It follows that without adjustment costs $q_t = 1$ always.

- Why? Since investment is frictionless, any situation with $q_t > 1$ triggers an immediate increase in investment until $q_t = 1$. 
Reviving Tobin’s q

- Tobin’s q theory sounds appealing, and intuitively right.

- Empirically we observe a positive (even though weak) correlation between \( \frac{I_t}{K_t} \) and average \( Q_t \), both at the aggregate and at the firm level.

- How can we modify the model to make it compatible with the \( q – theory \)?

- Adjustment costs!
  - They reduce the volatility of investment, and are consistent with the realistic feature that the cost of installing capital adds up to the cost of purchasing it.
Convex adjustment costs (1): Idea

- Convex adjustment costs: small investments can be easily integrated in the current structure of the firm, while big investments create larger disruptions.

- Idea: adjustment costs generate positive relation between $q$ and $I$.

  ▶ Suppose that $q = 1$ and suddenly future net revenues are expected to increase.

  ▶ $E_t \left[ \frac{\partial W(K_t)}{\partial K_t} \right]$ increases (one marginal unit of capital installed today generates more return in the future).

  ▶ Investment goes up, but spread over time to minimize adjustment costs.

  ▶ Since investments goes up a little today, $E_t \left[ \frac{\partial W(K_t)}{\partial K_t} \right]$ falls only little, and $q$ is still larger than 1.

  ▶ Therefore, $q$ and $I$ become positively related.
Convex adjustment costs (2): Problem

- Formally we write the problem with convex adjustments costs as:

\[
\max_{\{l_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \left( \frac{1}{R} \right)^t \left[ y_t - p_t l_t - C(l_t) \right] \right\}
\]

- It is equivalent to write the problem in terms of investment or capital.

- Adjustment costs \( C(l_t) \):
  
  - Convex function of investment \( (C_l > 0, C_{ll} > 0) \)
  
  - Measured in units of output.
  
  - We specialize to quadratic adjustment costs:

\[
C(l_t) \equiv \frac{\gamma}{2} l_t^2, \quad \gamma > 0
\]
Convex adjustment costs (3): Quadratic costs

- For simplicity we omit labour (think of a per capita production function):
  \[ y_t = \theta_t K_t^\alpha \]

- Substitute production function and adjustment costs to obtain:
  \[
  \max_{\{l_t\}} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \left( \frac{1}{R} \right)^t \left[ \theta_t K_t^\alpha - p_t l_t - \frac{\gamma}{2} l_t^2 \right] \right\}
  \]

- First order condition (with respect to \( K_t \)) is:
  \[
  \alpha \theta_t K_t^{\alpha-1} - p_t - \gamma l_t + \frac{1}{R} \left[ \mathbb{E}_t [p_{t+1}] (1 - \delta) + \gamma (1 - \delta) \mathbb{E}_t [l_{t+1}] \right] = 0
  \]

- Solving for \( l_t \):
  \[
  l_t = \frac{1}{\gamma} (F_{K_t} - UK_t) + \frac{1}{R} (1 - \delta) \mathbb{E}_t [l_{t+1}] \tag{12}
  \]

  where:
  \[
  UK_t = p_t - \frac{1 - \delta}{R} \mathbb{E}_t [p_{t+1}] \quad \text{and} \quad F_{K_t} = \alpha \theta_t K_t^{\alpha-1}
  \]
Convex adjustment costs (4): Policy

- Iterating (12) forward:

\[ I_t = \frac{1}{\gamma} \sum_{j=0}^{\infty} \left( \frac{1-\delta}{R} \right)^j \mathbb{E}_t \left[ F_{K_{t+j}} - UK_{t+j} \right] \]  

(13)

- Comparing this to the expression for \( q \) in (11) above, which read:

\[ q_t - 1 = \sum_{j=0}^{\infty} \left( \frac{1-\delta}{R} \right)^j \mathbb{E}_t \left[ F_{K_{t+j}} - UK_{t+j} \right] / p_t \]

we obtain that:

\[ I_t = \frac{1}{\gamma} (q_t - 1) p_t \]  

(14)

- Now investment is a positive function of \( q \), as argued by Tobin.
Convex adjustment costs (5): Intuition

