Aggregate Dynamics in Lumpy Economies*

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Abstract

We develop a new framework to analyze the aggregate implications of lumpiness in microeconomic adjustment, which is pervasive in many economic environments. We derive structural relationships between the steady state moments and the business cycle dynamics of lumpy economies, and we show how to discipline these relationships using micro panel data. As an application, we study capital misallocation and investment dynamics by implementing our tools on establishment–level data from Chile and Colombia. Our framework is very flexible and can accommodate a large set of inaction models, stochastic processes, and higher order dynamics.

JEL: D30, D80, E20, E30

Keywords: inaction, lumpiness, transitional dynamics, adjustment costs, aggregate shocks, deviations from steady state, sufficient statistics, Ss models.

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1 Introduction

Lumpiness in microeconomic adjustment is pervasive in many economic environments. For instance, capital investment, inventory management, consumption of durable goods, price setting, portfolio adjustment, and many other economic decisions faced by firms and households are characterized by periods of inaction followed by bursts of activity. Recurrent questions that arise in these environments relate to the macroeconomics consequences of this micro lumpiness. How does lumpiness in microeconomic adjustment affect aggregate dynamics? After a policy change or an aggregate shock, how long do transitions last until the lumpy economy reaches its new long-run equilibrium? Understanding these issues is key for the design and implementation of policies aiming at stabilizing the business cycle or generating long-run growth.

Our work contributes by developing new theoretical tools to study transitional dynamics in lumpy economies. We consider environments with heterogeneous agents that make decisions subject to adjustment frictions and idiosyncratic shocks. These frictions can take the form of fixed adjustment costs, random opportunities of adjustment, fixed dates of adjustments, among many other. The steady state of the economy is characterized through an invariant distribution. Starting from any arbitrary initial condition for the cross-sectional distribution, we characterize how it converges towards the invariant distribution by focusing on the transition of the moments of the distribution. Following Alvarez, Le Bihan and Lippi (2016), we measure transitional dynamics through the cumulative impulse-response function (CIR), which equals the cumulative deviations in the moment of interest with respect to its steady state value along the transition path. We contribute by formally characterizing three properties of the CIR that hold in any inaction model, whether it is state-dependent, time-dependent, or a hybrid of both.

The first property, aggregation, expresses transitional dynamics as the solution to a representative agent's recursive problem. The second property, representation, expresses the representative agent's problem as a function of steady state moments. The third property, observation, recovers the steady state moments and parameters using information about observable actions, such as the frequency and size of adjustments. Taking the three results in conjunction, our theory provides a tight link between observable actions (in micro panel data), steady state moments, and business cycle dynamics.

With our theoretical results at hand, we illustrate the potential of our framework with an application to capital misallocation and investment dynamics. For this purpose, we set-up a canonical model of lumpy investment à la Khan and Thomas (2008), Bachmann, Caballero and Engel (2013) and Winberry (2016), that features idiosyncratic productivity shocks, depreciation and productivity growth, asymmetric policies, and random capital adjustments costs. Defining the capital gap as the log difference between the current capital and its static optimum, the CIR of average capital depends on two cross-sectional steady state moments: the variance of capital gaps and the covariance between capital gaps and the time since their last adjustment. We show analytically how to measure these *unobservable* statistics using *observable* micro data on manufacturing plants in Chile and Colombia. A key contribution lies in demonstrating that certain micro statistics that were never computed in the data and ignored when calibrating models, such as the covariance above, are crucial determinants of aggregate dynamics. We show that the canonical investment model with adjustment costs misses these statistics. In this spirit, our tools can aid researchers in improving their models for the aggregate implications of inaction.

Let us now discuss in more detail the three properties of the CIR.

1. Aggregation. Transitional dynamics towards steady state are characterized as the solution to the recursive problem of a representative agent. By requiring certain degree of history independence in the processes and policies, we are able to characterize all ex-post heterogeneity due to idiosyncratic shocks through the problem of one agent. Intuitively, consider an environment in which upon adjustment, all agents adjust to bring their state to the same value. Then, agents become ex-ante identical at the moment of adjustment, and they will only differ ex-post due to the idiosyncratic shocks. Thus any ex-post heterogeneity due to different initial conditions or conditional dynamics can be summarized by one of these ex-ante agents. Importantly, our aggregation result *does not imply* that heterogeneity is irrelevant for aggregate dynamics; it says that all heterogeneity can be summarized in a compact way.

2. Representation. The solution to the representative agent's problem can be represented through a combination of steady state cross-sectional moments. The idea behind this result lies in that, in an approximation around the steady state, the ergodic moments encode information about agents' responsiveness to idiosyncratic shocks, and thus these moments inform us about the representative agent's policy. We show that in certain cases, the ergodic moments are not sufficient to fully characterize the CIR and additional information about micro-level elasticities is needed. In the case of the canonical lumpy investment model, this structural relation becomes a function of the two cross-sectional moments of the distribution discussed above.

3. Observation. Our previous results express transitional dynamics as a function of steady state moments. However, such moments, in many cases, may be hard to observe in the data. For instance, firms' markups are unobservable, since we only observe average cost and not marginal costs, and it is even harder to think about capital gaps, the distance between firms' capital and their optimal level. Then, how we can discipline steady state moments of these variables? Our third and most applicable result shows how to recover the steady state moments with observable actions, namely, adjustments and stopping times, both of which are very likely to be observable in micro datasets. The logic behind this mapping is that, by assuming a structure for the state's evolution during inaction and merging it with information revealed through agents' adjustments, we can back out ergodic moments and the process of shocks affecting them.

In our baseline analysis, we focus on the CIR of the first moment following a horizontal shift of the distribution, and assume that the idiosyncratic shocks are described by a random walk with drift. Then, we extend our results to study the transition of any moment of the distribution (first, second, etc.), for any initial condition away from the steady state (mean preserving spreads, etc.), and general stochastic processes (e.g. with mean-reversion). While the specific mappings between the micro data, the steady state moments and the aggregate dynamics change in each of these environments, the existence of a structural relationship among them continues to hold.

General equilibrium effects. Our analysis takes as a premise that the steady state policies hold along the transition path. This assumption is valid as long as the general equilibrium feedback from

the aggregate distribution to the individual policies though prices is quantitatively insignificant. There are several general equilibrium frameworks in which this is the case.[1] When general equilibrium effects are quantitative relevant, the tools developed in this paper do not fully characterize aggregate dynamics. Nevertheless, there exist lumpy environments in which the partial equilibrium mechanisms continue to hold in general equilibrium.

A concrete example of this logic is found in the context of pricing literature with Calvo-type adjustments. In a model with negligible first order general equilibrium effects, Alvarez, Le Bihan and Lippi (2016) show analytically that the effectiveness of monetary policy is a function of the average duration of pricing spells, independent of any type of heterogeneity. Following this result, Blanco and Cravino (2018) reach a similar conclusion in a model with large general equilibrium effects (arising from real rigidities) in the context of real-exchange dynamics. Therefore, the role of heterogeneity and inaction in shaping aggregate dynamics is not altered by general equilibrium forces.

Related literature. Aggregate dynamics in inaction models has been widely studied. The groundbreaking work of Caplin and Spulber (1987), Caballero and Engel (1991) and Caplin and Leahy (1991) provided theoretical guidelines in stylized models to understand the role of micro lumpiness in shaping aggregate dynamics. With the surge of micro data, more realistic models that incorporated idiosyncratic shocks were developed, such as Cooper and Haltiwanger (2006), Golosov and Lucas (2007), Midrigan (2011), Berger and Vavra (2015), Carvalho and Schwartzman (2015) and Álvarez, Lippi and Paciello (2016), with the objective of understanding how the interaction of heterogeneity and lumpiness mattered for aggregate dynamics. We contribute by providing novel theoretical insights and an empirical strategy that exploits the micro data to its maximum while imposing a minimum structure to the inaction model.

Our paper relates to the pioneer work in Hamermesh (1989), where it is shown how firms' labor decision can be rationalized in a fixed adjustment cost model in which the adjustment trigger depends on the labor gap, i.e. difference between current and static optimal labor. This strategy has been applied in Caballero and Engel (1993) and Caballero, Engel and Haltiwanger (1997) for analyzing the consequences of lumpiness for macroeconomic fluctuations. More recently, a similar approach has been applied in the analysis of cross-country productivity differences due to capital misallocation across firms (see Restuccia and Rogerson (2013) for a survey). While in theory this methodology appears adequate, the challenge lies in finding the optimal target to construct the gap. The standard approach, as in Hsieh and Klenow (2009), consists in specifying a production function at the micro level that allows to recover the optimal input demand, therefore the gap. However, as argued by Cooper and Willis (2004) and Oberfield (2013), such an approach suffers from a specification error, as it is hard to test the validity of the technological assumptions.

We propose an alternative way that circumvents this specification problem. Our approach—embedded in the observation property—consists in directly assuming a stochastic process for the unobserved marginal product of an input, thus its optimal static demand, and then adding discipline to the parameters of the

[1] For the effect of monetary shocks, see Woodford (2009), Golosov and Lucas (2007), and the vast literature that builds on them. For real exchange dynamics, see Carvalho and Nechio (2011) and Kehoe and Midrigan (2008). Regarding investment models, Bachmann, Caballero and Engel (2013) and Winberry (2016), building on Khan and Thomas (2008), show that partial equilibrium dynamics are not undone by general equilibrium effects whenever the model is calibrated to match the cyclical properties of aggregate investment or interest rates. Web Appendix B describes some of these frameworks.

stochastic process using observable micro data on investment that holds for *all* lumpy adjustment models. While this approach clearly depends on the assumptions on the stochastic process (e.g. mean-reversion vs. drift), the theory imposes cross-equation restrictions that allow us to validate such assumptions.

Our paper is closely related to the work by Alvarez, Le Bihan and Lippi (2016), who consider a multi-product menu cost model with random opportunities to freely adjust and Brownian innovations to markup gaps. In that setup, they study the real effects of monetary shocks. In our view, one of the key results in their paper is that the cumulative impulse-response (CIR) for average markup gaps—a measure of the real effects of a money shock—equals the kurtosis of price changes times the average duration of prices divided by 6. They show this result analytically for the case of one product ($n = 1$) and infinite products ($n = \infty$), and more generally, they construct power series of each of the terms in the equality and confirm numerically that the relationship holds. Our contribution lies in proving an alternative proof to their result in a more general framework using the structural relation between the CIR and the steady state moments. Moreover, our strategy allows to extend the results to richer environments.

Structure of the paper. Section 2 presents a standard model of lumpy investment that allows us to introduce the objects of interest. Section 3 develops the theory and explains the logic behind the aggregation, representation and observation properties of the CIR. Section 4 applies the theory using micro-level data. Section 5 generalizes and extends the results.

2 Baseline Model: Lumpy Investment

This section describes the economic environment in which we apply the theory we develop. We build a partial equilibrium lumpy investment model in the spirit of Khan and Thomas (2008), Bachmann, Caballero and Engel (2013) and Winberry (2016), with a few simplifications that are discussed below.

2.1 Environment

Time is continuous and infinite. There is a representative household and a continuum of ex-ante identical firms. There is no aggregate uncertainty and firms face two types of idiosyncratic shocks: productivity shocks and a random fixed cost to be paid with each capital adjustment. We denote with $\omega \in \Omega$ the full history of these two shocks and consider $(\Omega, P, \mathcal{F})$ to be a probability space equipped with the filtration $\mathcal{F} = \{\mathcal{F}_t : t \geq 0\}$. We use the notation $g_{\omega,t} : \Omega \times \mathbb{R} \to \mathbb{R}$ to denote an adapted process (a function $\mathcal{F}_t$-measurable for any $t \geq 0$) and $\mathbb{E}[g_{\omega,t}]$ to denote its expectation under P.

Firms. Firms operate in competitive markets. They produce output Y using capital K as the only input through a decreasing returns technology:

$$Y_{\omega,t} = E_{\omega,t}^{1-\alpha} K_{\omega,t}^{\alpha}, \tag{1}$$

where the log of idiosyncratic productivity E evolves according to a Brownian motion with drift μ and volatility σ, as follows:

$$d\log(E_{\omega,t}) = \mu dt + \sigma dW_{\omega,t}, \quad W_{\omega,t} \sim Wiener. \tag{2}$$

A firm chooses capital to maximize its expected stream of profits. For every capital adjustment, a firm pays a random fixed adjustment cost proportional to its productivity $\kappa_t E_{\omega,t}$, where κ_t is described by a compound Poisson process. In a period of length dt, the adjustment cost equals a constant $\kappa > 0$ with probability $1 - \lambda dt$, or it is given by random variable $\xi_{\omega,t}$ with probability λdt, where $\xi_{\omega,t}$ follows the distribution $H(\xi)$ with support $[0, \kappa]$. Letting $\tilde{N_{\omega,t}}$ describe a Poisson process with arrival rate λ, we write the adjustment cost $\kappa_{\omega,t}$ as

$$\kappa_{\omega,t} = \begin{cases} \kappa & \text{if } d\tilde{N}_{\omega,t} = 0 \\ \xi_{\omega,t} & \text{if } d\tilde{N}_{\omega,t} = 1. \end{cases} \tag{3}$$

Profits are discounted at the Arrow–Debreu time-zero price Q_t. Capital between adjustments depreciates at a constant rate ψ. With all the elements above, a firm's problem entering at time t consists in choosing a sequence of adjustment dates $(\tau_{\omega,i})$, and investment rates $(\Delta K_{\tau_{\omega,i}} = K_{\tau_{\omega,i}} - K_{\tau^-_{\omega,i}})$ that solve the following stopping-time problem:

$$\max_{\{\tau_{\omega,i}, \Delta K_{\tau_{\omega,i}}\}_{i=1}^{\infty}} \mathbb{E}\left[\int_0^\infty Q_s Y_{\omega,s}\, ds - \sum_{i=1}^{\infty} Q_{\tau_{\omega,i}} \left(\kappa_{\omega,\tau_{\omega,i}} E_{\omega,\tau_{\omega,i}} + \Delta K_{\tau_{\omega,i}}\right)\right], \tag{4}$$

where output, productivity, and fixed costs follow (1), (2), and (3), respectively, and capital follows

$$\log(K_{\omega,s}) = \log(K_{\omega,0}) - \psi(s-t) + \sum_{\tau_{\omega,i} \leq s} \Delta K_{\tau_{\omega,i}}. \tag{5}$$

Household. The household chooses the stochastic process for consumption to maximize its expected utility subject to a budget constraint. The household problem is given by

$$\int_0^\infty e^{-\rho t} C_t dt, \quad \text{subject to} \int_0^\infty Q_t \left(C_t - \Pi_t\right) dt = 0. \tag{6}$$

where $\Pi_t \equiv \mathbb{E}[\pi_{\omega,t}]$ denotes aggregate firm's profits and C_t denotes household's consumption.

Aggregate feasibility. Aggregate output Y_t is used for household's consumption C_t and firms investments I_t, which includes capital adjustments adjustment costs[2]:

$$\underbrace{\mathbb{E}\left[E_{\omega,t}^{1-\alpha} K_{\omega,t}^{\alpha}\right]}_{Y_t} = C_t + \underbrace{\mathbb{E}\left[\mathbb{1}_{\{\tau,t\}}\left[\kappa_{\omega,t} E_{\omega,t} + \Delta K_{\omega,t}\right]\right]}_{I_t}. \tag{7}$$

Equilibrium. Given an initial distribution of $\{K_{\omega,0}, E_{\omega,0}\}$, an equilibrium is a set of stochastic processes for prices $\{Q_t\}$, household's policy $\{C_t\}$, and firms' policies $\{\tau_{\omega,i}, \Delta K_{\omega,i}\}$ such that:

[2]Here $\mathbb{1}_{\{\tau,t\}} = \{\omega : \exists i \ s.t. \ \tau_{\omega,i} = t\}$ indicates the set of adjusters.

(i) Given $\{Q_t\}$, $\{C_t\}$ solve the household's problem (6).

(ii) Given $\{Q_t\}$, $\{\tau_{\omega,i}, \Delta K_{\omega,i}\}$ solve the firm's investment problem (4).

(iii) Goods market clears (7).

Price system dynamics. In principle, the price system may depend on two types of states variables: endogenous state variables, such as the distribution of capital holdings across firms, or exogenous state variables. The challenge under the first scenario, is that firms' values and policies would depend on the aggregate state (which in turn depends on their distribution) and thus steady state policies would not be adequate to characterize the transitional dynamics. To circumvent this issue in order to use steady state policies, we have specified a general equilibrium structure that generates a price system which is independent from the firm distribution. In this case, due to linearity of preferences, the only price in the economy is that of the Arrow security Q_t, which satisfies $Q_t = Q_0 e^{-\rho t}$ and is *independent* of the firm distribution. This assumption does not imply that the aggregate state is independent of the firm distribution, since for example, aggregate output does depend on the joint distribution of capital and productivity; but prices are.

Discussion of simplifying assumptions. Let us compare our environment with one of the benchmark models in this literature in Khan and Thomas (2008). First, in contrast to that paper, we do not consider labor as a factor of production. Given that we consider a partial equilibrium setting, and the labor decision is static in their model, this assumption is innocuous since adding labor would only affect the value of the output-capital elasticity. Second, all the investments in our model, regardless if these fall within a small range, require the payment of the fixed adjustment cost (in the language of these authors, we do not consider *unconstrained* investments). This assumption is quantitatively irrelevant for transitional dynamics, as in the calibration, most investments are constrained anyways due to the large size of idiosyncratic shocks relative to aggregate shocks.

Lastly, we consider a random-walk process for idiosyncratic productivity instead of mean reversion. This assumption is considered to simplify the exposition at this stage and it is relaxed in Section 5. Moreover, this assumption is also motivated by empirical observation: considering mean-reverting shocks generates a negative autocorrelation in investment rates at the firm level that is not observed in the data. A random-walk process generates *iid* investment rates which are more aligned with the data.

2.2 Characterization of investment policy

Firms' investment policy. The firm's policy is described by three objects. An inaction region, denoted with $\mathcal{C} = (\underline{k}, \overline{k})$, such that the firm adjusts with probability 1 if $k_t \notin \mathcal{C}$. An adjustment hazard, denoted with $\Lambda(k)$, that describes investments within the inaction region $\mathcal{C}$ that happen with a sufficiently low realization of the fixed cost. A reset capital gap, denoted with $\hat{k}$, that describes the new capital gap after investment. Proposition 1 characterizes these three objects. For convenience, let us define the total drift $\nu \equiv -(\psi + \mu)$ and the adjusted discount $\tilde{\rho} \equiv \rho + \lambda - \mu - \frac{\sigma^2}{2}$. Recall that $H(\xi)$ denotes the *cdf* of the adjustment costs.

Proposition 1. *Let $V(K,E)$ be the value of a firm with initial capital K and initial productivity E:*

$$V(K,E) = \mathbb{E}\left[\int_0^\tau e^{-\rho t} E_t^{1-\alpha} K_t^\alpha dt + e^{-\rho\tau}\left(\kappa_\tau E_\tau + \max_{K^*} V(K^*, E_\tau) - (K^* - K_\tau)\right)\right]. \tag{8}$$

Then we can reexpress the value as $V(K,E) = Ev\left(log\left(\frac{K}{E}\right)\right)$, where $v(k)$, the continuation region $\mathcal{C}$ and the reset capital $\hat{k}$ satisfy the Hamilton-Jacobi-Bellman equation

$$\tilde{\rho} v(k) = e^{\alpha k} + \nu v'(k) + \frac{\sigma^2}{2} v''(k) + \lambda \int \max\left\{v(\hat{k}) - \xi - (\hat{k} - k), v(k)\right\} dH(\xi) \quad \forall k \in \mathcal{C}, \tag{9}$$

together with the value matching conditions

$$v(\underline{k}) - e^{\underline{k}} = v(\overline{k}) - e^{\overline{k}} = v(\hat{k}) - e^{\hat{k}} - \kappa, \tag{10}$$

and the optimality for the reset capital and smooth pasting conditions

$$v'(z) = e^z \quad for \quad z \in \{\underline{k}\ \overline{k},\ \hat{k}\}. \tag{11}$$

The adjustment dates $\tau_{\omega,i}$ (when capital is reset to $k_{\tau_{\omega,i}} = \hat{k}$) are given by

$$\tau_{\omega,i+1} = \inf\left\{t \geq \tau_{\omega,i} : k_{\omega,t} \notin \mathcal{C} \ or \ N^k_{\omega,t} - N^k_{\omega,\tau_{\omega,i}} \geq 1\right\}, \qquad \tau_{\omega,0} = 0. \tag{12}$$

where $N^k_{\omega,t}$ is a Poisson process with arrival rate $\Lambda(k) = \lambda H(v(\hat{k}) - v(k) - (\hat{k} - k))$.

Capital gaps and aggregate variables. Given the firms investment policy, we are interested in characterizing the log deviations of aggregate capital from its steady state. To this end, we define three variables. First, we define the capital gap $k_{\omega,t} \equiv \log(K_{\omega,t}/E_{\omega,t})$ as the log of the ratio of a firm's capital to its productivity. Second, we define $k_{ss} \equiv \mathbb{E}\left[\log(K_\omega/E_\omega)\right]$ to be the average of capital gaps in the steady state.[3] Lastly, we define the *normalized capital gap* $x_{\omega,t} \equiv k_{\omega,t} - k_{ss}$ as a firm's capital gap minus the steady state average. Redefining the state in this way is convenient to characterize the investment policy, and moreover, it has useful properties to think about aggregate objects.[4]

With these definitions, we compute the aggregate capital log deviation from steady state, denoted with $\hat{K}_t$, which up to a first order approximation, it is equal to the average normalized capital gap $\mathbb{E}[x_{\omega,t}]$:

$$\hat{K}_t \equiv \mathbb{E}\left[\log(K_{\omega,t})\right] - \mathbb{E}\left[\log(K_\omega)\right] = \mathbb{E}\left[k_{\omega,t}\right] - k_{ss} = \mathbb{E}[x_{\omega,t}], \tag{13}$$

where we use the assumption that productivity distribution is in the steady state. Notice that, in this normalization, we first centralize the capital-gap distribution around its steady state average and then we aggregate across firms. By the previous analysis, we may shift the focus from aggregate capital to moments of the normalized capital gaps. Finally, note that the dynamics of other aggregate variables, such as output deviations from steady state $\hat{Y}_t$, can also be expressed in terms of moments of normalized

[3] The notation without time index t refers to moments computed with the steady state distribution.

[4] Note that we are able to transform the state from capital and productivity (K,E) to capital gaps x due to the homothetic production function and the shape of the fixed cost. For all the proofs in this example see Web Appendix C.

capital gaps:

$$\hat{Y}_t \equiv \mathbb{E}\left[\log(Y_{\omega,t})\right] - \mathbb{E}\left[\log(Y_\omega)\right] = \alpha \hat{K}_t = \alpha \mathbb{E}[x_{\omega,t}] \tag{14}$$

Law of motion of capital gaps. To derive the law of motion of capital gaps, we use the firm policy and its adjustment hazard from Proposition 1. Given the investment policy, we normalize the state to consider deviations from steady state: $(\underline{x}, \hat{x}, \overline{x}) \equiv (\underline{k} - k_{ss}, \hat{k} - k_{ss}, \overline{k} - k_{ss})$. The *uncontrolled* capital gaps—not considering any investments—follow the process

$$d\tilde{x}_{\omega,t} = \nu dt + \sigma dW_{\omega,t}, \tag{15}$$

where we use tildes to show explicitly that these variables evolve exogenously. The initial conditions $\tilde{x}_{\omega,0}$ are exogenously given. By the discussion above, the initial condition of the uncontrolled capital gap is by $\tilde{x}_{\omega,0} = k_{\omega,0} - k_{ss}$.

In contrast, the *controlled* capital gaps—taking into account investments—evolves as

$$x_{\omega,t} = \tilde{x}_{\omega,t} + \sum_{\tau_{\omega,i} \leq t} \Delta x_{\tau_{\omega,i}}, \tag{16}$$

where the adjustment dates are defined in (12) and the investment rates $\Delta x_{\tau_{\omega,i}}$ are defined implicitly by the difference in the capital stock between date $\tau_{\omega,i}$ and immediately before adjustment $\tau^-_{\omega,i}$:

$$\Delta x_{\tau_{\omega,i}} = \hat{x} - x_{\tau^-_{\omega,i}}. \tag{17}$$

2.3 Steady state and transitional dynamics

Steady state moments. Consider the steady state distribution of the controlled state $F(x)$. Define $G(a|x)$ the distribution of the time since x's last adjustment, which we refer to it as "age". For any numbers $k, l \in \mathbb{N}$, we define the ergodic cross-sectional moment of capital gaps and age as

$$\mathbb{E}[x^k a^l] \equiv \int_x \int_a x^k a^l \; dG(a|x) \; dF(x) \quad \forall k, l \in \mathbb{N}, \quad with \quad \mathbb{E}[x] = 0. \tag{18}$$

Transitional dynamics. Fix an initial distribution of the state $F_0(x) = \mathbb{E}\left[\mathbb{1}_{\{x_{\omega,0} \leq x\}}\right]$. We define the impulse-response function for the $m-th$ moment of the capital gap distribution under the initial distribution F_0, denoted by $IRF_{m,t}(F_0)$, as the difference between its time t value and its ergodic value:

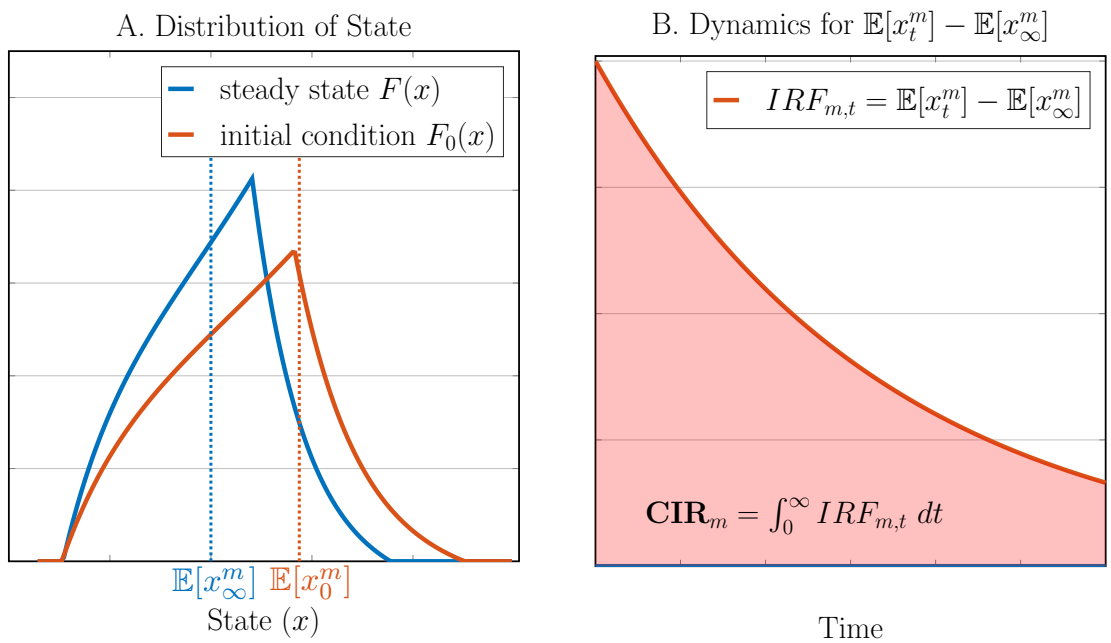
$$IRF_{m,t}(F_0, t) \equiv \underbrace{\mathbb{E}\left[x^m_{\omega,t}\right]}_{\text{transition}} - \underbrace{\mathbb{E}[x^m]}_{\text{steady state}}. \tag{19}$$

Following Alvarez, Le Bihan and Lippi (2016), we define the cumulative impulse-response (CIR), denoted by $\text{CIR}_m(F_0)$, as the area under the $IRF_{m,t}(F_0)$ curve across all dates $t \in (0, \infty)$:

$$\text{CIR}_m(F_0) \equiv \int_0^\infty IRF_{m,t}(F_0, t) \; dt. \tag{20}$$

Figure I illustrates these two objects. In the left panel, we plot the initial distribution F_0 and the steady state distribution F, and also highlight the m-th moment of capital gaps $\mathbb{E}[x_0^m]$ which will be tracked in its way towards steady state. In the right panel, the solid line represents the impulse-response of $\mathbb{E}[x_t^m]$, which is a function of time, and the area underneath it is the cumulative impulse-response or CIR.

Figure I – Cumulative Impulse-Response (CIR)



The CIR is our key measure of the convergence speed towards the steady state. The smaller is the CIR, the faster the convergence. The following Lemma expresses the CIR in a recursive way, and it is a generalization of the result in Alvarez, Le Bihan and Lippi (2016). Although at this stage we are working in a particular example, the result holds true for any moment of the distribution $m > 1$, for an arbitrary Markovian stopping policy, and for any Markovian law of motion of the uncontrolled state.

Lemma 1. *The CIR can be written recursively as:*

$$CIR_m(F_0) = \int v_m(x) dF_0(x). \tag{21}$$

where the value function for an agent with initial state x is given by:

$$v_m(x) \equiv \mathbb{E}^x \left[\int_0^\tau (x_t^m - \mathbb{E}[x^m])\, dt \right] \tag{22}$$

The idea behind Lemma 1 is to exchange the integral across agents (the cross-section) with the infinite time integral (the time-series).[5] Then, it is key to recognize that the first time a firm adjusts its capital it incorporates the deviations into its policy, and thus we only need to keep track until its *first* adjustment; any additional adjustment is driven by idiosyncratic conditions. The average of these

[5] This can be done due to the ergodic properties of the problem and the fact that moments are finite.

additional adjustments equals the ergodic moment $\mathcal{M}_m[x]$, implying that the value function $v_m(x)$ is equal to zero after the first adjustment. For that reason, the infinite time integral gets substituted for an integral between $t = 0$ and the stopping-time $t = \tau$.

3 The Three Properties in a Baseline Environment

This section derives the three properties of the CIR function. The first result—aggregation—approximates the transitional dynamics, measured via the CIR, as the stopping time problem of a representative firm that captures adjustments through the intensive and the extensive margins. The second result—representation—expresses the intensive and extensive margins in terms of moments of the state's ergodic distribution. Finally, the third result—observation—connects the state's ergodic moments with the distribution of policies Δx and τ, which are observable statistics in most micro-data sets.

Our baseline environment focuses on the inaction model of investment presented in Section 2. Here, we study transitional dynamics for average capital gaps, i.e. $m = 1$, when the initial condition consists of a mean translation of the steady state distribution. In Section 5, we extend the results for an arbitrary state space, arbitrary policies, and other additional features.

Initial conditions as δ-perturbations around steady state. For ease of exposition, we interpret the initial condition as a perturbation of the steady state distribution, that can be described in terms of one parameter δ. In particular, in the baseline case analyzed here, we consider a perturbation that translates horizontally the distribution of capital gaps. If $f(x-\delta)$ denotes the new density of capital gaps, and we approximate it as $f(x-\delta) \approx f(x) - \delta f'(x)$, we observe that it is equal to a right shift of the steady state density by $\delta f'(x)$. Afterwards, the distribution will evolve according to the agents' policies and will converge back to its steady state. Under this interpretation, we denote $\text{CIR}_m(F_0)$ with $\text{CIR}_m(\delta)$.

3.1 Aggregation

Starting from its recursive representation, Proposition 2 computes the cumulative CIR as the sum of intensive margin Γ and extensive margin Θ components, defined below.

Proposition 2. *To a first order, the transitional dynamics towards the steady state, measured through the CIR, are given by*

$$CIR_1(\delta) = \delta \times \left(\Gamma_1 + \Theta_1 - \frac{\sigma^2}{2\nu}\Theta_0\right) + o(\delta^2) \tag{23}$$

where the intensive margin component Γ_1 and the extensive margin components Θ_1 and Θ_0 are:

$$\Gamma_1 \equiv \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \varphi_1^\Gamma(x_t)dt\right]}{\mathbb{E}^{\hat{x}}[\tau]} \quad \textit{with} \quad \varphi_1^\Gamma(x) \equiv \frac{1}{\nu}\left(\mathbb{E}^x\left[x_\tau\right] - x\right) \tag{24}$$

$$\Theta_m \equiv \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \varphi_m^\Theta(x_t)dt\right]}{\mathbb{E}^{\hat{x}}[\tau]} \quad \textit{with} \quad \varphi_m^\Theta(x) \equiv \frac{1}{\nu}\left(\frac{\partial \mathbb{E}^x\left[x_\tau^{m+1}/(m+1)\right]}{\partial x} - \mathbb{E}^x\left[x_\tau^m\right]\right) \tag{25}$$

Equation (23) measures the total effect of the δ-perturbation as an area, with a height of δ, and a base given by two components: Γ_1, which measure of adjustments through the intensive margin, and

Θ_1, Θ_0, that measure adjustments through the extensive margin. In turn, each component is expressed through two nested HJB equations of a representative firm in (24) and (25), what is sometimes known as recursively squared. The inner HJBs denoted by $\varphi_1^{\Gamma}(x)$, and $\varphi_m^{\Theta}(x)$ track conditional dynamics for any initial condition x. They captures the evolution of the state from the initial condition towards the new steady state. The outer HJBs measures the mass of firms at each initial condition by computing the occupancy measure or local time spent at such state between any two adjustments. Together, the nested recursive problems capture the total deviations between the initial condition and the final destination, i.e., the steady state.[6]

The result uses four steps. The first step consists in a first order Taylor approximation of $\text{CIR}_1(\delta)$ together with integration by parts that delivers

$$\text{CIR}_1(\delta) \;=\; \int_{\underline{x}}^{\overline{x}} v(x) f(x-\delta) dx \;\approx\; \delta \times \int_{\underline{x}}^{\overline{x}} v'(x) f(x) dx. \tag{26}$$

where we have used that there is no mass at the boundary of the inaction region (or the regularity of the boundary of the continuation region).

The second step consists in decomposing the base into an intensive and an extensive margin. Since $\mathbb{E}[x] = 0$ by the normalization, we have that $v(x) = \mathbb{E}^x\left[\int_0^\tau x_t dt\right]$ and its derivative is $v'(x) = d\mathbb{E}^x\left[\int_0^\tau x_t dt\right]/dx$. Substitute this derivative into (26); then add and subtract the expectation of the derivative of the state with respect to the initial conditions, $\mathbb{E}^x\left[\int_0^\tau \frac{dx_t}{dx} dt\right]$, which equals $\mathbb{E}^x[\tau]$ to obtain[7]:

$$\text{CIR}_1(\delta) \;\approx\; \delta \underbrace{\int_{\underline{x}}^{\overline{x}} \mathbb{E}^x[\tau] f(x) dx}_{\Gamma_1 \;=\; \text{intensive margin}} \;+\; \delta \underbrace{\int_{\underline{x}}^{\overline{x}} \left(\frac{d\mathbb{E}^x\left[\int_0^\tau x_t dt\right]}{dx} - \mathbb{E}^x[\tau] \right) f(x) dx}_{\Theta_1 - \frac{\sigma^2}{2\nu}\Theta_0 \;=\; \text{extensive margin}}. \tag{27}$$

The first component measures aggregate adjustments through the intensive margin: changes in the path x_t due to the new initial condition x *keeping the duration fixed*. The second component measures the aggregate adjustments through the extensive margin: changes in duration due to the new initial condition *keeping the state's path fixed*. To see this clearly, suppose the state evolves deterministically, then the second term becomes $x_t \cdot d\tau/dx$.

The third step consists in finding an equivalent recursive representation for the conditional dynamics inside the integrals in (27). Let us show the steps for the intensive margin. Using the law of motion of x_t, we have that $dx_t = \nu dt + \sigma dW_t$. Then, integrating from 0 to τ and taking expectations with initial condition x, we have that $\mathbb{E}^x[x_\tau] = x + \nu\mathbb{E}^x[\tau] + \sigma\mathbb{E}^x\left[\int_0^\tau dW_t\right]$. By the Optional Sampling Theorem, the last term is equal to zero because it is a martingale with zero initial condition. Therefore, we can express $\mathbb{E}^x[\tau] = \frac{\mathbb{E}^x[x_\tau]-x}{\nu}$. With similar steps, we find the expression for the extensive margin.

In the final step, we focus on the unconditional dynamics, that is, the measure used to integrate the conditional dynamics outlined above. For this purpose, we use an alternative representation of the steady state distribution as the occupancy measure.[8] Intuitively, the mass of agent in x—given

[6] Convergence to the steady state is not needed for the aggregation result. For example, the property goes through in an Ss model without idiosyncratic shocks that features cycles, as in Caplin and Spulber (1987).

[7] $\mathbb{E}^x\left[\int_0^\tau \frac{dx_t}{dx} dt\right] = \mathbb{E}^x\left[\int_0^\tau \frac{d(x+\nu t+\sigma W_t)}{dx} dt\right] = \mathbb{E}^x\left[\int_0^\tau 1 dt\right] = \mathbb{E}^x[\tau]$.

[8] See Stokey (2009) for details.

by $f(x)dx$—is equal to the amount of time spent at x. Formally, we have the following equivalence: $F(x) \equiv \Pr[x_t \leq x] = \mathbb{E}^{\hat{x}}\left[\int_0^\tau \mathbb{1}_{\{x_t \leq x\}}\, dt\right] / \mathbb{E}^{\hat{x}}[\tau]$. Thus, we rewrite $\frac{\text{CIR}_1(\delta)}{\delta}$ in (27) as

$$\underbrace{\frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \overbrace{\left(\frac{\mathbb{E}^x[x_\tau]-x}{\nu}\right)}^{=\varphi_1^\Gamma(x_t)} dt\right]}{\mathbb{E}^{\hat{x}}[\tau]}}_{\Gamma_1} + \underbrace{\frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \overbrace{\frac{\partial \mathbb{E}^x\left[x_\tau^2/2\right]/\partial x - \mathbb{E}^x\left[x_\tau\right]}{\nu}}^{=\varphi_1^\Theta(x_t)} dt\right]}{\mathbb{E}^{\hat{x}}[\tau]}}_{\Theta_1} - \frac{\sigma^2}{2\nu}\underbrace{\frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \overbrace{\frac{\partial \mathbb{E}^x\left[x_\tau\right]/\partial x - 1}{\nu}}^{=\varphi_0^\Theta(x_t)} dt\right]}{\mathbb{E}^{\hat{x}}[\tau]}}_{\Theta_0} \tag{28}$$

Usefulness of aggregation. We have shown that transitional dynamics (measured through the CIR) can be written a the solution of a recursive problem for a single agent. The intuition lies in the fact that after adjustment, firms become identical, and then any differences in their dynamics arise due to differences in their initial conditions at the moment of the perturbation. All the information regarding these ex-post differences is neatly summarized in the representative agent problem.

The usefulness of the aggregation result in Proposition 2 is twofold. Within the scope of this paper, it allows us to derive a connection between the intensive and extensive margins and the steady state moments of the distribution as we show next. More generally, its usefulness lies in aiding researchers with an aggregation result in models with inaction in which the cross-sectional distribution is part of the state.

3.2 Representation

With a recursive expression for the transitional dynamics at hand, the second step consists in expressing the three components of the CIR, Γ_1 and Θ_1 and Θ_0, as a function of the steady state cross-sectional moments. We derive a mapping that does not depend on a particular inaction model, therefore different models will produce different aggregate effects if and only if they change the ergodic moments.

Characterization of intensive margin Γ_1. The first component of the CIR, Γ_1, measures the aggregate effects keeping any changes in aggregate duration fixed. Proposition 3 shows that the intensive margin equals the capital's average age across firms.

Proposition 3. *The intensive margin component is given by:*

$$\Gamma_1 = \mathbb{E}[a] \tag{29}$$

Technically, the proof consists on finding an equivalence between the average age $\mathbb{E}[a]$ and Γ_1 by using the occupancy measure, Itō's lemma, and the law of iterated expectations. First, by the equivalence between the ergodic distribution and the occupancy measure, we have that

$$\Gamma_1 = \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau (x_\tau - x_t)\, dt\right]}{\nu \mathbb{E}^{\hat{x}}[\tau]} = \frac{\mathbb{E}^{\hat{x}}[x_\tau \tau]}{\nu \mathbb{E}^{\hat{x}}[\tau]} - \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t\right]}{\nu \mathbb{E}^{\hat{x}}[\tau]} = \frac{\mathbb{E}^{\hat{x}}[x_\tau \tau]}{\nu \mathbb{E}^{\hat{x}}[\tau]}, \tag{30}$$

where we have used that $\frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t\right]}{\nu \mathbb{E}^{\hat{x}}[\tau]} = \frac{\mathbb{E}[x]}{\nu} = 0$ by the normalization. Lastly, following similar steps as before (using Itō's Lemma and the Optional Sampling Theorem), it is easy to show that $\frac{\mathbb{E}^{\hat{x}}[x_\tau \tau]}{\nu \mathbb{E}^{\hat{x}}[\tau]} = \frac{\mathbb{E}[x]}{\nu} + \mathbb{E}[a] = 0 + \mathbb{E}[a]$.Substituting this equivalence above, we have the result.

How do we understand the connection between the intensive margin and the capital's average age? Average age provides information about the speed at which the average firm adjusts to the perturbation from the steady state. The older is average capital in the economy, the longer the transition. Consider a frictionless limit in which all firms continuously invest to brings capital gaps to zero. Since capital in all firms would have age equal to zero, the economy reaches its steady state immediately. The reason is that any deviation from steady state is immediately absorbed into the representative firm's policy and there are no persistent deviations from steady state.

Characterization of extensive margin Θ_1 and Θ_0. Now, it is the turn to characterize the evolution of the extensive margin Θ_m in terms of ergodic cross-sectional moments. There are two main challenges. First, the extensive margin does not only depend on the *immediate* response of the aggregate adjustment frequency, but it also reflects all current and *future changes* in frequency. Second, even if we had the whole sequence of adjustment frequency that follows a perturbation, the extensive margin also depends on the capital-gaps of the particular set of firms selected to invest. This is clearly seen again in the deterministic case, where the extensive margin becomes $x_t \cdot d\tau/dx_0$, thus the change in frequency is scaled by the state, difficulting its characterization. Next, we develop a theory to discipline these two objects.

Proposition 4 presents a characterization of the extensive margin in terms of two objects: how investment responds to idiosyncratic initial conditions (a micro-elasticity) and the aggregate moments of the distribution. Together, these objects imply a macro-elasticity of the extensive margin with respect to the perturbation. Additionally, as known in the literature, we show that in time-dependent models, the extensive margin does not play a role.