- Intuition for expression (14):
  
  \[ l_t = \frac{1}{\gamma} (q_t - 1) p_t \]

- Rewrite as:
  
  \[ p_t + \gamma l_t = q_t p_t \]

  - Since \( C (l_t) = \frac{\gamma}{2} l_t^2 \), then the marginal cost is \( C' (l_t) = \gamma l_t \)
  
  - From the definition of \( q_t \), we have that: \( q_t p_t = \mathbb{E}_t \left[ \frac{\partial W(K_t)}{\partial K_t} \right] \)

- Therefore, expression (14) implies that optimal investment satisfies that:

  \[ p_t + C' (l_t) = \mathbb{E}_t \left[ \frac{\partial W(K_t)}{\partial K_t} \right] \]

  Marginal cost of installing one unit of capital

  Marginal profits expected from that unit of capital.
Convex adjustment costs (5): Transversality

- Consider the two equations derived before:

\[ I_t = \frac{1}{\gamma} (F_{K_t} - UK_t) + \frac{1}{R} (1 - \delta) \mathbb{E}_t [I_{t+1}] \]

\[ I_t = \frac{1}{\gamma} (q_t - 1) \rho_t \]

- Assume for simplicity that \( p_t = 1 \) and \( \delta = 0 \):

\[ \frac{1}{\gamma} (q_t - 1) = \frac{1}{\gamma} \left( F_{K_t} - 1 + \frac{1}{R} \right) + \frac{1}{R \gamma} \left( \mathbb{E}_t [q_{t+1}] - 1 \right) \]

- Simplifying:

\[ q_t = F_{K_t} + \frac{1}{R} \mathbb{E}_t [q_{t+1}] \]

- Substituting recursively forward:

\[ q_t = \sum_{j=0}^{\infty} \frac{1}{R^j} \mathbb{E}_t \left[ F_{K_{t+j}} \right] + \lim_{j \to \infty} \frac{1}{R^j} \mathbb{E}_t \left[ q_{t+j+1} \right] \]
Convex adjustment costs (5): Transversality

- Maximization requires the following **transversality condition**:
  \[
  \lim_{j \to \infty} \frac{1}{R^j} \mathbb{E}_t (q_{t+j+1}) = 0
  \]

- This is not an exogenous constraint (e.g. “no Ponzi scheme” condition).

- It is a condition required from profit maximization.
  
  - If this condition is not satisfied, \( q \) grows too fast over time.
  
  - This cannot be compatible with profit maximization, it would be optimal to invest more.
Lagrange Multiplier Method

• We can also solve the model using the Lagrange multiplier (LM) method.

• Recall the investment equation:

\[ K_t = I_t + (1 - \delta) K_{t-1} \]

• The Lagrange multiplier is the shadow value of capital:

  ▶ What is the increase in firm value \( V \) when we relax constraint by 1 unit?
  ▶ 1 unit of investment good has a value of \( p \) outside the firm and of \( pq \) inside the firm.
  ▶ Therefore the LM associated to this constraint will measure exactly \( pq \)!
• Set the Lagrangian, using as multiplier $\lambda_t = q_t p_t$:

$$\max_{\{I_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \left( \frac{1}{R} \right)^t \left[ \theta_t K_t^\alpha - p_t I_t - \frac{\gamma}{2} I_t^2 + q_t p_t (I_t + (1 - \delta) K_{t-1} - K_t) \right] \right\}$$

• FOC with respect to $I_t$:

$$- p_t - \gamma I_t + q_t p_t = 0$$

• And we recover back our expression for investment:

$$I_t = \frac{1}{\gamma} (q_t - 1) p_t$$
Implications of the $q$ model

- Investment is a linear function of marginal $q_t$

$$I_t = \frac{1}{\gamma} (q_t - 1) p_t$$

where $q_t$ is a sufficient statistic: summarizes all relevant information about the future that is relevant for $I_t$

$$q_t = 1 + \sum_{j=0}^{\infty} \left[ \frac{1}{R} (1 - \delta) \right]^j \mathbb{E}_t \left[ F_{K_{t+j}} - UK_{t+j} \right] p_t$$

- Implications:
  - $I_t$ is much smoother than $F_{K_t} - UK_t$
  - $I_t$ reacts to $\mathbb{E}_t \left[ F_{K_{t+j}} - UK_{t+j} \right]$ even if $F_{K_t} - UK_t$ does not change.
Marginal \( q \) is the value of one marginal unit of capital in the firm divided by its purchasing price.