Proposition 4. *Let $g_m(x)$ be a smooth function such that for all m*

$$g_m(x) = \mathbb{E}^{\hat{x}+x}[(\hat{x} - \Delta x)^m] - \mathbb{E}^{\hat{x}}\left[(\hat{x} - \Delta x + x)^m\right], \tag{31}$$

and define the micro-elasticities as $\theta_{m,j} \equiv \frac{1}{\nu} \sum_{k \geq j}^{\infty} \frac{\hat{x}^{k-j}}{k!j!} \left[\frac{d^{k+1} g_{m+1}(0)/m+1}{dx^{k+1}} - \frac{d^k g_m(0)}{dx^k}\right]$. Then, the extensive margin is given by

$$\Theta_m = \sum_{j=0}^{\infty} \theta_{m,j} \mathbb{E}[x^j]. \tag{32}$$

Moreover, if τ is independent of x, then $g_m(x) = \theta_{m,j} = 0$ for all m and $\Theta_m = 0$ as well.

First, let us describe each object in equation (31). Recall that the expected capital gap at the moment of adjustment is equal to $x_\tau = \hat{x} - \Delta x$. Now, the first term, given by $\mathbb{E}^{\hat{x}+x}[(\hat{x} - \Delta x)^m]$, equals the expected capital gap at the moment of adjustment when the initial condition is $\hat{x} + x$; while the second term, given by $\mathbb{E}^{\hat{x}}\left[(\hat{x} - \Delta x + x)^m\right]$, equals the expected capital gap at the moment of adjustment plus a deterministic increase of size x when the initial condition is $\hat{x}$. The difference between these two functions of x provides

information of how the *stopping time* depends on the initial condition and how it correlates with the state. To see this more clearly, notice that we can re-express $g_m(x)$ in the following way

$$g_m(x) = \mathbb{E}\left[(\hat{x} + x - \nu\tau^{\hat{x}+x} - \sigma W_{\tau^{\hat{x}+x}})^m\right] - \mathbb{E}\left[(\hat{x} + x - \nu\tau^{\hat{x}} - \sigma W_{\tau^{\hat{x}}})^m\right], \tag{33}$$

where τ^z is the stopping time with initial condition z. In equation (33) we observe that if τ is independent of the initial condition, i.e. $\tau^{\hat{x}+x} = \tau^{\hat{x}}$, as in time-dependent models, then $g_m(x) = 0$ for all x. Thus, $g_m(x)$ provides a micro-elasticity of firms idiosyncratic response to the new initial conditions though changes in its stopping-time τ. To construct the aggregate elasticity, we aggregate the micro-elasticities with weights equal to the capital-gap's ergodic moments, which encode information about the state's distribution.

Constructing $g_m(x)$ from the data. To construct the micro-elasticities, we need to ask: what is observable in the data and what is not? The object $\mathbb{E}^{\hat{x}}\left[(\hat{x} - \Delta x + x)^m\right]$ is an observable statistic as it depends on the steady state investment rates.[9] The objects $\mathbb{E}[x^j]$ and $\hat{x}$ can also be recovered from the data as we show in the next section. Therefore, the only object that might not be directly observable is $\mathbb{E}^{\hat{x}+x}\left[(\hat{x} - \Delta x)^m\right]$, which measures the elasticity of investment with respect to changes in initial conditions. Guided by the theory, we suggest that this elasticity is the key object that future research should focus on computing, both in the data and in the models.

There are two papers that use an adequate methodology and data to construct the micro-elasticities. First, in the pricing literature, Karadi and Reiff (2014) study the immediate monthly price response to a change in the VAT with Hungarian CPI. Since a change in the VAT proxies a cost-push shock, the experiment is equivalent to an increase in firm's markup in the same proportion; this design would allow to compute the micro-elasticity of the expected price change to initial conditions. Second, in the investment literature, Zwick and Mahon (2017) exploit shifts in accelerated depreciation to estimate the effect of temporary tax incentives on equipment investment; such a design would allow to compute the micro-elasticity of expected investment to initial conditions.

The role of micro-elasticities: two examples. Are the micro-elasticities necessary to discipline the extensive margin? Moreover, can we provide an analytic value for the infinite sum in Θ_m, that can be related to ergodic moments? The answer to these questions is model-dependent as we illustrate with one example and one counterexample. First, we show that in the random fixed cost model presented in Section 2 (known in the pricing literature as the CalvoPlus model, see Nakamura and Steinsson (2008)), the micro-elasticities are not needed to compute the CIR, since there exists a one-to-one mapping from ergodic moments to the CIR. Second, we provide a counterexample where two different models that generate the same ergodic moments have different CIR; thus, in this case, micro-elasticities play a key role in determining the CIR through the extensive margin.

Example 1. *In the random fixed cost model, where $H(\xi) = 1$ for all ξ, the CIR is equal to (up to first*

[9] If idiosyncratic volatility is large enough with respect to aggregate volatility, then steady state micro-statistics can be recovered with average statistics in a model with business cycles. See Blanco (2015) for a verification of this statement in the context of firm pricing decisions.

order):

$$CIR_1(\delta)/\delta \approx \frac{\mathbb{V}ar[x] - \nu\mathbb{C}ov[a,x]}{\sigma^2}. \tag{34}$$

Equation (34) shows that there exists a one-to-one mapping from the ergodic cross-sectional moments to the CIR, which is determined by the steady state variance of the capital-gaps, normalized by the shock volatility, minus the covariance between capital age (or vintage) and the capital gap. Consequently, under this particular type of adjustment cost, micro-elasticities are not needed to characterize transitional dynamics.

To build the intuition for this result, consider the case $\nu = 0$ so that the CIR is given exclusively by the (ex-post) dispersion of capital gaps $\mathbb{V}ar[x]$ normalized by the (ex-ante) volatility of shocks. This dispersion encodes information about agents' *responsiveness* to idiosyncratic shocks (the higher the ratio the less responsive), and in turn, the responsiveness determines the speed of convergence to the steady state. For instance, high levels of capital misallocation or large price dispersion (normalized by the volatility of idiosyncratic shocks) signal little responsiveness to idiosyncratic shocks, from where we infer that there is also slow adjustment to aggregate shocks. In the case $\nu \neq 0$, the covariance between capital vintage and the investment rate (which is negative in this model) appears as a way to correct for the effects of the drift.

From Example 1 a natural question arises: Is this a general result? Are the micro-elasticities presented in Proposition 4 actually not relevant? The answer to both questions is no, as we show with the following counterexample.[10]

Example 2. *Let $T \equiv \mathbb{E}^{\hat{x}}[\tau]$ denote average duration. Consider an inaction model with adjustments at fixed dates (Taylor-type) and a standard Ss model; assume away idiosyncratic shocks ($\sigma = 0$) and allow for a non-zero drift ($\nu \neq 0$). In these two models there exists a steady state with a uniform distribution of capital-gaps and an investment distribution with an atom at $-\nu T$; thus they produce the same ergodic moments.*[11] *Now, let us study transitional dynamics for $\delta < 0$. As stated by the theory, in both models the intensive margin is equal to the average age: $\Gamma_1 = T/2$. Since the Taylor model is time-dependent, $\Theta_1 = 0$ and its CIR equals: $CIR^{Taylor}(\delta)/\delta \approx T/2$. In the Ss model, the extensive margin is equal to $\Theta_1 = \theta_{1,0}\mathbb{E}[x^0] = -T/2$ (as $\theta_{1,j} = 0$ for $j > 0$), and its CIR equals $CIR^{Ss}(\delta)/\delta \approx T/2 - T/2 = 0$. This result mirrors the classic money neutrality outcome in Caplin and Spulber (1987) in lumpy adjustment models.*

The previous counterexample illustrates that two models may produce the same steady state statistics, but nevertheless, they can exhibit completely different transitional dynamics. Our explanation lies in the differences in micro-elasticities, zero in the Taylor-type model and $-\Gamma_1$ in the Ss model. Therefore, there exist cases for which the micro-elasticities are relevant objects for characterizing the extensive margin, and our theory can guide researchers in finding experiments or exogenous variation to compute them.

[10] See Web Appendix E.4 for the proof.

[11] These two models have several ergodic time-varying distributions that depend on the initial condition. To generate an unique ergodic distribution for any initial condition, we add a small and random probability of free adjustment. Besides generating a unique ergodic distribution, it gives differentiability to the CIR at $\delta = 0$ in the Ss model.

3.3 Observation

In the third set of results, we express the ergodic cross-sectional moments of the state distribution and the structural parameters in terms of the investment distribution Δx and adjustment dates τ. The relevance of this result lies in that in many applications, the state x is likely to be unobservable, but the adjustments Δx and τ are. This is the case in our example, as capital gaps are hard to observe but investment rates are readily available in the data.[12] As a consequence of the results in the following Proposition 5, we can track unobservable states using observable statistics.

Proposition 5. *Let $(\Delta x, \tau)$ be a panel of observations of adjustment size and inaction duration. Denote with $\mathbb{E}^{\hat{x}}[\cdot], \mathbb{C}ov^{\hat{x}}[\cdot]$ and $\mathbb{CV}^{\hat{x}}[\cdot]^2 = \mathbb{V}^{\hat{x}}[\cdot]/\mathbb{E}^{\hat{x}}[\cdot]^2$, respectively, the cross-sectional average, covariance and coefficient of variation squared conditional on adjustment.*

1. *The reset capital gap is given by:*

$$\hat{x} = \frac{\mathbb{E}^{\hat{x}}[\Delta x]}{2}\left(1 - \mathbb{CV}^{\hat{x}}[\tau]^2\right) + \frac{\mathbb{C}ov^{\hat{x}}[\tau, \Delta x]}{\mathbb{E}^{\hat{x}}[\tau]}. \tag{35}$$

2. *The drift and volatility of the capital gap process are recovered as:*

$$\nu = -\frac{\mathbb{E}^{\hat{x}}[\Delta x]}{\mathbb{E}^{\hat{x}}[\tau]} \quad ; \quad \sigma^2 = \frac{\mathbb{E}^{\hat{x}}[\Delta x^2]}{\mathbb{E}^{\hat{x}}[\tau]} + 2\nu\hat{x}. \tag{36}$$

3. *For any $m \leq 1$, the steady state moments are given by:*

$$\mathbb{E}[x^m] = \frac{1}{m+1}\left\{\frac{\hat{x}^{m+1} - \mathbb{E}^{\hat{x}}\left[(\hat{x} - \Delta x)^{m+1}\right]}{\mathbb{E}^{\hat{x}}\left[\Delta x\right]} - \frac{\sigma^2 m(m+1)}{2\nu}\mathbb{E}[x^{m-1}]\right\}, \tag{37}$$

$$\mathbb{E}[a] = \frac{1}{2}\mathbb{E}^{\hat{x}}[\tau]\left(1 + \mathbb{CV}^{\hat{x}}[\tau]^2\right), \tag{38}$$

$$\mathbb{E}[x^m a] = \frac{1}{\nu(m+1)}\left\{\frac{\mathbb{E}^{\hat{x}}\left[\tau\left(\hat{x} - \Delta x\right)^{m+1}\right]}{\mathbb{E}^{\hat{x}}[\tau]} - \mathbb{E}[x^{m+1}] - \frac{\sigma^2 m(m+1)}{2}\mathbb{E}[x^{m-1}a]\right\}, \tag{39}$$

Equation (35) shows the observability property for the reset state $\hat{x}$, which is derived from the cross-equation restriction imposed by the normalization of the ergodic mean $\mathbb{E}[x] = 0$. The expression has two components: the first one mainly reflects the effect of the drift in the reset state, while the second one mainly reflects the asymmetry in the policies.

To explain the drift component consider an Ss model without idiosyncratic shocks and a negative drift. In such a model, τ and Δx have a degenerate distribution and the capital-gap distribution is uniform in the domain $[\hat{x} - \Delta x, \hat{x}]$. Our formula implies a reset state of $\hat{x} = \frac{\mathbb{E}^{\hat{x}}[\Delta x]}{2} > 0$, which centers the distribution around zero. By compensating the negative drift, the positive reset state ensures that the

[12] Our formulas require us to compute the change in log capital gaps Δx in the data. Due to the continuity assumption for the idiosyncratic productivity, the changes in the capital-gap equal the observed investment rates: $\Delta x_{\omega,\tau} = \lim_{t\uparrow\tau_i} \log(K_{\omega,\tau_i}/K_{\omega,t}) - \lim_{t\uparrow\tau_i} \log(E_{\omega,\tau_i}/E_{\omega,t}) = \lim_{t\uparrow\tau_i} \log(K_{\omega,\tau_i}/K_{\omega,t})$. Therefore, we can compute the changes in the *capital gap* using changes in the *capital stock*.

ergodic mean is zero. To explain the second component, set the drift to zero and consider an asymmetric inaction region $|\underline{x} - \hat{x}| > |\overline{x} - \hat{x}|$ such that the upper trigger is closer to $\hat{x}$ than the lower trigger. In this case, the capital-gap distribution is skewed towards values lower than $\hat{x}$. Our formula implies a reset state given by $\hat{x} = \mathbb{C}ov^{\hat{x}}[\tau, \Delta x]/\mathbb{E}^{\hat{x}}[\tau] > 0$.[13] By reflecting the bias in the policy, the positive reset state shifts the distribution to the right to ensure that the ergodic mean is zero.

Expressions in (36), which extend those in Alvarez, Le Bihan and Lippi (2016) for the case with drift, provide a guide to infer the parameters of the stochastic process. The first expression shows how to infer the drift from the average investment rate in the data, scaled by the adjustment frequency; and the second expression shows how to infer the volatility from the dispersion in investment rates, scaled by the frequency and corrected by the drift (which also generates dispersion, but not due to fundamental volatility).

Equation (38) relates average age to the average and the dispersion in duration, measured through the coefficient of variation. The relationship with the average duration is straightforward. To understand why the dispersion of duration affects average age, it is important to recall a basic property in renewal theory: the probability that a random firm has an expected time between capital changes of τ is increasing in τ, i.e. larger stopping times are more representative in the capital-gap distribution.[14] Therefore, dispersion in duration means there are firms that take a long time to adjust, and on top of that, those firms are more representative in the economy; this rises the average age.

Lastly, equation (37) provides a recursively formula to compute the centralized moments using observed investment rates. By assuming a stochastic process for the uncontrolled capital gaps ($dx_t = \nu dt + \sigma dW_t$), together with Itō's lemma and the Optional Sampling Theorem, we can connect the average *slope* of moment $m+1$ to the *level* of the previous moments m and $m-1$ as follows:

$$\underbrace{\mathbb{E}_t\left[dx_t^{m+1}\right]}_{slope} = \nu(m+1)\underbrace{x_t^m}_{level}dt + \frac{\sigma^2}{2}(m+1)m\underbrace{x_t^{m-1}}_{level}dt + \sigma(m+1)\underbrace{\mathbb{E}_t\left[x_t^m dW_t\right]}_{=0} \tag{40}$$

Therefore, the average investment to the power $m+1$, gives information about the centralized moment m of capital gaps. To see this clearly, set $\hat{x} = 0$ and $m = 2$, then equation (37) reads $\mathbb{E}[x^2] = \frac{\mathbb{E}^{\hat{x}}\left[(\Delta x)^3\right]}{3\mathbb{E}^{\hat{x}}[\Delta x]}$, relating the dispersion of capital gaps in the LHS to the skewness of investment rates in the RHS. A similar argument holds for (39).

[13] The covariance is positive since the longer the duration, the higher is the probability to hit the lower trigger and to do an upward adjustment.

[14] This property has been widely studied in labor economics when thinking about long-term unemployment. For example, Mankiw (2014)'s textbook Principles of Macroeconomics mentions that: "[...] many spells of unemployment are short, but most weeks of unemployment are attributable to long-term unemployment".

4 Application: Investment Dynamics

In this section, we revisit the investment model from Section 2 and apply our tools using establishment-level data to gauge the magnitude of capital misallocation in steady state as well as the transitional dynamics of capital gaps.

4.1 Data description

Sources. We use yearly micro data on the cross-section of manufacturing plants in Chile. Information on depreciation rates and price deflators, used to construct the capital series, comes from Penn World Tables and National Accounts. We report statistics for the total capital stock as well as for structures, a capital category that represents approximately 30% of all investment in the manufacturing sector and features the strongest lumpy behavior.[15] We consider all plants that appear in the sample for at least 10 years (more than 60% of the sample).

Capital stock and investment rates. We construct the capital stock series through the perpetual inventory method (PIM).[16] Let firm's ω stock of capital of type j on year t be given by:

$$K_{\omega,j,t} = (1-\delta_j)K_{\omega,j,t-1} + I_{\omega,j,t}/D_{j,t} \quad \text{for } K_{\omega,j,t_0} \text{ given,} \tag{41}$$

where the depreciation rate δ_j is a type-specific time-invariant depreciation rate; price deflators $D_{j,t}$ are gross fixed capital formation deflators by capital type; and initial capitals K_{i,j,t_0} are given by the firms' self-reported nominal stock of capital of type j at current prices on the first year in which they report a non-negative capital stock. Gross nominal investment $I_{\omega,j,t}$ is based on the information on purchases, reforms, improvements, and sales of fixed assets reported by each plant in the survey:

$$I_{\omega,j,t} = puchases_{\omega,j,t} + reforms_{\omega,j,t} + improvements_{\omega,j,t} - sales_{\omega,j,t} \tag{42}$$

Once we construct the investment and capital stock series, we define the investment rate $i_{\omega,j,t}$ as the ratio of gross investment to the capital stock:

$$i_{\omega,j,t} \equiv I_{\omega,j,t}/K_{\omega,j,t}. \tag{43}$$

Table I presents descriptive statistics on investment rates. Inaction is defined as investment below 1% in absolute value; positive spikes are investments above 20% and negative spikes below −20%. Besides the statistics for Chile, we include the numbers reported by Cooper and Haltiwanger (2006) for 7,000 US manufacturing plants between 1972 and 1988.[17] Structures presents an inaction rate of 76% and a large fraction of positive spikes of 31%. Investment rates are very asymmetric (the frequency of positive investments is larger than the frequency of negative investments) and serially uncorrelated, as in the US

[15] The Online Data Appendix presents all the details on the data, construction of variables, and analysis. We repeat the results for each capital category separately: structures, machinery, equipment and vehicles.

[16] See Section A.2 for details on the PIM method and several checks on the data.

[17] The Data Online Appendix shows numbers reported in Zwick and Mahon (2017) from tax records for US firms. In particular, the weighted inaction rate across firms is 30% in their unbalanced panel.

data. The zero correlation of investment rates in the data is consistent with the model we developed in the previous section.

Table I – Investment Rates Statistics

	Structures	**Total**	**US**
Average Investment	10.3	18.3	12.2
Positive Fraction ($i > 1\%$)	22.8	56.7	81.5
Negative Fraction ($i < -1\%$)	1.4	4.0	10.4
Inaction rate ($\|i\| <= 1\%$)	75.8	39.3	8.1
Spike rate ($\|i\| > 20\%$)	10.2	24.3	20.4
Positive spikes ($i > 20\%$)	9.7	23.2	18.6
Negative spikes ($i < -20\%$)	0.5	1.0	1.8
Serial correlation	0.0	0.0	0.1

Sources: Own calculations using plant-level data for Chile. The time period is 1979-2011 for plants that appear in the sample for at least 10 years. US refers to data in Cooper and Haltiwanger (2006) which covers manufacturing plants in the US from 1972 to 1988. Investment rates reported in this table are computed as investment divided by initial capital. We use perpetual inventories to compute capital stock. We eliminate investment rates below the 1st percentile and above the 99th percentile of the investment rate distribution.

4.2 Construction of capital gaps and duration

To apply our formulas, for each firm ω and each inaction spell k, we record the capital gap change upon action $\Delta x_{\omega,k}$ and the spell's duration $\tau_{\omega,k}$. Recall that the capital gap change is given by the log difference in the capital stock between an adjustment date $t_{\omega,k}$ and immediately before adjustment $t^{-}_{\omega,k}$:

$$\Delta x_{\omega,k} = \hat{x} - x_{t^{-}_{\omega,k}} = \log\left(K_{t_{\omega,k}}/K_{t^{-}_{\omega,k}}\right) = \log\left(1 + i_{\omega,t}\right) \tag{44}$$

Using the information on investment rates, we construct capital gap changes as:

$$\Delta x_{\omega,k} = \begin{cases} \log\left(1 + i_{\omega,k}\right) & \text{if} \quad |i_{\omega,k}| > \underline{i} \\ 0 & \text{if} \quad |i_{\omega,k}| < \underline{i}, \end{cases} \tag{45}$$

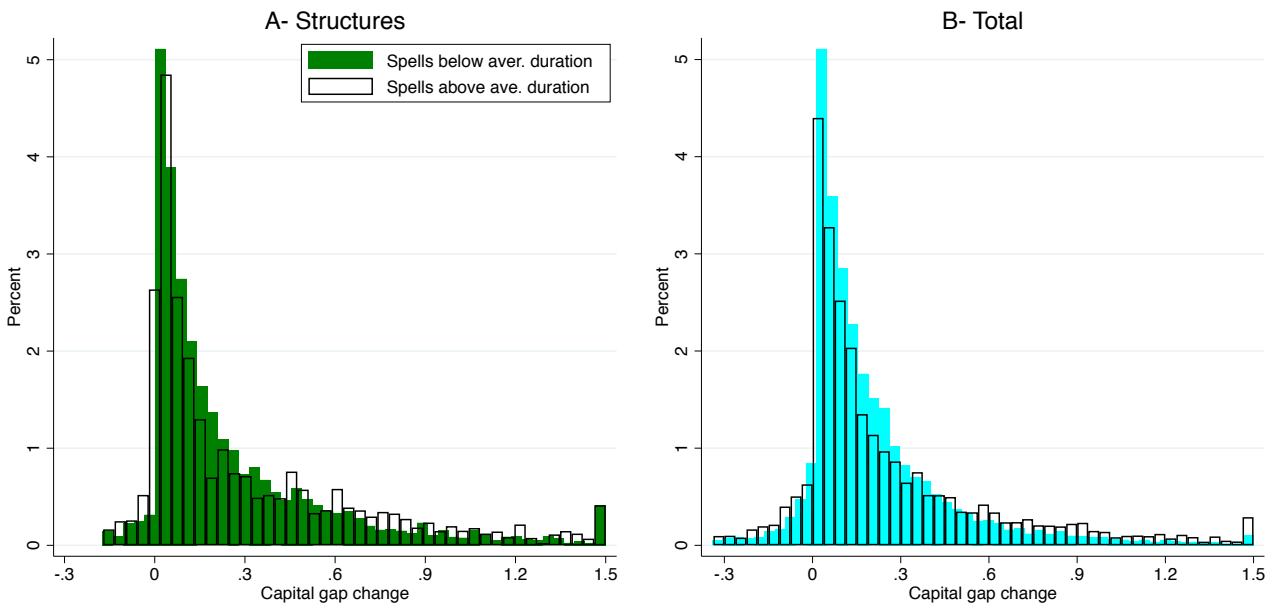
where $\underline{i} > 0$ is a parameter that captures the idea that small maintenance investments do not incur the fixed cost. Following Cooper and Haltiwanger (2006), we set $\underline{i} = 0.01$, such that all investments smaller than 1% in absolute value are excluded and considered as inaction. Finally, we compute a spell's duration as the difference between two adjacent adjustment dates:[18]

$$\tau_{\omega,k} = t_{\omega,k} - t_{\omega,k-1}. \tag{46}$$

[18] See the Online Data Appendix for corrections due to duration dependence and unobserved heterogeneity.