- Problem: It is not observable!

Average \( Q \) is the value of the whole firm divided by the replacement value of its assets.

- Solution: Average \( Q \) can be estimated

\[
\hat{Q} = \frac{\text{Stock market value}}{\text{Book value}}
\]
Tobin’s q: Empirical Evidence (2)

- Hayashi (1982) provides a set of sufficient conditions for $q = Q$:

1. The firm is price-taker (additional units of output do not imply selling at a lower price).

2. Production technology and adjust. costs with constant returns to scale.

   For example: $y(K) = \theta K$, $C(i, K) = c(I/K)K$

3. Adjustment costs are convex in ratio $I/K$.

   For example: $c\left(\frac{I}{K}\right) = \frac{\gamma}{2} \left(\frac{I}{K}\right)^2$
Tobin’s q: Empirical Evidence (3)

• Assume $Q = q$ holds. Model implies that:

$$\frac{I}{K} = \beta_0 + \beta_1 Q + \varepsilon, \quad \varepsilon \sim \text{is an observational error}$$

• Obtain an empirical counterpart for $Q$, called $\hat{Q}$, usually:

$$\hat{Q} = \text{stock market value} / \text{book value}$$

• Estimate with OLS the following regression using aggregate data

$$\left(\frac{I}{K}\right)_t = \beta_0 + \beta_1 \hat{Q}_t + \hat{\varepsilon}_t$$
Tobin’s q: Empirical Evidence (4)

- Estimate \((\frac{I}{K})_t = \beta_0 + \beta_1 \hat{Q}_t + \hat{\varepsilon}_t\) with yearly data.
- \(\beta > 0\), but \(Q_t\) is not a sufficient statistic for \(\frac{I}{K}\)
• Even worse at a quarterly frequency.
Tobin’s q: Empirical Evidence (6)

- More and more good firm level data available.

- Estimate a panel regression with firm (or plant) level data:

\[
\left( \frac{I}{K} \right)_{it} = \beta_{i0} + \beta_1 \hat{Q}_{it} + \beta_2 \left( \frac{X_{it}}{K_{it}} \right) + \nu_{it}
\]

  - \( \hat{Q}_{it} \) is an empirical counterpart for \( Q_{it} \)
  - Individual fixed effects \( \beta_{i0} \)
  - Other firm characteristics \( X_{it} \) (should be insignificant)

- Results:
  - \( \beta_1 \) is positive \( \Rightarrow \) Evidence for q-theory
  - \( \beta_2 \) not zero \( \Rightarrow \) \( Q_t \) is not a sufficient statistic for \( I/K \)
  - \( \beta_2 > 0 \) when \( X_{it} \) is cash flow \( \Rightarrow \) evidence of financing constraints
Why is the $q$ model rejected?

- Possible explanations for empirical rejection:

1. $\hat{Q}_t$ is a noisy measure of $Q_t$.
2. Assumptions that make $Q_t = q_t$ do not hold
   - Imperfect competition (Cooper and Ejarque, 2001)
   - Decreasing returns to scale
3. Adjustment costs not quadratic
   - Model misspecification
4. Finance matters (borrowing constraints, capital market imperfections)

- The literature has explored these possibilities both at the aggregate and at the firm level and has found strong evidence for all the points above.
- Example, cash flow and acceleration of output predict investment very well.
Other factors beyond $q$?

- Cash flow and acceleration of business output explain much better investment (PDE: producers durable equipment)
Two main avenues

- The empirical failure of the Q model started two fields of research:
  1. Optimal Investment with financing constraints
  2. Optimal Investment with non convex adjustment costs

- Plenty of evidence has been found that both factors matter at firm level, but still unclear whether they matter at the aggregate level.

- Active debate about the Q model, especially in Finance, where cash flow, credit lines and other measures of liquidity affect investment beyond $q$

- In Macro, the debate is whether or not we should care about non-convex adjustment costs when we model aggregate investment.
  - Bachmann, Caballero and Engel, AEJ: Macro (2013): we should care.