Figure II plots the cross-sectional distribution of capital gap changes for structures and the total capital stock. In each figure, we show the distribution for two subsamples: observations with spell duration above the average (dark bars) and spell duration below the average (white bars). Interestingly, the two populations present the same behavior in terms of adjustment size: there is a large mass concentrated at low levels of capital gap changes and there are a few firms that have very large adjustments, and the distribution is asymmetric. Moreover, given that the two distributions lie on top of one another, there is no apparent correlation between adjustment size and duration.

Figure II – Distribution of capital gap changes by duration



As the next step, we put the theory to work by computing the cross-sectional statistics of the capital gap changes and duration in order to back out the parameters of the stochastic process as well as the ergodic moments. With these objects at hand, we then study the transitional dynamics via the CIR implied by different models of inaction.

4.3 Putting the theory to work

The relationships derived in Proposition 5 tell us how to use cross-sectional data on capital gaps and duration to pin down the parameters of the productivity process, the reset point, as well as the ergodic moments, which in turn are used to construct the CIR. Table XVI summarizes the statistics calculated from the micro data which serve as inputs into the formulas, as well as the theory's output.

Table XVI shows results by capital type and is divided in two parts. In the upper part, we show the inputs from the data for our observation formulas: cross-sectional statistics for frequency, capital gaps, and covariances between them. The lower part of the table shows the outputs from our theory: estimated parameters $\nu, \sigma^2, \hat{x}$ and ergodic moments $\mathbb{V}ar[x], \mathbb{C}ov[x, a]$, as well as the implied $\text{CIR}_1(\delta)$.

Table II – Inputs from Micro Data and Outputs from the Theory

	Inputs from Data			Outputs from Theory	
	Structures	**Total**		**Structures**	**Total**
Frequency			**Parameters**		
$\mathbb{E}^{\hat{x}}[\tau]$	2.441	1.714	ν	-0.111	-0.129
$\mathbb{CV}^2[\tau]$	1.093	0.855	σ^2	0.076	0.067
			$\hat{x}$	0.013	0.039
Capital Gaps			**Steady State Moments**		
$\mathbb{E}^{\hat{x}}[\Delta x]$	0.271	0.220	$\mathbb{V}ar[x]$	0.228	0.149
$\mathbb{E}^{\hat{x}}[\Delta x^2]$	0.192	0.132	$\mathbb{E}[a]$	2.554	1.590
$\mathbb{E}^{\hat{x}}[(\hat{x}-\Delta x)^3]$	-0.186	-0.099	$\mathbb{C}ov[a,x]$	0.914	0.417
Covariances			**Transitional Dynamics**		
$\mathbb{C}ov^{\hat{x}}[\tau,\Delta x]$	0.063	0.040	$\mathbb{V}ar[x]/\sigma^2$	3.014	2.240
$\mathbb{E}^{\hat{x}}[\tau(\hat{x}-\Delta x)^2]$	0.534	0.254	$-\nu\mathbb{C}ov[a,x]/\sigma^2$	1.340	0.803
			$\mathrm{CIR}_1(\delta)$	4.354	3.043

Sources: Authors' calculations using establishment-level survey data for Chile.

Inputs from Micro Data. Consider first the distribution of expected times τ. We obtain an average expected time to adjustment of $\mathbb{E}^{\hat{x}}[\tau] = 2.4$ years with a large dispersion, suggesting substantial heterogeneity in adjustment times across establishments. Now consider the distribution of capital gaps; it has an average of $\mathbb{E}^{\hat{x}}[\Delta x] = 0.27$ and a second moment of $\mathbb{E}^{\hat{x}}[\Delta x^2] = 0.19$, and it is right-skewed. The covariance between adjustment size and expected time is almost zero $\mathbb{C}ov^{\hat{x}}[\tau, \Delta x] = 0.06$. This zero covariance is surprising, as one would expect a positive covariance: the longer the inaction period, the stronger the effect of the drift; consequently, upon taking action, the investment rate should be larger. We discuss this finding at the end of this section.

Outputs from theory: parameters. Let us know explain the parameter values implied by our formulas. From (36), the implied drift, which captures the depreciation rate, productivity growth, and changes in relative prices, equals

$$\nu = -\frac{\mathbb{E}^{\hat{x}}[\Delta x]}{\mathbb{E}^{\hat{x}}[\tau]} = -\frac{0.271}{2.441} = -0.111, \tag{47}$$

and the volatility of idiosyncratic shocks equals

$$\sigma = \sqrt{\underbrace{\frac{\mathbb{E}^{\hat{x}}[\Delta x^2]}{\mathbb{E}^{\hat{x}}[\tau]}}_{0.08} + \underbrace{2\nu\hat{x}}_{0.002}} = 0.27. \tag{48}$$

Note that the main component that drives the volatility estimate is the second moment of capital gap changes, normalized by expected duration, whereas the drift term is negligible. The calibration for the volatility of innovations in the literature falls within a very wide range, from 0.052 in Khan and

Thomas (2008) to 0.117 in Winberry (2016) to 0.202 in Bachmann, Caballero and Engel (2013).[19] It is worth noting that these calibrations are done jointly with the fixed adjustment cost within a particular inaction model. In contrast, our volatility estimate is pinned down directly through our model–indepedent mapping between data and parameters. Lastly, the observation formula for the reset capital gap in (35) implies that, upon adjustment, capital gaps are reset on average 1% above the average capital gap:

$$\hat{x} = \underbrace{\frac{\mathbb{E}^{\hat{x}}[\Delta x]}{2}}_{0.14} \underbrace{\left(1 - \mathbb{CV}^{\hat{x}}[\tau]^2\right)}_{-0.23} + \underbrace{\frac{\mathbb{C}ov^{\hat{x}}[\tau, \Delta x]}{\mathbb{E}^{\hat{x}}[\tau]}}_{0.02} = 0.013 \tag{49}$$

Recall the drift of minus 11%. As capital gaps are reset 1.3% above the average capital gap, the part of the drift that is not compensated by the reset state (9.7%) must be accommodated through an asymmetric policy.

Output from theory: ergodic moments. According to the observation formula (37), the steady state dispersion of capital gaps $\mathbb{V}ar[x]$—a notion of misallocation—can be expressed in terms of capital gap changes and the reset point as follows:

$$\mathbb{V}ar[x] = \frac{\hat{x}^3 - \mathbb{E}^{\hat{x}}[(\hat{x} - \Delta x)^3]}{3\mathbb{E}^{\hat{x}}[\Delta x]} = 0.23, \tag{50}$$

where the cubic powers capture asymmetries in the distribution. Note that the ratio $\mathbb{V}ar[x]/\sigma^2 = 2.8$ is quantitatively close to the expected duration $\mathbb{E}^{\hat{x}}[\tau] = 2.4$. This suggests that the dispersion—a measure of ex-post heterogeneity—is almost equal to the fundamental volatility σ^2 times expected duration, even if there is a large negative drift of -11%.

The average age $\mathbb{E}[a]$ is recovered using information about the average and the dispersion of adjustment times from (38). Following our earlier discussion on renewal theory—larger stopping times are more representative in the sample—the heterogeneity in expected times increases the average age:

$$\mathbb{E}[a] = \underbrace{\mathbb{E}^{\hat{x}}[\tau]/2}_{1.2} \underbrace{\left(1 + \mathbb{CV}^{\hat{x}}[\tau]^2\right)}_{2.1} = 2.52. \tag{51}$$

Lastly, equation (39) implies that the covariance between age and adjustment size suggested by the data is positive:

$$\mathbb{C}ov[x, a] = \frac{1}{2\nu}\left(\underbrace{\frac{\mathbb{E}^{\hat{x}}\left[\tau\left(\hat{x} - \Delta x\right)^2\right]}{\mathbb{E}^{\hat{x}}[\tau]}}_{0.22} - \underbrace{\mathbb{V}ar[x]}_{0.23} - \underbrace{\sigma^2\mathbb{E}[a]}_{0.20}\right) = 0.914. \tag{52}$$

This positive covariance between capital gaps and capital age means that the capital holdings of plants that have not adjusted in a long time (their capital is old) are *above* the average.

[19] The original numbers used in those papers are 0.022 and 0.049, respectively. Since we abstract from labor and our productivity is rescaled, we must adjust their volatilities by a factor $1/1-\alpha$ in order to make their numbers comparable to ours. We assuming a labor share of $\alpha = 0.58$ and obtain the numbers above. Additionally, for Bachmann, Caballero and Engel (2013) and Winberry (2016), we convert their quarterly volatilities $\sigma^q = 0.021, 0.047$ to yearly taking into account the persistence as follows: $\sigma^a = \sigma^q\sqrt{1 + \rho + \rho^2 + \rho^3}$), with $\rho = 0.94, 0.86$. Lastly, for Bachmann, Caballero and Engel (2013), we only consider the idiosyncratic shocks (excluding the sectorial shocks). Recall that we only consider here structures.

Discussion. The data throws two surprising results: a tiny covariance between adjustment sizes and duration, $\mathbb{C}ov[\tau, \Delta x] \approx 0$, and a positive covariance between capital gaps and capital age $\mathbb{C}ov[x, a] \approx 1$. While the value of these covariances appears counterintuitive, we want to stress that they are coming directly from the data. Let us recall the only two assumptions we make to establish the link between data and ergodic moments: (i) capital gaps follow a Brownian motion, and (ii) the reset state $\hat{x}$ is *iid*. To be clear, we have not assumed any particular inaction model or parametric restriction of the plants' state besides those imposed to the capital gaps x. Clearly, plants may have other drivers of their investments besides capital gaps, and we do not impose any structure on those.

As a consequence, if a particular model generates a covariance that does not square with the data, one of the two assumptions above must be changed. One option would be to abandon the Brownian motion assumption for capital gaps x, for instance, by introducing mean reversion into productivity. However, this alternative assumption is problematic given the zero autocorrelation of investment rates, as shown at the bottom of Table I. Another option would be to abandon the *iid* reset state, for instance, through the introduction of convex adjustment costs or information frictions that avoid full adjustment. In this case, the substantial lumpy behavior that we document sets a limit to the role of these alternative frictions in explaining the investment behavior.

In Section 4.6, we show that increasing adjustment hazards, such as the one predicted by the random fixed cost model, always generate negative covariances between capital gaps and capital age. This holds for any calibration. Therefore, such a model by its own cannot replicate the micro-data.

4.4 Business Cycle Dynamics.

Now we study the transitional dynamics of the first moment of capital gaps, i.e. CIR_1. For this purpose, we consider an unanticipated permanent aggregate productivity shock that shifts horizontally the distribution of idiosyncratic productivity of all firms. The transitional dynamics are model dependent, so we provide results for two specifications.

If the random fixed cost model is true, where both intensive and extensive margins are active, we obtain that

$$\frac{\text{CIR}_1(\delta)}{\delta} \approx \underbrace{\frac{\mathbb{V}ar[x]}{\sigma^2}}_{3.014} - \underbrace{\frac{\nu\ \mathbb{C}ov[a, x]}{\sigma^2}}_{1.340} = 4.354. \tag{53}$$

In order to interpret these numbers, assume that the IRF is exponential $\exp(-\lambda t)$. Then the CIR equals $1/\lambda$. Its half-life is equal to the date T such that $\exp(-T * \lambda) = 0.5$ or $T = -\log(0.5)/\lambda$. Therefore, $T = \log(2) * \text{CIR} = 1.31$ in the random fixed cost model, i.e. it takes more than 5 quarters for half of the effect of the shock to vanish.

4.5 Heterogeneity concerns

One concern regarding the empirical analysis is how heterogeneity can affect the cross-sectional statistics. As a robustness check, we repeat the analysis splitting the sample by subsectors within manufacturing as well as by plant size.

We consider 8 major subsectors within the manufacturing sector: (1) Food and beverages; (2) Textile, clothing and leather; (3) Wood and furniture; (4) Paper and printing; (5) Chemistry, petroleum, rubber and plastic; (6) Manufacture of non-metallic mineral products; (7) Basic metal; (8) Metal products, machinery and equipment. We find that, besides textiles and chemicals, all other sectors present very similar investment rate distributions.

Table III summarizes the investment rate statistics by subsector for the total capital stock.[20] We observe that, besides the textile sector, there are no significant differences in the size of investments, the relative frequencies of positive, negative and zero investments, or in the spike rates. This suggests that heterogeneity across sectors is not a concern when computing the cross-sectional statistics.

We define plant size in terms of the average number of workers during the sample period and then consider four quartiles: small plants (0-25%, S), medium plants (25-50%, M), large plants (50-75%, L), and very large plants (75-100%, XL). Table III shows statistics by plant size for the total capital stock.[21] In all capital categories, average investment, the frequency of non-zero investments, and the fraction of spikes increase with plant size. In contrast, the inaction rate decreases with size. Given the heterogeneity, we compute below the median and above the median.

[20] Table XV provides more details by sector for each capital type.

[21] Table XIII in the Online Data Appendix presents more detail by capital-type and plant size.

Table III – Inputs from Micro Data and Outputs from the Theory by Size and Sector

	Inputs from Data				Outputs from Theory		
	Plants size		Excluding		Plants size		Excluding
	Small	Large	Textile/Chemical		Small	Large	Textile/Chemical
Frequency				**Parameters**			
$\mathbb{E}^{\hat{x}}[\tau]$	2.862	2.300	2.445	ν	-0.111	-0.111	-0.110
$\mathbb{CV}^2[\tau]$	1.167	1.012	1.096	σ^2	0.077	0.078	0.075
				$\hat{x}$	0.004	0.013	0.011
Capital Gaps				**Steady State Moments**			
$\mathbb{E}^{\hat{x}}[\Delta x]$	0.317	0.256	0.270	$\mathbb{V}ar[x]$	0.220	0.244	0.228
$\mathbb{E}^{\hat{x}}[\Delta x^2]$	0.222	0.185	0.190	$\mathbb{E}[a]$	3.101	2.314	2.563
$\mathbb{E}^{\hat{x}}[(\hat{x}-\Delta x)^3]$	-0.209	-0.187	-0.184	$\mathbb{C}ov[a,x]$	0.849	1.016	0.922
Covariances				**Transitional Dynamics**			
$\mathbb{C}ov^{\hat{x}}[\tau,\Delta x]$	0.088	0.033	0.059	$\mathbb{V}ar[x]/\sigma^2$	2.870	3.145	3.021
$\mathbb{E}^{\hat{x}}[\tau(\hat{x}-\Delta x)^2]$	0.769	0.455	0.531	$-\nu\mathbb{C}ov[a,x]/\sigma^2$	1.227	1.454	1.350
				$\mathrm{CIR}_1(\delta)$	4.097	4.599	4.371

Sources: Authors' calculations using establishment-level survey data for Chile.

4.6 Is the random fixed cost model consistent with the data?

Now we study the quantitative implications of the random fixed cost model we presented in Example 1, where $H(\xi) = 1$. We have two parameters to calibrate: the size of the adjustment cost $\kappa = 0.45$ and the arrival rate of free adjustment opportunities $\lambda = 0.71$. We set these parameters in order to match the average expected time to adjustment $\mathbb{E}^{\hat{x}}[\tau]$ and the variance of capital gaps $\mathbb{V}ar[x]$. The parameters for the stochastic process of the capital-gaps are taken from the data and our formulas above $(\mu, \psi, \sigma) = (0.016, 0.0620, 0.146)$. The rest of the parameters, the discount $\rho = 0.04$ and the capital share $\alpha = 0.58$, are set externally. The moments produced by the model are reported in the last column of Table **??**.

We find that the model is able to match most moments from the data, except for two moments: the covariance $\mathbb{C}ov^{\hat{x}}[\tau, \Delta x]$ is equal to 0.193 (almost zero in the data), and the $\mathbb{E}^{\hat{x}}[\tau(\hat{x}-\Delta x)^2]$ equal to 0.091 (and 2/3 smaller in the data). In particular, missing the second covariance has important implications for the CIR through its effect on $\mathbb{C}ov[x, a]$. When the model calibrated to match the first component of the CIR, equal to the normalized steady state misallocation $\mathbb{V}ar[x]/\sigma^2$, it dramatically misses the second component that includes the covariance between age and capital gap $-\nu\ \mathbb{C}ov[a, x]/\sigma^2 > 0$. Since this second component is positive (the negative drift times the negative covariance), the implied CIR from the model is substantially below the one implied by our formulas.

Proposition **??** shows formally that in the random fixed adjustment cost model, for all inaction regions and all parametrizations, the covariance between capital gap and capital age is always positive. This fact closes the door for the random fixed cost model to adequately describe the micro data on investment.

5 Extensions and Generalization

In the previous sections we specified parametric restrictions to the inaction model and to the firms' state space. Such assumptions exclude from our analysis models with fixed adjustment dates as in Taylor (1980), models with observation costs as in Álvarez, Lippi and Paciello (2011), and several others. Nevertheless, it is possible to extend our theory to accommodate richer models. In this section, we generalize our results to consider any stopping-time model or state space, explaining the assumptions on policies and processes that are key to apply our tools.

Second, we extend the analysis in three directions, to consider: (i) transitions of higher moments ($m > 1$) of the distribution; (ii) transitions starting from any general initial condition F_0; and (iii) transitions for a mean-reverting process. In each case, we focus on the one property that delivers the most interesting mechanism.[22] We denote conditional distributions as $Z|Y$, conditional expectations with initial condition z as $\mathbb{E}^z[Z]$, and the minimum between two stopping times as $t \wedge s \equiv \min\{t, s\}$.

5.1 Generalization

Let $(\Omega, P, \mathcal{F})$ be a probability space equipped with a filtration $\mathcal{F} = (\mathcal{F}_t; t \geq 0)$. We consider an economy populated by a continuum of agents indexed with $\omega \in \Omega$, where agent ω's information set at time t is the filtration $\mathcal{F}_t$. Each agent's uncontrolled state is given by $\tilde{S}_t(\omega) = [\tilde{x}_t(\omega), S_t^{-x}(\omega)] \in \mathbb{R}^{1+K_{-x}}$. The state is split between a *main state* $\tilde{x}$ and a set of *complementary states* $\tilde{S}_t^{-x}$. The main state follows a Brownian motion $d\tilde{x}_t(\omega) = \sigma dW_t(\omega)$. Agent's policies consist of a sequence of adjustment dates $\{\tau_k\}_{k=1}^{\infty}$ and adjustments sizes $\{\Delta S_{\tau_k}\}_{k=1}^{\infty}$, measurable with respect to $\mathcal{F}_t$. Given these policies $\{\tau_k(\omega), \Delta S_{\tau_k}(\omega)\}_{k=1}^{\infty}$, the controlled state $S_t(\omega)$ evolves as the sum of the uncontrolled state plus the adjustments: $S_t(\omega) = \tilde{S}_t(\omega) + \sum_{\tau_k(\omega) \leq t} \Delta S_{\tau_k(\omega)}$.

The first premise for our theory is a recursive representation of the conditional CIR, both between and within stopping dates. This demands $S_t(\omega)$ to be a sufficient statistic for the conditional CIR, which in turn requires that the policy is history independent. Formally, this mean that

$$\mathbb{E}\left[\int_{\tau_i \wedge t(\omega)}^{\tau_{i+1}} f(x_t)dt|\mathcal{F}_{\tau_i \wedge t(\omega)}\right] = \mathbb{E}\left[\int_0^{\tau} f(x_t)dt|S_{\tau_i \wedge t(\omega)}\right] = v^f(S_{\tau_i \wedge t(\omega)}), \quad \text{for all} \quad t(\omega) \leq \tau_{i+1}.$$

Since the main state follows a Brownian motion, the burden of this requirement falls completely on the complementary state and the policy. Assumption 1 and 2 formalize these requirements.

Assumption 1 (**Markovian complementary state**). *The complementary state $\tilde{S}_t^{-x}$ follows a Strong Markov process:*

$$\tilde{S}^{-x}_{(t\wedge\tau_k)+h}(\omega)|\mathcal{F}_{t\wedge\tau_k} \;=\; \tilde{S}^{-x}_h(\omega)|\tilde{S}_{(t\wedge\tau_k)}(\omega), \qquad \forall k. \tag{54}$$

To understand this assumption, consider a history ω such that $t < \tau_k(\omega)$. In this case, the complementary state's law of motion depends only on its current value; thus it is independent of its own history. Additionally, the complementary state is an homogenous process, since its law of motion at date t is

[22] The Web Appendix presents the full characterization and analysis of the three properties.

equivalent to its law of motion at zero, given an initial condition. In the complementary case $t \geq \tau_k(\omega)$, these properties continue to hold, thus the stopping policy does not reveal new information about the complementary state's law of motion.

Assumption 2 (**Markovian policies**). *Policies satisfy the following conditions:*

$$\tau_{k+1}|\mathcal{F}_{\tau_k+h} \;=\; \tau_1|S_{\tau_k+h} \quad \textit{for all} \quad h \in [0, \tau_{k+1} - \tau_k]. \tag{55}$$

A second premise in our theory is that we can characterize the CIR with the *first* stopping time of every agent. This means that, upon taking action, agents fully adjust to include any deviations from their steady state behavior and come back to the steady state process. This would imply that S_{τ_k} is *iid* across time and independent of the history previous to the adjustment. The challenge with stochastic *iid* resets is that is makes it more difficult to identify the parameters of the stochastic process, e.g. differentiating the fundamental volatility σ from the volatility arising from a random reset state. Therefore, in order for the reset state to be sufficiently informative, we ask that it is a constant $x_{\tau_i} = \hat{x}$.[23]

Assumption 3 (**Constant reset state**). *The reset state is constant: $x_{\tau_k} = \hat{x}$ for all k.*

It is straightforward to check that the previous assumptions hold in the investment example developed in Section 2. For Assumption 1, the complementary state is given by the arrival of free adjustment opportunities N_t, which is assumed to be a Poisson counter process and thus a Strong Markov process. The requirements in Assumption 2 and 3 are also satisfied. We showed that the reset capital gap is constant; and since the stopping policy is an inaction set with respect to the controlled state, the stopping policy is history independent within and between adjustments.

Finally, in order to apply the Optional Sampling Theorem, we require several stopping processes to be well-defined (finite moments at the stopping-time).[24]

Assumption 4 (**Well-defined stopping processes**). *The processes $\left(\left\{\int_0^t s^j x_s^m \, dB_s\right\}_t, \tau\right)$ for all m and $j = 0, 1$, are well-defined stopping processes.*

The previous Markovian requirements are enough in order to characterize the aggregation, representation of the intensive margin, and observation properties; however, in order to apply the representation property to the extensive margin, we must require one additional assumption. There must exist an equivalent representation of the extensive margin as a function exclusively of the main state x. For this, we require that there exists a stopping policy τ^* that only depends on the main state x and can fully describe the extensive margin by itself. For instance, a stopping policy given by a Poisson counter with hazard $\Lambda(x)dt$ satisfies this requirement.

Assumption 5 (**Hazard**). *Assume that there exist a stopping policy τ^* s.t.*

$$\mathbb{E}^{\hat{x}}\left[\int_0^{\tau}\left(\frac{\partial \mathbb{E}^S\left[x_\tau^{m+2}/m+2\right]}{\partial x} - \mathbb{E}^S\left[x_\tau^{m+1}\right]\right)dt\right] = \mathbb{E}^{\hat{x}}\left[\int_0^{\tau}\left(\frac{\partial \mathbb{E}^x\left[x_{\tau^*}^{m+2}/m+2\right]}{\partial x} - \mathbb{E}^x\left[x_{\tau^*}^{m+1}\right]\right)dt\right], \tag{56}$$

[23] In this paper, we ignore ex-ante heterogeneity across agents (that could be reflected in different reset states and policies), but this can be relaxed. Nevertheless, it remains crucial that history is erased at the moment of reseting the state.

[24] See Web Appendix A for a formal definition of a well-defined stopping process.

and there exist a smooth function $g_m(x)$ such that

$$g_m(x) = \mathbb{E}^{\hat{x}+x}\left[(\hat{x} - \Delta x)^m\right] - \mathbb{E}^{\hat{x}}\left[(\hat{x} - \Delta x + x)^m\right], \quad \forall m. \tag{57}$$

where Δx is under the policy τ^*.

5.2 Extensions

Now that we have stated the formal requirements needed to apply our theory, we proceed to develop the three extensions. To highlight the new mechanisms, in all the extensions we focus on the driftless case $\nu = 0$, but the proofs are straightforward to extend to consider a non-zero drift.

Extension I: Transitional dynamics for higher moments We first consider the transitional dynamics for higher moments of the distribution ($m \geq 1$). The initial condition remains to be a mean translation of the steady state distribution. In this case, we focus the discussion on the representation of the intensive margin.

Proposition 6. *Assume $d\tilde{x}_t = \sigma dW_t$. To a first order, the transitional dynamics of the m-th moment are given by*

$$CIR_m(\delta) = \delta \times (\Gamma_m + \Theta_m - \mathbb{E}[x^m]\Theta_0) + o(\delta^2) \tag{58}$$

where the intensive margin relates to ergodic moments as follows:

$$\Gamma_m = m\mathbb{E}[x^{m-1}, a], \tag{59}$$

$$\mathbb{E}[x^{m-1}, a] = \frac{2}{m(m+1)}\left[\frac{\mathbb{E}^{\hat{x}}\left[\tau\left(\hat{x} - \Delta x\right)^{m+1}\right]}{\mathbb{E}^{\hat{x}}\left[\Delta x^2\right]} - \frac{\mathbb{E}[x^{m+1}]}{\sigma^2}\right]. \tag{60}$$

To focus on the intensive margin, assume $\Theta_m = 0$ for all m and consider the transitional dynamics for the state's first three moments by setting $m = 1, 2, 3$. We have that

$$\mathrm{CIR}_1(\delta)/\delta = \Gamma_1 = \mathbb{E}[a] \tag{61}$$

$$\mathrm{CIR}_2(\delta)/\delta = \Gamma_2 = 2\mathbb{E}[xa] \tag{62}$$

$$\mathrm{CIR}_3(\delta)/\delta = \Gamma_3 = 3\mathbb{E}[x^2 a] \tag{63}$$

As discussed earlier, the dynamics of the first moment ($m = 1$)—average capital gaps—are fully driven by the state's average age. The dynamics of the second moment ($m = 2$)—dispersion of capital gaps or misallocation—are driven by the covariance between the age and the size of capital gaps. If this covariance is zero, then the distribution's second moment remains constant along the transition path. Asymmetry in in the agents' investment policy, which generates a skewed ergodic distribution, is one way to generate a non-zero covariance. This interaction between the business cycle dynamics of capital misallocation and the asymmetry of the ergodic capital distribution is studied by Ehouarne, Kuehn and Schreindorfer (2016) and Jo and Senga (2014). Finally, the dynamics of the third moment ($m = 3$)—skewness of capital gaps—are driven by the covariance between age and the square of capital gaps. Note

that if the ergodic distribution features excess kurtosis, then the skewness of the distribution will change along the transition.

Proposition 6 provides formulas for the CIR of the m-th moment. Additionally, these formulas have two useful applications. They can be used to (i) derive bounds for the dynamics of functions of the m-th moments, and (ii) study transitions of any arbitrary function of the state. To illustrate the first application, let us consider the transitional dynamics for the variance. Using Jensen's inequality, we derive an upper bound on the variance's CIR:[25]

$$\mathrm{CIR}(\mathbb{V}ar[x]) \equiv \int_0^\infty \left(\mathbb{V}_t\left[x\right] - \mathbb{V}\left[x\right]\right) dt \;\leq\; \mathrm{CIR}_2(\delta) - \mathrm{CIR}_1^2(\delta). \tag{64}$$

To illustrate the second application, consider a smooth function of the state $f(x)$. For example, in many models the aggregate welfare criteria can be written in this form. Using a Taylor approximation around zero, we write the CIR of the $f(x)$ function in terms of the state's CIR, weighted by the Taylor factors.

$$\mathrm{CIR}(f(x)) = \int_0^\infty \mathbb{E}_t[f(x)] - \mathbb{E}[f(x)]dt \;= \sum_{j=1}^{\infty} \frac{df^j(0)}{dx^j} \frac{\mathrm{CIR}_j(\delta)}{j!}. \tag{65}$$

Extension II: General initial conditions This extension considers transitional dynamics for general initial conditions. For instance, since the work on uncertainty shocks by Bloom (2009), there has been a large literature interested in the macroeconomic consequences of uncertainty in the business cycle. Within our framework, these aggregate uncertainty shocks can be studied by setting the initial distribution as a mean-preserving spread of the steady state distribution. Moreover, the interaction between first and second moment shocks, as studied by Aastveit, Natvik and Sola (2013), Vavra (2014), Caggiano, Castelnuovo and Nodari (2014), Castelnuovo and Pellegrino (2018), and Baley and Blanco (2019), can be accommodated as well.

For simplicity, we consider perturbations that can be expressed via a single parameter δ. The initial distribution is described through a function $\mathcal{G}(x,\delta)$, such that $F_0(x) = F(\mathcal{G}^{-1}(x,\delta))$. To make progress, we impose certain smoothness and differentiability properties to the function $\mathcal{G}$.[26] Additionally, we focus on perturbations to the first and second moments. Since this extension does not affect steady state moments, we omit the characterization of the observation property as it remains as before.

Proposition 7. *Assume $d\tilde{x}_t = \sigma dW_t$ and let $\mathcal{G}(x,\delta)$ be a function that satisfies the following properties:*

1. $\mathcal{G}(x,0) = x$.
2. $\exists z > 0$ *such that* $\forall \epsilon \in (-z,z)$, $\mathcal{G}(\cdot,\epsilon)$ *is bijective.*
3. $\frac{\partial \mathcal{G}(\mathcal{G}^{-1}(y,0),0)}{\partial \delta} = -(\mathcal{G}_0 + \mathcal{G}_1 y)$ *with* $\mathcal{G}_0^2 + \mathcal{G}_1^2 = 1$.

[25] $\mathrm{CIR}(\mathbb{V}ar[x]) \equiv \int_0^\infty \left(\mathbb{V}_t\left[x\right] - \mathbb{V}\left[x\right]\right) dt = \int_0^\infty \left(\mathbb{E}_t\left[x^2\right] - \mathbb{E}\left[x^2\right]\right) dt - \int_0^\infty \mathbb{E}_t\left[x\right]^2 dt \leq \mathrm{CIR}_2(\delta) - \mathrm{CIR}_1^2(\delta)$.

[26] See proof in Web Appendix D.3 for details.

To a first order, the CIR is given by:

$$CIR_1(\mathcal{G}) = \delta \times \left(\underbrace{\mathcal{G}_0 \left(\Gamma_{1,0} + \Theta_{1,0}\right)}_{\textit{1st moment shock}} + \underbrace{\mathcal{G}_1 \left(\Gamma_{1,1} + \Theta_{1,1}\right)}_{\textit{2nd moment shock}} \right) + o(\delta^2) \tag{66}$$

$$\Gamma_{1,i} = (i+1)\mathbb{E}[x^i a] \tag{67}$$

$$\Theta_{1,i} = \sum_{j=0}^{\infty} \theta_{1,j} \mathbb{E}[x^{j+i}] \tag{68}$$

with $\theta_{1,j}$ are the micro-elasticities.

Proposition 7 points towards the moments that are crucial to characterize the dynamics for a particular type of initial condition. As long as there exists enough differentiability in the perturbation of the initial condition, we can find ergodic moments that perfectly describe the dynamics of the model. Interestingly, the micro-elasticities needed to compute the extensive margin are independent of the number of moments that are shocked.

As an example, consider $\mathcal{G}$ to be a mean preserving spread of the steady state distribution F_0. This means that $\mathcal{G}(x, \delta) = x(1+\delta)$ and therefore $\mathcal{G}_0 = 0$ and $\mathcal{G}_1 = 1$. Again, let us focus only in the intensive margin by setting $\Theta_{1,i} = 0$ for all i. Then the CIR is approximated as:

$$\frac{\text{CIR}_1(\delta)}{\delta} \approx \mathcal{G}_1 \Gamma_{1,1} = \mathbb{C}ov[x, a].$$

Thus mean-preserving perturbations have first order effects if and only if the covariance between age and the state is different from zero. A non-zero covariance is consistent with the data presented in Section 4. Therefore, suggesting that uncertainty shocks (in the form of mean-preserving spreads of the capital gap distribution) would have effects on average investment.

Extension III: Mean-reversion This extension considers a mean-reverting process for the uncontrolled state. This type of process is wildly used due to its empirical relevance and because it ensures the existence of an ergodic distribution. For this application, we focus on the observation properties.

Proposition 8. *Assume the uncontrolled state follows a Ornstein–Uhlenbeck process $d\tilde{x}_t = \rho \tilde{x}_t dt + \sigma dW_t$. Then, the reset state and structural parameters are recovered through a system of equations:*

$$\hat{x} = \frac{\mathbb{E}^{\hat{x}}[e^{-\rho\tau}\Delta x]}{\mathbb{E}^{\hat{x}}[e^{-\rho\tau}] - 1} \tag{69}$$

$$\frac{\sigma^2}{\rho} = 2\frac{\hat{x}^2 - \mathbb{E}^{\hat{x}}\left[e^{-2\rho\tau}(\hat{x} - \Delta x)^2\right]}{\mathbb{E}^{\hat{x}}\left[e^{-2\rho\tau}\right] - 1} \tag{70}$$

$$erf\left(\frac{\hat{x}}{\sqrt{\sigma^2/\rho}}\right) = \mathbb{E}^{\hat{x}}\left[erf\left(\frac{\hat{x} - \Delta x}{\sqrt{\sigma^2/\rho}}\right)\right] \tag{71}$$

where $erf(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ is the Gauss error function.

To gain some intuitions about the observation formulas above, let us consider the limiting case $\rho \to 0$. Using the approximation $e^{-\rho\tau} \approx 1 - \rho\tau$, it is easy to show that equations (69) and (70) converge to our baseline observations expressions in (35) and (36) with $\nu = 0$ (no mean-reversion):

$$\hat{x} \to_{\rho\to 0} \frac{\mathbb{E}^{\hat{x}}[\tau\Delta x]}{\mathbb{E}^{\hat{x}}[\tau]}, \qquad \sigma^2 \to_{\rho\to 0} \frac{\mathbb{E}^{\hat{x}}\left[\Delta x^2\right]}{\mathbb{E}^{\hat{x}}[\tau]}. \tag{72}$$

Therefore, as long mean reversion is "sufficiently small", the mappings between the data and the reset state, and between the data and idiosyncratic volatility do not change.

Let us make a deeper comparison of how $\hat{x}$ is determined with and without mean-reversion. With *iid* shocks, we can write (35) as a weighted sum of investment rates across firms:

$$\hat{x}^{iid} = \mathbb{E}^{\hat{x}}[\eta(\tau)\Delta x], \quad with \quad \eta(\tau) \equiv \frac{\tau}{\mathbb{E}^{\hat{x}}[\tau]} > 0, \quad \mathbb{E}^{\hat{x}}[\eta(\tau)] = 1,$$

where the weights $\eta(\tau)$ are increasing in τ, i.e. more weight is given to the investment rate of firms with large periods of inaction (with "old" capital). In order to understand this result, note that conditional of surviving, the distribution of the state is more centered around the reset state for "young" capital vintages, which cannot reflect policy asymmetries. The opposite happens for firms with "old" vintages, as the distribution of the state is more centered around *the domain's middle point*, reflecting the policy asymmetries. Thus investment rates associated with large stopping times are more informative about these asymmetries.

The opposite happens when we consider a mean-reverting process. An analogous decomposition yields

$$\hat{x}^{mr} = \mathcal{R}\mathbb{E}^{\hat{x}}[\eta'(\tau)\Delta x], \quad with \quad \eta'(\tau) \equiv \frac{e^{-\rho\tau}}{\mathbb{E}^{\hat{x}}[e^{-\rho\tau}]} > 0, \quad \mathbb{E}^{\hat{x}}[\eta'(\tau)] = 1, \quad \mathcal{R} \equiv \frac{\mathbb{E}^{\hat{x}}[e^{-\rho\tau}]}{\mathbb{E}^{\hat{x}}[e^{-\rho\tau}] - 1} < 0,$$

where now the weights are decreasing in duration and it is preceded by a negative number. As the inaction period of increases, the mean-reverting productivity process goes back to its zero long-run mean, and the distribution gets centered *around zero* on its own, so there is no need to correct for policy asymmetries with the initial condition.

6 Conclusion

This paper provides a structural relation in model of inaction between the CIR (a measure of persistence for aggregate dynamics) and micro-data. This relation holds for any moment of the distribution, any inaction model, and any initial condition. In the same way we apply our tools to a model of lumpy investment, we foresee applications in models with labor adjustment costs, inventory models, portfolio management, government debt management, among others.

For developing our theory, we assume that upon taking action, agents fully adjust to include any deviations from their steady state behavior. Thus our results do not accommodate partial adjustments which are due, for instance, to imperfect information or convex adjustment costs. One example of these frameworks is the menu cost model with information frictions in Baley and Blanco (2019). We leave for future research the application of the tools developed here to that type of frameworks.

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A Auxiliary Theorems

The following three theorems are repeatedly used in the proofs.

Theorem 1. ***[Optional Sampling Theorem (OST)]*** *Let Z be a (sub) martingale on the filtered space $(\Omega, \mathcal{P}, \mathcal{F})$ and τ an stopping time. If $(\{Z_t\}_t, \tau)$ is a well-define stopping process, then*

$$\mathbb{E}[Z_\tau](\geq) = \mathbb{E}[Z_0] \tag{A.1}$$

Proof. See Theorem 4.4 in Stokey (2009). □

Theorem 2. ***[Ergodic distribution and occupancy measure]*** *Let S be a strong Markov process and $g : S \to \mathbb{R}$ a function of S. Denote with F the ergodic distribution of S and with R the renewal distribution, i.e. the distribution conditional on adjustment. Assume $\int g(S)dF(S) = \lim_{T\to\infty} \frac{\int_0^T g(S_t)dt}{T}$ for all initial conditions S_0. Then the following relationship holds:*

$$\int g(S)dF(S) = \frac{\int \mathbb{E}^S \left[\int_0^\tau g(S_t)dt\right] dR(S)}{\int \mathbb{E}^S [\tau]\, dR(S)} \tag{A.2}$$

Moreover, with homogenous resets, i.e. $\Pr[S = \hat{S}] = 1$ under the renewal distribution R, then

$$\int g(S)dF(S) = \frac{\mathbb{E}^{\hat{S}} \left[\int_0^\tau g(S_t)dt\right]}{\mathbb{E}^{\hat{S}} [\tau]} \tag{A.3}$$

Proof. The numbers in the proof refer to the equalities below. (1) Start from the ergodicity assumption and (2) write T as the sum of n stopping times. (3) Take conditional expectations with respect to the filtration at time t. Since S is a strong Markov process, there is history independence across stopping times, and thus (4) we can write $\mathbb{E}\left[\int_{\tau_i}^{\tau_{i+1}} g(S_t)dt|\mathcal{F}_{\tau_i}\right] = \mathbb{E}^{S_{\tau_i}}\left[\int_0^{\tau_1} g(S_t)dt\right]$ for each τ_i. (5) Exchange the order between the outer expectation and the limit and divide by n; then use the definition of the renewal distribution to (6) substitute the infinite sum $\lim_{n\to\infty} \frac{\sum_{i=1}^n \mathbb{E}^{S_{\tau_i}}\left[\int_0^{\tau_1} g(S_t)dt\right]}{n}$ with $\int \mathbb{E}^S \left[\int_0^\tau g(S_t)dt\right] dR(S)$ and we reach the first result.

$$\begin{aligned}
\int g(S)dF(S) \quad &=^{(1)} \quad \lim_{T\to\infty} \frac{\int_0^T g(S_t)dt}{T} =^{(2)} \lim_{n\to\infty} \frac{\sum_{i=1}^n \int_{\tau_i}^{\tau_{i+1}} g(S_t)dt}{\sum_{i=1}^n \int_{\tau_i}^{\tau_{i+1}} 1dt} =^{(3)} \lim_{n\to\infty} \frac{\mathbb{E}\left[\sum_{i=1}^n \mathbb{E}\left[\int_{\tau_i}^{\tau_{i+1}} g(S_t)dt|\mathcal{F}_{\tau_i}\right]\right]}{\mathbb{E}\left[\sum_{i=1}^n \mathbb{E}\left[\int_{\tau_i}^{\tau_{i+1}} 1dt|\mathcal{F}_{\tau_i}\right]\right]} \\
&=^{(4)} \quad \frac{\lim_{n\to\infty} \mathbb{E}\left[\sum_{i=1}^n \mathbb{E}^{S_{\tau_i}}\left[\int_0^{\tau_1} g(S_t)dt\right]\right]}{\lim_{n\to\infty} \mathbb{E}\left[\sum_{i=1}^n \mathbb{E}^{S_\tau}[(\tau_1)]\right]} =^{(5)} \frac{\mathbb{E}\left[\lim_{n\to\infty} \frac{\sum_{i=1}^n \mathbb{E}^{S_{\tau_i}}\left[\int_0^{\tau_1} g(S_t)dt\right]}{n}\right]}{\mathbb{E}\left[\lim_{n\to\infty} \frac{\sum_{i=1}^n \mathbb{E}^{S_\tau}[(\tau_1)]}{n}\right]} \\
&=^{(6)} \quad \frac{\mathbb{E}\left[\int \mathbb{E}^S \left[\int_0^\tau g(S_t)dt\right] dR(S)\right]}{\mathbb{E}\left[\int \mathbb{E}^S [\tau]\, dR(S)\right]}
\end{aligned}$$

If the renewal distribution has a mass point at $\hat{S}$, then the last expression simplifies to:

$$\frac{\mathbb{E}\left[\int \mathbb{E}^S \left[\int_0^\tau g(S_t)dt\right] dR(S)\right]}{\mathbb{E}\left[\int \mathbb{E}^S [\tau]\, dR(S)\right]} = \frac{\mathbb{E}\left[\mathbb{E}^{\hat{S}} \left[\int_0^\tau g(S_t)dt\right]\right]}{\mathbb{E}\left[\mathbb{E}^{\hat{S}} [\tau]\right]} = \frac{\mathbb{E}^{\hat{S}} \left[\int_0^\tau g(S_t)dt\right]}{\mathbb{E}^{\hat{S}} [\tau]}$$

□

B Appendix: Proofs

Proposition 1. *Assume that:*

- *The uncontrolled state follows* $d\tilde{x}_t = \nu dt + \sigma dW_t$*, with* W_t *a Wiener process;*
- $\left(\left\{\int_0^t x_s^m s^n \; dW_s\right\}_t, \tau\right)$ *are a well-defined stopping processes for any* m *and* $n = 0, 1$*; and*
- *The moments of adjustment size can be decomposed as follows:*

$$g_m(x) = \mathbb{E}^{\hat{x}+x}\left[(\hat{x} - \Delta x)^m\right] - \mathbb{E}^{\hat{x}}\left[(\hat{x} - \Delta x + x)^m\right],$$

1. ***Aggregation:*** *To a first order, the CIR is given by*

$$\mathcal{A}_m(\delta) = \delta \times (\mathcal{Z}_m - \mathcal{M}_m[x]\Theta_0) + o(\delta^2) \tag{B.4}$$

where the intensive and extensive margin are given by

$$\mathcal{Z}_m = \Theta_m + \Gamma_m - \frac{\sigma^2 m}{2\nu}\mathcal{Z}_{m-1} \tag{B.5}$$

$$\Gamma_m \equiv \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \varphi_m^\Gamma(x_t)dt\right]}{\mathbb{E}^{\hat{x}}[\tau]} \quad ; \quad \varphi_m^\Gamma(x_t) =\equiv \frac{1}{\nu}\left(\mathbb{E}^x\left[x_\tau^m\right] - x_t^m\right) \tag{B.6}$$

$$\Theta_m \equiv \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \varphi_m^\Theta(x_t)dt\right]}{\mathbb{E}^{\hat{x}}[\tau]} \quad ; \quad \varphi_m^\Theta(S_t) \equiv \frac{1}{\nu}\left[\frac{\partial \mathbb{E}^x\left[x_\tau^{m+1}/(m+1)\right]}{\partial x} - \mathbb{E}^x\left[x_\tau^m\right]\right] \tag{B.7}$$

2. ***Representation for the intensive margin:***

$$\Gamma_m = m\mathcal{M}_{m-1,1}[x,a] + \frac{\mathbb{1}_{\{m\geq 2\}}\sigma^2 m(m-1)}{2\nu}\mathcal{M}_{m-2,1}[x,a] \tag{B.8}$$

3. ***Representation for the extensive margin:***

$$\Theta_m = \sum_{j=0}^{\infty}\theta_{m,j}\mathcal{M}_j[x] \quad \textit{with} \quad \theta_{m,j} \equiv \frac{2}{\sigma^2(m+1)}\sum_{k\geq j}^{\infty}\frac{\hat{x}^{k-j}}{k!j!}\left[\frac{d^{k+1}g_{m+2}(0)}{dx^{k+1}}/m+2 - \frac{d^k g_{m+1}(0)}{dx^k}\right]. \tag{B.9}$$

   - *If* $\tau|x_t \sim \tau$*,* $g_m(x) = \theta(m,j) = 0$ *for all* m, i*.*

4. ***Observation:*** *The reset state* $\hat{x}$ *and structural parameters* (ν, σ) *are recovered as*

$$\hat{x} = \mathbb{E}[\Delta x]\left(\frac{1 - \mathbb{CV}^2[\tau]}{2}\right) + \frac{Cov[\tau, \Delta x]}{\mathbb{E}[\tau]}, \qquad \nu = -\frac{\mathbb{E}[\Delta x]}{\mathbb{E}[\tau]}, \qquad \sigma^2 = \frac{\mathbb{E}[\Delta x^2]}{\mathbb{E}[\tau]} + 2\nu\hat{x} \tag{B.10}$$

and the ergodic moments are recovered as:

$$\mathcal{M}_m[x] = \frac{\hat{x}^{m+1} - \mathbb{E}[(\hat{x} - \Delta x)^{m+1}]}{\mathbb{E}[\Delta x](m+1)} - \frac{\sigma^2 m}{2\nu}\mathcal{M}_{m-1}[x], \tag{B.11}$$

$$\mathcal{M}_{m,1}[x,a] = \frac{\mathbb{E}\left[\tau/\mathbb{E}[\tau]\left(\hat{x} - \Delta x\right)^{m+1}\right] - \mathcal{M}_{m+1}[x]}{\nu(m+1)} - \frac{\sigma^2 m}{2\nu}\mathcal{M}_{m-1,1}[x,a] \tag{B.12}$$

with initial conditions $\mathcal{M}_1[x] = 0$ *and* $\mathcal{M}_{0,1}[x,a] = \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]}$*.*

The proof is divided into 4 Lemmas for clarity.

Lemma 1. ***[Aggregation]*** *To a first order, the transitional dynamics of the m-th moment are given by*

$$\mathcal{A}_m(\delta) = \delta \times (\mathcal{Z}_m - \mathcal{M}_m[x]\Theta_0) + o(\delta^2) \tag{B.13}$$

where the intensive and extensive margin are given by

$$\mathcal{Z}_m = \Theta_m + \Gamma_m - \frac{\sigma^2 m}{2\nu}\mathcal{Z}_{m-1} \tag{B.14}$$

$$\Gamma_m \equiv \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \varphi_m^\Gamma(x_t)dt\right]}{\mathbb{E}^{\hat{x}}[\tau]} \quad ; \quad \varphi_m^\Gamma(x_t) \equiv \frac{1}{\nu}\left(\mathbb{E}^x\left[x_\tau^m\right] - x_t^m\right) \tag{B.15}$$

$$\Theta_m \equiv \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \varphi_m^\Theta(x_t)dt\right]}{\mathbb{E}^{\hat{x}}[\tau]} \quad ; \quad \varphi_m^\Theta(S_t) \equiv \frac{1}{\nu}\left[\frac{\partial \mathbb{E}^x\left[x_\tau^{m+1}/(m+1)\right]}{\partial x} - \mathbb{E}^x\left[x_\tau^m\right]\right] \tag{B.16}$$

Proof. The proof consists of 6 steps.

Characterization of CIR as the recursive problem of a representative agent. Fix an $m \in \mathbb{N}$. Start from the CIR's definition:

$$\mathcal{A}_m(\delta) = \mathbb{E}\left[\int_0^\infty \left(x_t(\omega|\delta)^m - \mathcal{M}_m[x]\right)dt\right], \tag{B.17}$$

where the expectation is taken across agents ω. Let $\{\tau_i\}_{i=1}^\infty$ be the sequence of stopping times after the arrival of the perturbation. In (1), we write the CIR as the cumulative deviations between time $t = 0$ and the first stopping time τ_1 plus the sum of deviations between all future stopping times. In (2), we use the Law of Iterated Expectations to condition on the information set $\mathcal{F}_{\tau_i}$. In (3), we use the Strong Markov Property of the Brownian motion, the assumption of homogenous resets and that $\hat{x}$ is independent of δ for $i \geq 1$ to change the conditioning from $x_{\tau_i+h}|\mathcal{F}_{\tau_i}$ to $x_h|\hat{x}$ and write the problem recursively. To get (4), we show that *every element* inside the infinite sum is equal to zero. For this purpose, recall the relationship between ergodic moments and expected duration derived in Auxiliary Theorem 2, $\mathcal{M}_m[x] = \mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t(\delta|\omega)^m\right]/\mathbb{E}^{\hat{x}}[\tau]$, and thus we are left with the simple expression in the fourth line (we also relabel τ_1 as τ):

$$\begin{aligned}
\mathcal{A}_m(\delta) &=^{(1)} \mathbb{E}\left[\int_0^{\tau_1}\left(x_t(\delta|\omega)^m - \mathcal{M}_m[x]\right)dt + \sum_{i=1}^\infty \int_{\tau_i}^{\tau_{i+1}}\left(x_t(\delta|\omega)^m - \mathcal{M}_m[x]\right)dt\right] \\
&=^{(2)} \mathbb{E}\left[\int_0^{\tau_1}\left(x_t(\delta|\omega)^m - \mathcal{M}_m[x]\right)dt + \sum_{i=1}^\infty \mathbb{E}\left[\int_{\tau_i}^{\tau_{i+1}}\left(x_t(\delta|\omega)^m - \mathcal{M}_m[x]\right)dt\Big|\mathcal{F}_{\tau_i}\right]\right] \\
&=^{(3)} \mathbb{E}\left[\int_0^{\tau_1}\left(x_t(\delta|\omega)^m - \mathcal{M}_m[x]\right)dt\right] + \mathbb{E}\left[\sum_{i=1}^\infty \underbrace{\mathbb{E}\left[\int_0^\tau x_t(\delta|\omega)^m\,dt\Big|\hat{x}\right] - \mathcal{M}_m[x]\mathbb{E}[\tau|\hat{x}]}_{=0}\right] \\
&=^{(4)} \mathbb{E}\left[\int_0^\tau \left(x_t(\delta|\omega)^m\right)dt\right] - \mathcal{M}_m[x]\mathbb{E}[\tau].
\end{aligned}$$

As a final step, define the following value function conditional on a particular initial condition x:

$$v^m(x) \equiv \mathbb{E}^x\left[\int_0^\tau x_t(\omega)^m dt\right] - \mathcal{M}_m[x]\mathbb{E}^x[\tau], \tag{B.18}$$

and notice that $\mathcal{A}_m(\delta)$ is equal to the average of $v^m(x)$ across all initial conditions after the perturbation, given by the shift in the ergodic distribution ($F_0(x) = F(x-\delta)$):

$$\mathcal{A}_m(\delta) = \int v^m(x)dF(x-\delta). \tag{B.19}$$

2. State's support. Since Brownian motions are continuous in t, and initial conditions are identical across agents (by the assumption of homogeneous resets), the ergodic set is connected. Thus, the support of x is given by an interval $[\underline{x}, \overline{x}]$.

3. Taylor approximation to $\mathcal{A}_m(\delta)$ and decomposition into two terms. We do a first order Taylor approximation of $\mathcal{A}_m(\delta)$ around zero: $\mathcal{A}_m(\delta) = \mathcal{A}_m(0) + \mathcal{A}_m'(0)\delta$. Since $\mathcal{A}_m(0) = 0$ by definition, we have that: $\mathcal{A}_m(\delta) = \delta\mathcal{A}_m'(0)$, which we now characterize. Start from the representation in (B.19), expressed in terms of the marginal density of x:

$$\mathcal{A}_m(\delta) = \int v^m(x)f(x-\delta)dx. \tag{B.20}$$

The derivative with respect to δ, at $\delta = 0$, is given by:

$$\begin{aligned}\mathcal{A}'_m(0) &= \frac{\partial}{\partial\delta}\int v^m(x)f(x-\delta)dx\Big|_{\delta=0} = -\int v^m(x)f'(x)dx = -\,v^m(x)f(x)|_{\underline{x}}^{\overline{x}} + \int \frac{d}{dx}v^m(x)f(x)dx \\ &= \int \frac{d}{dx}v^m(x)f(x)dx\end{aligned}$$

where in the third equality we do integration by parts, and in the fourth equality we use the result that there is no mass at the endpoints (or $Pr^{x=\underline{x}}[\tau=0] = Pr^{x=\overline{x}}[\tau=0] = 1$). The previous expression says that the effect of the perturbation is equivalent to the changes in the stopping time problem of one agent when her initial conditions change (derivative of v^m with respect to x), averaged across all the possible initial conditions (the steady state distribution). In turn, as we show next, changes in the stopping time problem are reflected by alterations in the state paths and by shifts in duration.

From v^m's definition in (B.18), take its derivative with respect to initial conditions and substitute it back into $\mathcal{A}'_m(0)$

$$\mathcal{A}'_m(0) = \int \frac{\partial}{\partial x}\mathbb{E}^x\left[\int_0^\tau x_t(\omega)^m dt\right]dF(x) - \mathcal{M}_m[x]\int \frac{\partial\mathbb{E}^S[\tau]}{\partial x}dF(x)$$

Lastly, by adding and subtracting the term $\int \mathbb{E}^x\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right]dF(x)$, we re-express $\mathcal{A}'_m(0)$ as the sum of three terms Γ_m, Θ_m, and Θ_0 defined in the brackets.

$$\mathcal{A}'_m(0) = \underbrace{\int \mathbb{E}^x\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right]dF(x)}_{\mathcal{B}_m} - \mathcal{M}_m[x]\underbrace{\int \frac{\partial\mathbb{E}^x[\tau]}{\partial x}dF(x)}_{\Theta_0} + \underbrace{\int\left(\frac{\partial}{\partial x}\mathbb{E}^x\left[\int_0^\tau x_t^m dt\right] - \mathbb{E}^x\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right]\right)dF(x)}_{\mathcal{C}_m}. \quad \text{(B.21)}$$

Now we further characterize each of these terms. Note that for various extensions, the proof up to this point is exactly the same. The results change from this point onwards as we make use of the particular stochastic process for the uncontrolled state.

4. Characterize $\mathcal{B}_m$. Since $x_t = x + \nu t + \sigma W_t$, for all $t \le \tau$ we have that

$$\mathcal{B}_m \equiv \int \mathbb{E}^x\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right]dF(x) = \int \mathbb{E}^x\left[\int_0^\tau m x_t^{m-1}dt\right]dF(x).$$

Applying Itō's Lemma to x_t^m we have $dx_t^m = \nu m x_t^{m-1}dt + \sigma m x_t^{m-1}dW_t + \frac{\sigma^2}{2}m(m-1)x_t^{m-2}dt$, and integrating both sides from 0 to τ and taking expectations with initial condition x we get

$$\mathbb{E}^x\left[x_\tau^m\right] - x^m = m\sigma\underbrace{\mathbb{E}^x\left[\int_0^\tau x_t^{m-1}dW_t\right]}_{=0 \text{ by OST}} + \nu\mathbb{E}^x\left[\int_0^\tau m x_t^{m-1}dt\right] + \frac{\sigma^2 m}{2}\mathbb{E}^x\left[\int_0^\tau (m-1)x_t^{m-2}dt\right].$$

Given that $\int_0^t x_s^m dW_t$ is a martingale with zero initial condition and it is well-defined by assumption, we apply the Optional Sampling Theorem (OST) to conclude that $\mathbb{E}^x\left[\int_0^\tau x_t^{m-1}dW_t\right] = 0$. Solve for $\mathbb{E}^x\left[\int_0^\tau m x_t^{m-1}dt\right]$:

$$\mathbb{E}^x\left[\int_0^\tau m x_t^{m-1}dt\right] = \frac{\mathbb{E}^x\left[x_\tau^m\right] - x_t^m}{\nu} - \frac{\sigma^2 m}{2\nu}\mathbb{E}^x\left[\int_0^\tau (m-1)x_t^{m-2}dt\right].$$

Integrating both sides across all initial conditions, defining $\varphi_m^\Gamma \equiv \frac{1}{\nu}\left(\mathbb{E}^x\left[x_\tau^m\right] - x_t^m\right)$ and $\Gamma_m \equiv \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \varphi_m^\Gamma(x_t)dt\right]}{\mathbb{E}^{\hat{x}}[\tau]}$, and recognizing $\mathcal{B}_m$ and $\mathcal{B}_{m-1}$ we get:

$$\mathcal{B}_m = \Gamma_m - \frac{\sigma^2 m}{2\nu}\mathcal{B}_{m-1}, \qquad \Gamma_0 = 0, \quad \text{(B.22)}$$

where we used the Auxiliary Theorem 2, exchanging the ergodic distribution for the local occupancy measure.

5. Characterize $\mathcal{C}_m$. With similar steps as in the previous point, we characterize $\mathcal{C}_m$ as follows.

$$\mathcal{C}_m \equiv \int \underbrace{\frac{\partial}{\partial x}\mathbb{E}^x\left[\int_0^\tau x_t^m dt\right]}_{A} - \underbrace{\mathbb{E}^x\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right]}_{B}dF(x).$$

First we get an expression for the term A. Applying Itō's Lemma to x_t^{m+1} we have $dx_t^{m+1} = (m+1)\nu x_t^m dt + \sigma(m+1)x_t^m dW_t + \frac{\sigma^2}{2}m(m+1)x_t^{m-1}dt$. Integrating both sides from 0 to τ, taking expectations with initial condition x, using the OST, and rearranging we get: $\mathbb{E}^x\left[\int_0^\tau x_t^m dt\right] = \frac{1}{\nu(m+1)}\left(\mathbb{E}^x\left[x_\tau^{m+1}\right] - x_t^{m+1}\right) - \frac{\sigma^2 m}{2\nu}\mathbb{E}^x\left[\int_0^\tau x_t^{m-1}dt\right]$, and its derivative with respect to

initial condition x:

$$A \equiv \frac{\partial}{\partial x}\mathbb{E}^x\left[\int_0^\tau x_t^m dt\right] = \frac{1}{\nu}\left(\frac{\partial \mathbb{E}^x\left[x_\tau^{m+1}\right]/(m+1)}{\partial x} - x_t^m\right) - \frac{\sigma^2 m}{2\nu}\frac{\partial \mathbb{E}^x\left[\int_0^\tau x_t^{m-1}dt\right]}{\partial x}$$

Now, for the term B, recall from the characterization of Γ_m that

$$B \equiv \mathbb{E}^x\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right] = \frac{1}{\nu}\left(\mathbb{E}^x\left[x_\tau^m\right] - x_t^m\right) - \frac{\sigma^2 m}{2\nu}\mathbb{E}^x\left[\int_0^\tau \frac{\partial x_t^{m-1}}{\partial x}dt\right].$$

Subtract the equations for A and B and simplify to obtain:

$$\frac{\partial \mathbb{E}^x\left[\int_0^\tau x_t^m dt\right]}{\partial x} - \mathbb{E}^x\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right] = \underbrace{\frac{1}{\nu}\left(\frac{\partial \mathbb{E}^x\left[x_\tau^{m+1}\right]/(m+1)}{\partial x} - \mathbb{E}^x\left[x_\tau^m\right]\right)}_{\varphi_m^\Theta(x_t)} - \frac{\sigma^2 m}{2\nu}\left\{\frac{\partial \mathbb{E}^x\left[\int_0^\tau x_t^{m-1}dt\right]}{\partial x} - \mathbb{E}^x\left[\int_0^\tau \frac{\partial x_t^{m-1}}{\partial x}dt\right]\right\}$$

Integrating with the ergodic distribution and using the definition of Θ_m in (D.10) and recognizing $\mathcal{C}_m$ and $\mathcal{C}_{m-1}$ we get:

$$\mathcal{C}_m = \Theta_m - \frac{\sigma^2 m}{2\nu}\mathcal{C}_{m-1}, \qquad \mathcal{C}_{-1} = 0. \tag{B.23}$$

Define $\mathcal{Z}_m \equiv \mathcal{B}_m + \mathcal{C}_m$, which implies $\mathcal{Z}_m = \Gamma_m + \Theta_m - \frac{\sigma^2 m}{2\nu}\mathcal{Z}_{m-1}$. Combine the results in (D.16), (D.17) and (D.20) to obtain (D.7): $\mathcal{A}_m'(0) = (\mathcal{Z}_m - \mathcal{M}_m[x]\Theta_0)$.

6. Characterize Θ_0. We corroborate that the expression $\int \frac{\partial \mathbb{E}^x[\tau]}{\partial x}dF(x)$ is equal to Θ_0. By the OST, we have $\mathbb{E}^x[x_\tau] - x = \nu\mathbb{E}^x[\tau]$. Thus $\frac{\partial \mathbb{E}^x[\tau]}{\partial x} = \frac{1}{\nu}\left[\frac{\partial \mathbb{E}^x[x_\tau]}{\partial x} - 1\right]$. Substituting and using Auxiliary Theorem 2 we recover the expression for Θ_0 in the definition of Θ_m:

$$\Theta_0 \equiv \int \frac{\partial \mathbb{E}^x[\tau]}{\partial x}dF(x) = \int \frac{1}{\nu}\left[\frac{\partial \mathbb{E}^x[x_\tau]}{\partial x} - 1\right]dF(x)$$

□

Lemma 2. ***[Representation for intensive margin]*** *The intensive margin Γ_m defined as*

$$\Gamma_m \equiv \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \varphi^\Gamma(x_t)dt\right]}{\mathbb{E}^{\hat{x}}[\tau]}, \qquad \textit{with} \qquad \varphi_m^\Gamma(x_t) = \frac{1}{\nu}\left(\mathbb{E}^x\left[x_\tau^m\right] - x_t^m\right).$$

can be represented as a function of steady state moments as:

$$\Gamma_m = m\mathcal{M}_{m-1,1}[x,a] + \frac{\mathbb{1}_{\{m\geq 2\}}\sigma^2 m(m-1)}{2\nu}\mathcal{M}_{m-2,1}[x,a].$$

Proof. Start in (1) from the definition of Γ_m and $\varphi_m^\Gamma(S)$, then (2) exchange the time integral with the expectation conditional on adjustment $\mathbb{E}^{\hat{x}}[\cdot]$, which introduces an indicator $\mathbb{1}_{\{t\leq\tau\}}$. Use the law of iterated expectations in (3) to condition on the set $\{t \leq \tau\}$.

$$\begin{aligned}
\nu\mathbb{E}^{\hat{x}}[\tau]\Gamma_m &=^{(1)} \mathbb{E}^{\hat{x}}\left[\int_0^\tau \mathbb{E}^{x_t}\left[x_\tau^m\right] - x_t^m dt\right] =^{(2)} \int_0^\infty \mathbb{E}^{\hat{x}}\left[\left(\mathbb{E}^{x_t}\left[x_\tau^m\right] - x_t^m\right)\mathbb{1}_{\{t\leq\tau\}}\right]dt \\
&=^{(3)} \int_0^\infty \mathbb{E}^{\hat{x}}\left[\mathbb{E}\left\{\left(\mathbb{E}^{x_t}\left[x_\tau^m\right] - x_t^m\right)\mathbb{1}_{\{t\leq\tau\}}\middle| t\leq\tau\right\}\right]dt =^{(4)} \int_0^\infty \mathbb{E}^{\hat{x}}\left[\mathbb{E}\left\{\mathbb{E}^{x_t}\left[x_\tau^m\right] - x_t^m\middle| t\leq\tau\right\}\right]dt \\
&=^{(5)} \int_0^\infty \mathbb{E}^{\hat{x}}\left[\mathbb{E}\left[x_\tau^m - x_t^m\middle| t\leq\tau\right]\right]dt =^{(6)} \int_0^\infty \mathbb{E}^{\hat{x}}\left[\mathbb{E}\left\{\left(x_\tau^m - x_t^m\right)\mathbb{1}_{\{t\leq\tau\}}\middle| t\leq\tau\right\}\right]dt \\
&=^{(7)} \mathbb{E}^{\hat{x}}\left[\int_0^\tau \left(x_\tau^m - x_t^m\right)dt\right] =^{(8)} \mathbb{E}^{\hat{x}}\left[x_\tau^m\int_0^\tau dt\right] - \mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^m dt\right]
\end{aligned} \tag{B.24}$$

We now characterize $\mathbb{E}^{\hat{x}}\left[x_\tau^{m+1}\int_0^\tau dt\right]$. Applying Ito's lemma followed by the OST to $Y_t^m \equiv x_t^m\int_0^t ds$

$$dY_t^m = x_t^m dt + \mathbb{1}_{\{m\geq 1\}}\nu m x_t^{m-1}\int_0^t dsdt + \mathbb{1}_{\{m\geq 1\}}\sigma m x_t^{m-1}\int_0^t dsdW_t + \mathbb{1}_{\{m\geq 2\}}\frac{m(m-1)\sigma^2}{2}x_t^{m-2}\int_0^t dsdt \tag{B.25}$$

$$\mathbb{E}^{\hat{x}}\left[Y_\tau^m\right] = \mathbb{E}^{\hat{x}}\left[x_\tau^m\int_0^\tau dt\right] = \underbrace{\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^m dt\right]}_{\mathcal{M}_m[x]\mathbb{E}^{\hat{x}}[\tau]} + \mathbb{1}_{\{m\geq 1\}}\nu m\underbrace{\frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^{m-1}tdt\right]}{\mathbb{E}^{\hat{x}}[\tau]}}_{\mathcal{M}_{m-1,1}[x,a]} + \mathbb{1}_{\{m\geq 2\}}\frac{m(m-1)\sigma^2}{2}\underbrace{\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^{m-2}tdt\right]}_{\mathcal{M}_{m-2,1}[x,a]} \tag{B.26}$$

Using equations (B.24) and (B.26), we have that

$$\Gamma_m = \mathbb{1}_{\{m\geq 1\}} m\mathcal{M}_{m-1,1}[x,a] + \frac{\mathbb{1}_{\{m\geq 2\}}\sigma^2 m(m-1)}{2\nu}\mathcal{M}_{m-2,1}[x,a]$$

□

Lemma 3. ***[Representation for extensive margin]*** *Assume the moments of the adjustment size can be written as:*

$$g_m(x) = \mathbb{E}^{\hat{x}+x}\left[(\hat{x}-\Delta x)^m\right] - \mathbb{E}^{\hat{x}}\left[(\hat{x}-\Delta x+x)^m\right] \tag{B.27}$$

Then the extensive margin given by $\Theta_m \equiv \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \varphi_m^\Theta(x_t)dt\right]}{\mathbb{E}^{\hat{x}}[\tau]}$ *with* $\varphi_m^\Theta(x) \equiv \frac{1}{\nu}\left(\frac{\partial \mathbb{E}^x\left[x_\tau^{m+1}/(m+1)\right]}{\partial x} - \mathbb{E}^x\left[x_\tau^m\right]\right)$ *can be represented as a function of steady state moments as follows:*

$$\Theta_m = \sum_{j=0}^{\infty}\theta_{m,j}\mathcal{M}_j[x] \quad \textit{with} \quad \theta_{m,j} = \sum_{k\geq j}^{\infty}\frac{1}{\nu}\frac{\left[\frac{d^{k+1}g_{m+1}(x)}{dx^{k+1}}/m+1-\frac{d^k g_m(x)}{dx^k}\right]\Big|_{x=0}\hat{x}^{k-j}}{k!j!}. \tag{B.28}$$

- *If* $\tau|x_t \sim \tau$, $g_m(x) = \theta(m,j) = 0$ *for all* m, i.

Proof. Using a change of variable in assumption (B.27), we have that :

$$\mathbb{E}^y\left[x_\tau^m\right] = g_m(y-\hat{x}) + \mathbb{E}^{\hat{x}}\left[(y-\hat{x}+x_\tau)^m\right]. \tag{B.29}$$

Using the previous equation we have that

$$\begin{aligned}
\Theta_m &= \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau\left(\frac{\partial\mathbb{E}^S[x_\tau^{m+1}/(m+1)]}{\partial x} - \mathbb{E}^S[x_\tau^m]\right)dt\right]}{\nu\mathbb{E}^{\hat{S}}[\tau]} \qquad \text{(B.30)}\\
&= \frac{1}{\nu}\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau\left[\left[\frac{dg_{m+1}(y-\hat{x})}{dy}/(m+1) + \mathbb{E}^{\hat{x}}\left[(y-\hat{x}+x_\tau)^{m+1}\right]\right] - \left[g_m(y-\hat{x}) + \mathbb{E}^{\hat{x}}\left[(y-\hat{x}+x_\tau)^m\right]\right]\right]dt\right]}{\mathbb{E}^{\hat{S}}[\tau]}\\
&= \frac{1}{\nu}\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau\left[\frac{dg_{m+1}(y-\hat{x})}{dy}/(m+1) - g_m(y-\hat{x})\right]dt\right]}{\mathbb{E}^{\hat{S}}[\tau]}\\
&= \frac{1}{\nu}\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau\left[\sum_{j=0}^\infty\frac{d^j}{j!dx^j}\left[\frac{dg_{m+1}(x)}{dx}/(m+1) - g_m(x)\right]_{x=0}(y-\hat{x})^j\right]dt\right]}{\mathbb{E}^{\hat{S}}[\tau]}\\
&= \frac{1}{\nu}\sum_{j=0}^\infty\sum_{z=0}^j\frac{\left[\frac{d^{j+1}g_{m+1}(0)}{dx^{j+1}}/m+1-\frac{d^j g_m(0)}{dx^j}\right]\hat{x}^z}{j!}\begin{pmatrix} j\\ z\end{pmatrix}\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{j-z}dt\right]}{\mathbb{E}^{\hat{S}}[\tau]}\\
&= \frac{1}{\nu}\sum_{j=0}^\infty\sum_{z=0}^j\frac{\left[\frac{d^{j+1}g_{m+1}(0)}{dx^{j+1}}/m+1-\frac{d^j g_m(0)}{dx^j}\right]\hat{x}^z}{j!}\begin{pmatrix} j\\ z\end{pmatrix}\mathcal{M}_{j-z}[x]\\
&= \sum_{j=0}^\infty\sum_{z=0}^j\underbrace{\frac{1}{\nu}\frac{\left[\frac{d^{j+1}g_{m+1}(0)}{dx^{j+1}}/m+1-\frac{d^j g_m(0)}{dx^j}\right]\hat{x}^z}{z!(j-z)!}}_{=\mathcal{H}_{m,j,z}}\mathcal{M}_{j-z}[x]\\
&= \sum_{h=0}^\infty\theta_{m,h}\mathcal{M}_h[x],\ \textit{with}\ \theta_{m,h} = \sum_{k\geq h}^\infty\mathcal{H}_{m,k,k-h}. \qquad \text{(B.31)}
\end{aligned}$$

If $\tau|x_t \sim \tau$, then we have that

$$\begin{aligned}
g_m(x) &= \mathbb{E}^{\hat{x}+x}\left[(\hat{x}-\Delta x)^m\right] - \mathbb{E}^{\hat{x}}\left[(\hat{x}-\Delta x+x)^m\right]\\
&= \mathbb{E}\left[(\hat{x}+x+\nu\tau^{x+\hat{x}}+\sigma W_{\tau^{x+\hat{x}}})^m\right] - \mathbb{E}\left[(\hat{x}+x+\nu\tau^{\hat{x}}+\sigma W_{\tau^{\hat{x}}})^m\right]\\
&= \mathbb{E}\left[(\hat{x}+x+\nu\tau+\sigma W_\tau)^m\right] - \mathbb{E}\left[(\hat{x}+x+\nu\tau+\sigma W_\tau)^m\right]\\
&= 0 \qquad \text{(B.32)}
\end{aligned}$$

□

Lemma 4. ***[Observation]****The reset state $\hat{x}$ and structural parameters (ν, σ) are recovered as*

$$\hat{x} = \mathbb{E}[\Delta x]\left(\frac{1-\mathbb{CV}^2[\tau]}{2}\right) + \frac{Cov[\tau, \Delta x]}{\mathbb{E}[\tau]}, \qquad \nu = -\frac{\mathbb{E}[\Delta x]}{\mathbb{E}[\tau]}, \qquad \sigma^2 = \frac{\mathbb{E}[\Delta x^2]}{\mathbb{E}[\tau]} + 2\nu\hat{x}$$

and the ergodic moments are recovered as:

$$\mathcal{M}_m[x] = \frac{\hat{x}^{m+1} - \mathbb{E}[(\hat{x}-\Delta x)^{m+1}]}{\mathbb{E}[\Delta x](m+1)} - \frac{\sigma^2 m}{2\nu}\mathcal{M}_{m-1}[x],$$
$$\mathcal{M}_{m,1}[x,a] = \frac{\mathbb{E}\left[\tau/\mathbb{E}[\tau]\left(\hat{x}-\Delta x\right)^{m+1}\right] - \mathcal{M}_{m+1}[x]}{\nu(m+1)} - \frac{\sigma^2 m}{2\nu}\mathcal{M}_{m-1,1}[x,a]$$

with initial conditions $\mathcal{M}_1[x] = 0$ and $\mathcal{M}_{0,1}[x,a] = \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]}$.

Proof. The basis of the proof is the application of Itō's lemma and the OST.

- **Average adjustment size.** We show that $\frac{\mathbb{E}^{\hat{x}}[\Delta x]}{\mathbb{E}^{\hat{x}}[\tau]} = -\nu$. From the law of motion $x_t = \hat{x} + \nu t + \sigma W_t$, we find the following equalities: $\sigma W_\tau = -\nu\tau + x_\tau - \hat{x} = -\nu\tau - \Delta x$. Taking expectations on both sides, we have $\sigma\mathbb{E}[W_\tau] = -\nu\mathbb{E}^{\hat{x}}[\tau] - \mathbb{E}[\Delta x]$. Since W_τ is a martingale, $\mathbb{E}[W_\tau] = W_0 = 0$ by the OST. Therefore, $\nu = -\frac{\mathbb{E}[\Delta x]}{\mathbb{E}[\tau]}$ as well.
- **Observation of fundamental volatility:** For characterizing σ define $Y_t = x_t - \nu t$ with initial condition $Y_0 = \hat{x}$. With similar steps as before we have that

$$\sigma^2 = \frac{\mathbb{E}^{\hat{x}}\left[\Delta Y_\tau^2\right]}{\mathbb{E}^{\hat{x}}[\tau]} = \frac{\mathbb{E}^{\hat{x}}\left[(x_\tau - \nu\tau - x_\tau + \hat{x})^2\right]}{\mathbb{E}^{\hat{x}}[\tau]} = \frac{\mathbb{E}^{\hat{x}}\left[(\nu\tau + \Delta x)^2\right]}{\mathbb{E}^{\hat{x}}[\tau]} \tag{B.33}$$

  or equivalently

$$\sigma^2 = \frac{\mathbb{E}^{\hat{x}}[\Delta x^2]}{\mathbb{E}^{\hat{x}}[\tau]} + 2\nu\left(\frac{\mathbb{E}^{\hat{x}}[\Delta x\tau]}{\mathbb{E}^{\hat{x}}[\tau]} + \nu\frac{\mathbb{E}^{\hat{x}}[\tau^2]}{\mathbb{E}^{\hat{x}}[\tau]}\right)$$

  Applying the formula (B.35) we have the result.
- **Observation of reset state:** For the reset state $\hat{x}$, we apply Itō's lemma to x_t^2 to obtain $d(x_t^2) = 2x_t dx_t + (dx_t)^2 = \left(2\nu x_t + \sigma^2\right)dt + 2\sigma x_t dW_t$. Using the OST $\mathbb{E}^{\hat{x}}[\int_0^\tau x_s dW_s] = 0$. Moreover, given that $\mathbb{E}^{\hat{x}}[\int_0^\tau x_s ds] = \mathcal{M}_1[x]\mathbb{E}^{\hat{x}}[\tau] = 0$, we have that

$$\mathbb{E}^{\hat{x}}[x_\tau^2] = \hat{x}^2 + \sigma^2\mathbb{E}^{\hat{x}}[\tau] \tag{B.34}$$

  Completing squares $\mathbb{E}^{\hat{x}}[x_\tau^2] = \mathbb{E}^{\hat{x}}[(\hat{x} - (\hat{x} - x_\tau))^2] = \mathbb{E}^{\hat{x}}[\Delta x^2] - 2\hat{x}\mathbb{E}^{\hat{x}}[\Delta x] + (\hat{x})^2$, we get

$$\begin{aligned}\hat{x} &= \frac{1}{2\mathbb{E}^{\hat{x}}[\Delta x]}\left[\mathbb{E}^{\hat{x}}[\Delta x^2] - \sigma^2\mathbb{E}^{\hat{x}}[\tau]\right] \\ &= \frac{1}{2\mathbb{E}^{\hat{x}}[\Delta x]}\left[\mathbb{E}^{\hat{x}}[\Delta x^2] - \left(\mathbb{E}^{\hat{x}}[\Delta x^2] + 2\frac{\mathbb{E}^{\hat{x}}[\Delta x]\mathbb{E}^{\hat{x}}[\Delta x\tau]}{\mathbb{E}^{\hat{x}}[\tau]} + \frac{\mathbb{E}^{\hat{x}}[\Delta x]^2\mathbb{E}^{\hat{x}}[\tau^2]}{\mathbb{E}^{\hat{x}}[\tau]^2}\right)\right] \\ &= \frac{\mathbb{E}^{\hat{x}}[\Delta x\tau]}{\mathbb{E}^{\hat{x}}[\tau]} - \frac{\mathbb{E}^{\hat{x}}[\Delta x]\mathbb{E}^{\hat{x}}[\tau^2]}{2\mathbb{E}^{\hat{x}}[\tau]^2}.\end{aligned} \tag{B.35}$$

  Applying the formula for the covariance $\mathbb{E}^{\hat{x}}[\tau\Delta x] + \mathbb{E}^{\hat{x}}[\tau]\mathbb{E}^{\hat{x}}[\Delta x] = Cov[\tau, \Delta x]$ and coefficient of variation square $\mathbb{CV}^2[\tau] = \frac{\mathbb{V}^{\hat{x}}[\tau]}{\mathbb{E}^{\hat{x}}[\tau]^2}$, we have the result.
- **Observation of ergodic moments with respect to the state:** For observability of ergodic moments of x, apply Itō's lemma to x^{m+1} and get $dx_t^{m+1} = (m+1)x_t^m\nu dt + (m+1)x_t^m\sigma dW_t + \frac{\sigma^2}{2}m(m+1)x_t^{m-1}dt$. Integrating from 0 to τ, using the OST to eliminate martingales, and rearranging:

$$\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^m dt\right] = \frac{1}{\nu(m+1)}\left(\mathbb{E}^{\hat{x}}[x_\tau^{m+1}] - \hat{x}^{m+1}\right) - \frac{\sigma^2}{2\nu}m\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^{m-1}dt\right] \tag{B.36}$$

  Substituting the equivalences $\mathcal{M}_m[x] = \mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^m dt\right]/\mathbb{E}^{\hat{x}}[\tau]$ and $\mathbb{E}^{\hat{x}}[\Delta x] = -\nu\mathbb{E}^{\hat{x}}[\tau]$ yields:

$$\mathcal{M}_m[x] = \frac{\hat{x}^{m+1} - \mathbb{E}^{\hat{x}}[(\hat{x}-\Delta x)^{m+1}]}{\mathbb{E}^{\hat{x}}[\Delta x](m+1)} - \frac{\sigma^2 m}{2\nu}\mathcal{M}_{m-1}[x], \qquad \mathcal{M}_1[x] = 0 \tag{B.37}$$

- **Observation of ergodic moments with respect to the joint moments of state and age:** For observability of ergodic moments of $x^m a$, where a stand for the duration of the last action, we use Itō's lemma and the OST on $x_t^{m+1}t$:

$$\mathbb{E}^{\hat{x}}\left[\tau\left(\hat{x}-\Delta x\right)^{m+1}\right] = \mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^{m+1}dt\right] + (m+1)\nu\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^m t dt\right] + \frac{\sigma^2 m(m+1)}{2}\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^{m-1}t dt\right] \tag{B.38}$$

and therefore

$$\mathcal{M}_{m,1}[x,a] = \frac{\mathbb{E}^{\hat{x}}\left[\tau\left(\hat{x}-\Delta x\right)^{m+1}\right]}{\nu(m+1)\mathbb{E}^{\hat{x}}[\tau]} - \frac{\mathcal{M}_{m+1}[x]}{\nu(m+1)} - \frac{\sigma^2 m}{2\nu}\mathcal{M}_{m-1,1}[x,a] \tag{B.39}$$

with initial condition $\mathcal{M}_{0,1}[x,a] = \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]}$.

□

Aggregate Dynamics in Lumpy Economies

Isaac Baley and Andrés Blanco

Web Appendix: Not for Publication

A Definitions

Let $(\Omega, \mathcal{P}, \mathcal{F})$ be a probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t; t \geq 0)$ and let $\tilde{S}_t(\omega) \in \mathbb{R}^n$ be an stochastic process $\mathcal{F}_t$ measurable in this probability space. Here we state all the definitions used in the main text.

Definition 1 (Stopping time)**.** *A function $\tau : \Omega \to [0, \infty]$ is called a stopping time with respect to $\mathcal{F}_t$ if $\{\omega; \tau(\omega) \leq t\} \in \mathcal{F}_t$ for all t.*

Definition 2 (Strong Markov process for Itō diffusions)**.** *Let f be a bounded Borel function on $\mathbb{R}^n$, τ a stopping time with respect to $\mathcal{F}_t$ and $\tau < \infty$ a.s.. Then a process has the Strong Markov property <u>under τ</u> if*

$$\mathbb{E}^x[f(X_{\tau+h})|\mathcal{F}_\tau] = \mathbb{E}^{X_\tau}[f(X_{\tau+h})] \quad \textit{for all } h > 0$$

A process has the Strong Markov property if it has the Strong Markov property under all τ.

Definition 3 (Well-defined stopping process)**.** *We say that $(\{X_t\}_t, \tau)$ is a well-defined stopping process if τ is bounded almost surely, or the following conditions hold:*

1. $\Pr^{\mathcal{P}}[\tau < \infty] = 1$.
2. $\Pr^{\mathcal{P}}[|X_\tau|] < \infty$.
3. $\lim_{t\to\infty} \mathbb{E}^{\mathcal{P}}\left[|X_t \mathbb{1}_{\{\tau > t\}}|\right] = 0$.

B Economic Frameworks With Small GE Effects

This section describes general equilibrium frameworks that feature small general equilibrium feedback into agents' policies, and therefore, the tools developed in this paper can be directly applied.

B.1 Monetary Shocks: Golosov and Lucas (2007)

There is a representative consumer, a continuum of firms that operate in monopolistic competition, and a monetary authority. We study the transitional dynamics to steady state for an exogenously given initial condition of the distribution of idiosyncratic states.

Money Supply The economy is subject to monetary shocks, which we summary in the money supply M_t. The log money supply is assumed to follow a Brownian motion with drift μ_m and volatility σ_m,

$$d\log(M_t) = \mu_m dt + \sigma_m dB_t^m. \tag{B.1}$$

Representative Household The household has the following preferences over consumption C_t, labor N_t, and real money holding M_t/P_t, where P_t is the aggregate price level and the future is discounted at rate $\rho > 0$:

$$\mathbb{E}_0\left[\int_0^\infty e^{-\rho t}\left(\frac{C_t^{1-\gamma}}{1-\gamma} - \alpha N_t + \log\left(\frac{M_t}{P_t}\right)\right)\, dt\right] \tag{B.2}$$

Consumption consists of a CES aggregator as in Woodford (2009), Midrigan (2011), and **?** with demand elasticity η. The household has access to complete financial markets. The budget includes labor earnings $W_t N_t$, profits Π_t from the ownership of all firms, and the opportunity cost of holding cash $R_t M_t$, where R_t is the nominal interest rate. Let D_t be the stochastic discount factor; with complete financial markets, the time-0 Arrow-Debreu budget constraint reads:

$$\mathbb{E}_0\left[\int_0^\infty D_t\left(P_t C_t + R_t M_t - W_t N_t - \Pi_t\right) dt\right] \leq M_0. \tag{B.3}$$

The household chooses consumption, labor supply and money holdings to maximize (B.2) subject to (B.3). The household's first order conditions establish nominal wages as a proportion of the (constant) money stock $W_t = \alpha(\rho + \mu_m)M_t$; thus the stochastic process of money supply transfers one-to-one to the process of nominal wages.

Monopolistic Firms On the production side, there is a continuum of firms indexed by $z \in [0,1]$ who operate in a monopolistically competitive market. Each firm maximizes its expected stream of profits, discounted at D_t. It chooses a price and then satisfies all its demand. For every price change, it must pay a menu cost θ measured in units of labor. Production uses a linear technology with labor as its only input: producing $y_t(z)$ units requires $l_t(z) = y_t(z)A_t(z)$ units of labor, so that the marginal nominal cost is $A_t(z)W_t$. Given the consumer's demand $c_t(z)$ for product z, the instantaneous profit can be written as a function of prices:

$$\Pi_t(p,z) = c_t(p,z)\Big(p - A_t(z)W_t\Big). \tag{B.4}$$

The firm's problem is to choose a sequence of adjustment dates $(\tau_i(z))$ and reset prices $(p_i(z))$ that solve the following stopping-time problem:

$$\max_{\{\tau_i, p_i\}_{i=1}^\infty} \mathbb{E}_0\left[\sum_{i=0}^\infty D_{\tau_{i+1}} W_{\tau_{i+1}}\theta + \sum_{i=0}^\infty \int_{\tau_i}^{\tau_{i+1}} D_t \Pi_t(p_i, z) dt\right], \tag{B.5}$$

where $\Pi_t(p,z)$ satisfies (B.4) and $\tau_0 = 0$. Firm z's log idiosyncratic cost $a_t(z) \equiv \ln A_t(z)$ evolves according to a diffusion process which is idiosyncratic and independent across z:

$$da_t(z) = \sigma dB_t^a(z), \tag{B.6}$$

We define firms' markups as $\mu_t(z) \equiv p_t(z)/(A_t(z)W_t)$ and aggregate markup as μ_t. With this definition we can rewrite the firm problem as a function of individual and aggregate markups:

$$\max_{\{\tau_i, \mu_{\tau_i}\}_{i=1}^\infty} \mathbb{E}_0\left[\sum_{i=0}^\infty e^{-\rho\tau_{i+1}}\theta + \sum_{i=0}^\infty \int_{\tau_i}^{\tau_{i+1}} e^{-\rho t}\mu_t^{\eta-1/\sigma}\mu_t(z)^{-\eta}(\mu_t(z)-1)dt\right], \tag{B.7}$$

where idiosyncratic markups has the following stochastic process

$$\log(\mu_t(z)) = \log\left(\mu_{\tau_i}(z)\right) - \mu_m(t-\tau_i) - \sigma_m(B_t^m - B_{\tau_i}^m) - \sigma(B_t^a - B_{\tau_i}^a) \quad \forall t \in [\tau_i, \tau_{i+1}] \tag{B.8}$$

Steady state equilibrium Given the exogenous stochastic processes for idiosyncratic productivity ($B_t^a(z)$), an equilibrium is defined by a set of stochastic processes for (i) consumption strategies $c_t(z)$, labor supply N_t, and money holdings M_t for the household, (ii) markup policies functions $\mu_t(z)$ for the firms, (iii) prices P_t, W_t, D_t, and (iv) a fixed distribution over firm states $F(\mu)$ such that the household and the firms optimize, markets clear at each date, and the distribution is consistent with actions.

In a steady state equilibrium with no money shocks, the price index, nominal wages, and nominal interest rates grow at rate μ_m, as there is no aggregate uncertainty. Nominal expenditure is constant and equal to the nominal wage, and by market clearing, aggregate output equals aggregate consumption and also the real wage, which in turn is equal to the inverse of the aggregate markup $Y = \mu^{-1/\sigma}$.

Transition Dynamics for Aggregate Markups Fix an initial distribution of markups—this initial distribution could be the outcome of an one-time unanticipated change in the money supply. Proposition 7 in **?** shows that steady-state decision rules, as derived in partial equilibrium from solving (B.7) with steady state aggregate markups $\mu = \eta/(\eta-1)$, provide an accurate approximation of the policies during the transition in a general equilibrium. Thus solving

$$V(\mu) = \max_{\{\tau_i,\mu_i\}_{i=1}^{\infty}} \mathbb{E}_0 \left[\sum_{i=0}^{\infty} e^{-\rho\tau_{i+1}}\theta + \sum_{i=0}^{\infty} \int_{\tau_i}^{\tau_{i+1}} e^{-\rho t} \left(\frac{\eta}{\eta-1}\right)^{\eta-1/\sigma} \mu_t(z)^{-\eta}(\mu_t(z)-1)dt \right] \tag{B.9}$$

provides a good approximation of the firm policy. We can conclude that the CIR of this general equilibrium model can be approximate with the partial equilibrium policies.

B.2 Real Exchange Dynamics: Blanco and Cravino (2018)

The world economy consists of two symmetric countries, i and n, each inhabited by a government, a monetary authority, a representative household, a producer of final goods and a continuum of monopolistic intermediate producers indexed by $\omega \in [0,2]$.

Representative Household Each household in country i has preferences given by

$$U_i = \mathbb{E}_0 \left[\sum_{t=0} \beta^t e^{\eta_t^\beta} \left[\eta_{i,t}^C \log(C_{i,t}) - N_{i,t} \right] \right] \tag{B.10}$$

where $C_{i,t}$ and $N_{i,t}$ denote consumption and labor, $\eta_{i,t}^C$ is a taste shock to the utility of consumption and η_t^β is a symmetric discount factor shock. The time 0 intertemporal budget constraint is:

$$\mathbb{E}_0 \left[\sum_{t=0} D_{i,t} \left(P_{i,t}C_{i,t} - W_{i,t}N_{i,t} - \Upsilon_{i,t} - T_{i,t} \right) \right] = 0 \tag{B.11}$$

Here $P_{i,t}$, $W_{i,t}$, $\Upsilon_{i,t}$, and $T_{i,t}$ respectively denote the price of consumption, nominal wages, firm's profits, and government transfers, all denominated in the currency of country i. $D_{i,t}$ is the time 0 local-currency price of an Arrow security that pays one unit of the local currency at time t. By definition, the nominal exchange rate satisfies: $\epsilon_{in,t} = \frac{\epsilon_{in,0}D_{i,t}}{D_{n,t}}$.

Monetary and Fiscal Policy As in Carvalho and Nechio (2011), we define the monetary policy in an implicit way, and assume that aggregate nominal expenditures in each country are exogenous and given by $Z_{i,t} = P_{i,t}C_{i,t}$. Government expenditures G_t are also exogenous, and taxes $T_{i,t}$ balance the budget every period. This is without loss of generality, since the Ricardian equivalence holds.

Final goods producer The final good in country i, $Y_{i,t}$, is produced according to:

$$Y_{i,t} = \left[\mu^{\frac{1}{\xi}} Y_{ii,t}^{\frac{\xi-1}{\xi}} + [1-\mu]^{\frac{1}{\xi}} Y_{ni,t}^{\frac{\xi-1}{\xi}} \right]^{\frac{\xi}{\xi-1}}, \quad where \tag{B.12}$$

$$Y_{ni,t} = \left[\int \left[Y_{ni,t}(\omega)/E_{in,t}(\omega) \right]^{\frac{\eta-1}{\eta}} d\omega \right]^{\frac{\eta}{\eta-1}}, \qquad Y_{ii,t} = \left[\int \left[Y_{ii,t}(\omega)E_{in,t}(\omega) \right]^{\frac{\eta-1}{\eta}} d\omega \right]^{\frac{\eta}{\eta-1}}$$

Here $Y_{ni,t}(\omega)$ denotes the quantity of intermediate good ω produced in country n and consumed in country i, μ is the share of domestic goods in absorption in the symmetric steady state, and ξ and η are the elasticities of substitution between domestic and foreign goods and across varieties, respectively. $E_{in,t}(\omega)$ is a quality shock as in Woodford (2009).

Intermediate Good Producer Intermediate producers behave as monopolistic competitors and set prices. Importantly, producers set prices in the currency of the country where they sell. The probability that a producer has the option to change its price in any period is given by $1-\theta_p$. The production function for intermediate goods is

$$Y_{ni,t}(\omega) = E_t(\omega) N_{ni,t}(\omega). \tag{B.13}$$

Here $E_t(\omega)$ and $N_{ni,t}(\omega)$ denote the idiosyncratic productivity and labor input, respectively.

Let $S_t = (n_t, a_t)$ be the idiosyncratic state of the firm composed by a Poisson counter with arrival rate λ, n_t, and the time since the last adjustment, a_t, following the stochastic process $da_t = dt$. The initial conditions after an adjustment are given by $\hat{S} = (0,0)$. Define the stopping time $\tau = \inf_t\{t \geq 0 : a_t \geq T \ or \ n_t \geq 1\}$. The profit maximizing price for an intermediate producer that gets to adjust prices satisfies

$$\hat{p}_{in,t} = \arg\max_{p_{in,t}} \mathbb{E}\left[\int_t^{t+\tau} \frac{D_s}{D_t}\left[\frac{p_{in,t}}{E_{in,s}} - \frac{W_{i,s}}{A_{in,s}}\right] Y_{n,t}\left(\frac{A_{in,t}p_{in,t}}{P_{in,s}}\right)^{-\eta}\left(\frac{P_{in,t}}{P_{n,s}}\right)^{-\xi}\right]. \tag{B.14}$$

Here $\frac{D_s}{D_t}$ is country i's nominal discount factor between dates t and $t+s$, $E_{in,s}$ is the nominal exchange rate, expressed in units of currency n per-unit of currency i. P_{in} is the price level of the goods produced in i and consumption in n and P_n is the price level in n country.

Stochastic process for aggregate shocks $\eta^C_{i,t}$, $\eta^C_{n,t}$, $Z_{i,t}$ and $Z_{n,t}$ follow a Brownian process with no drift in logs. We leave without specification the stochastic process for government expenditures, $G_{i,t}$ and $G_{n,t}$, and discount factor shocks, η^β_t.

Equilibrium An equilibrium is a set of allocations for the households $\{C_{i,t}, W_{i,t}(h)\}_{\forall i,t}$, production decisions for final good producers $\{Y_{i,t}, Y_{in,t}, \{Y_{in}(\omega)\}_\omega\}_{\forall n,t}$, and price policy functions for intermediate producers $\left\{\bar{P}_{in,t}\right\}_{\forall i,n,t}$, such that given prices: (i) households maximize (B.10) subject to (B.11); (ii) final good producers minimize cost according to equations (B.12); (iii) intermediate producers maximize profits according to equation (B.14); and (iv) labor and goods markets clear.

Nominal and real exchange rates The assumption of complete markets and the process for nominal exchange rate implies that $d\log(E_{in,t}) = dZ_t$, where $Z_{i,t}$ is a Weiner process given by $dZ_t = d\log(\eta^C_{i,t}) + d\log(Z_{n,t}) - \left(d\log(\eta^C_{n,t}) + d\log(Z_{i,t})\right)$. Additionally, the complete market assumption and the labor-leisure condition implies that $d\log(W_{i,t}) = d\log(Z_{i,t}) - d\log(\eta^C_{i,t})$ and $d\log(W_{n,t}) = d\log(Z_{n,t}) - d\log(\eta^C_{n,t})$. Define aggregate markups as:

$$\mu_{in,t} = \log\left(\frac{\eta}{\eta-1}\frac{A_{in,t}}{W_i E_{in}}\right) \ ; \ \mu_{ii,t} = \log\left(\frac{\eta}{\eta-1}\frac{A_{ii,t}}{W_i}\right)$$
$$\mu_{ni,t} = \log\left(\frac{\eta}{\eta-1}\frac{A_{ni,t}E_{in}}{W_n}\right) \ ; \ \mu_{nn,t} = \log\left(\frac{\eta}{\eta-1}\frac{A_{nn,t}}{W_n}\right)$$

Under the observation that multiplicative term in the profits function are only second order, up to a first order, we have that the optimality conditions over the reset markup $\hat{\mu}_{in}$ are given by

$$0 = \mathbb{E}\left[\int_t^{t+\tau} \mu_{in,s} ds\right], \tag{B.15}$$

and thus, independent of general equilibrium feedback. Finally, define rer as the log deviation from the steady state of the real exchange. Then real exchange dynamics are given by

$$rer_{in,t} = p_i + e_{in,t} - p_n = \mu\mu_{ii} + (1-\mu)\mu_{ni} - ((1-\mu)\mu_{in} + \mu\mu_{nn}),$$

where we used that $w_i + e_{in,t} - w_{n,t} = 0$.

C Lumpy Investment: Proofs

This section provides additional steps for characterizing the investment model in Section 2.

C.1 Characterization of Equilibrium Transition Dynamics

We skip the description of the environment for this case.

Optimal policy. In equilibrium, due to linear preferences in consumption, the time-zero Arrow-Debreu price is $Q_t = Q_0 e^{-\rho t}$. Since the law of motion of the price system is independent of the distribution of firms, the firms' state only depends on the idiosyncratic states K and E. Let $V(K,E)$ the present discounted value of the optimal plan with the state (K,E) measure in time t consumption units.

Define the inaction region as $\mathcal{R} = \{K, E : \underline{K}(E) < K < \overline{K}(E)\}$. For all $(K,E) \in \mathcal{R}$, $V(\cdot)$ satisfies the HJB given by

$$(\rho+\lambda)\,V(K,E) = E^{1-\alpha}K^{\alpha} - \psi K\frac{\partial V(K,E)}{\partial K} + \frac{\sigma^2 E^2}{2}\frac{\partial^2 V(K,E)}{\partial E^2} + (\mu+\frac{\sigma^2}{2})E\frac{\partial V(K,E)}{\partial E} + \dots \tag{C.1}$$

$$\dots + \lambda\mathbb{E}_{\xi}\left[\max\left\{\max_{K^*} V(K^*,E) - \xi E - (K^*-K), V(K,E)\right\}\right], \tag{C.2}$$

together with the value matching conditions:

$$\begin{aligned} V(\underline{K}(E),E) &= \max_{K^*} V(K^*,E) - \kappa E - (K^* - \underline{K}(E)) \\ V(\overline{K}(E),E) &= \max_{K^*} V(K^*,E) - \kappa E - (K^* - \overline{K}(E)), \end{aligned} \tag{C.3}$$

and the smooth pasting conditions:

$$1 = \left.\frac{\partial V(K,E)}{\partial K}\right|_{K=\underline{K}(E)} = \left.\frac{\partial V(K,E)}{\partial K}\right|_{K=\overline{K}(E)} \tag{C.4}$$

$$\frac{\partial V(K^*,E)}{\partial E} - \kappa = \left.\frac{\partial V(K,E)}{\partial E}\right|_{K=\underline{K}(E)} = \left.\frac{\partial V(K,E)}{\partial E}\right|_{K=\overline{K}(E)}. \tag{C.5}$$

For additional details over the HJB equations, value matching and smooth pasting conditions see **?** and Baley and Blanco (2019). The next proposition shows that we can consider the normalized capital $\tilde{K} \equiv \frac{K}{E}$ as a state.

Proposition C.1. *Let V the solution of (C.2) to (C.5). Also define the total drift $\nu \equiv -(\psi+\mu)$ and the adjusted discount $\tilde{\rho} \equiv \rho + \lambda - \mu - \frac{\sigma^2}{2}$. Then $V(K,E) = Ev\left(\frac{K}{E}\right)$ where v satisfies the following HJB, value matching and smooth-pasting conditions:*

$$\tilde{\rho}v\left(\tilde{K}\right) = \tilde{K}^{\alpha} - (\psi+\mu+\frac{\sigma^2}{2})\tilde{K}v'\left(\tilde{K}\right) + \frac{\sigma^2}{2}\tilde{K}^2 v''\left(\tilde{K}\right) + \lambda\mathbb{E}_{\xi}\left[\max\left\{\max_{\tilde{K}^*} v(\tilde{K}^*) - \xi - (\tilde{K}^* - \tilde{K}), v(\tilde{K})\right\}\right] \tag{C.6}$$

$$v\left(\underline{\tilde{K}}\right) - \underline{\tilde{K}} = v\left(\tilde{K}^*\right) - \kappa - \tilde{K}^* \tag{C.7}$$

$$v\left(\overline{\tilde{K}}\right) - \overline{\tilde{K}} = v\left(\tilde{K}^*\right) - \kappa - \tilde{K}^*, \tag{C.8}$$

$$v'\left(\underline{\tilde{K}}\right) = v'\left(\overline{\tilde{K}}\right) = v'\left(\tilde{K}^*\right) = 1. \tag{C.9}$$

Proof. We guess and verify that $V(K,E) = Ev\left(\frac{K}{E}\right)$. Note that the following relations hold:

$$\frac{\partial V(K,E)}{\partial K} = v'\left(\frac{K}{E}\right), \qquad \frac{\partial V(K,E)}{\partial E} = v\left(\frac{K}{E}\right) - \frac{K}{E}v'\left(\frac{K}{E}\right), \qquad \frac{\partial^2 V(K,E)}{\partial E^2} = \frac{K^2}{E^3}v''\left(\frac{K}{E}\right). \tag{C.10}$$

Using these results we have that (C.2) can be written as

$$\begin{aligned}(\rho+\lambda)Ev\left(\frac{K}{E}\right) = E\left(\frac{K}{E}\right)^{\alpha} - \psi\frac{K}{E}v'\left(\frac{K}{E}\right) + \frac{\sigma^2}{2}\left(\frac{K}{E}\right)^2 v''\left(\frac{K}{E}\right) + (\mu+\frac{\sigma^2}{2})\left[v\left(\frac{K}{E}\right) - \frac{K}{E}v'\left(\frac{K}{E}\right)\right] \\ +\lambda E\mathbb{E}_{\xi}\left[\max\left\{\max_{K^*} v\left(\frac{K^*}{E}\right) - \xi - \left(\frac{K^*}{E} - \frac{K}{E}\right), v\left(\frac{K}{E}\right)\right\}\right]\end{aligned}$$

Thus the HJB is given by

$$\tilde{\rho}v\left(\tilde{K}\right) = \tilde{K}^{\alpha} - (\psi+\mu+\frac{\sigma^2}{2})\tilde{K}v'\left(\tilde{K}\right) + \frac{\sigma^2}{2}\tilde{K}^2 v''\left(\tilde{K}\right) + \lambda\mathbb{E}_{\xi}\left[\max\left\{\max_{\tilde{K}^*} v(\tilde{K}^*) - \xi - (\tilde{K}^* - \tilde{K}), v(\tilde{K})\right\}\right]. \tag{C.11}$$

The value-matching conditions can be expressed as

$$v\left(\underline{\tilde{K}}\right) = \max_{\tilde{K}^*} v\left(\tilde{K}^*\right) - \kappa - (\tilde{K}^* - \underline{\tilde{K}})$$
$$v\left(\overline{\tilde{K}}\right) = \max_{\tilde{K}^*} v\left(\tilde{K}^*\right) - \kappa - (\tilde{K}^* - \overline{\tilde{K}}). \tag{C.12}$$

and the smooth pasting conditions as

$$v'\left(\underline{\tilde{K}}\right) = v'\left(\overline{\tilde{K}}\right) = 1 \tag{C.13}$$

and the smooth pasting conditions for the productivity are given by

$$v\left(\tilde{K}^*\right) - \tilde{K}^* v'\left(\tilde{K}^*\right) - \kappa = v\left(\underline{\tilde{K}}\right) - \underline{\tilde{K}} v'\left(\underline{\tilde{K}}\right)$$
$$v\left(\tilde{K}^*\right) - \tilde{K}^* v'\left(\tilde{K}^*\right) - \kappa = v\left(\overline{\tilde{K}}\right) - \overline{\tilde{K}} v'\left(\overline{\tilde{K}}\right). \tag{C.14}$$

Using the optimality condition we have that $v'\left(\tilde{K}^*\right) = 1$ and using the value matching (C.13) we have

$$v\left(\tilde{K}^*\right) - \tilde{K}^* - \kappa = v\left(\underline{\tilde{K}}\right) - \underline{\tilde{K}}$$
$$v\left(\tilde{K}^*\right) - \tilde{K}^* - \kappa = v\left(\overline{\tilde{K}}\right) - \overline{\tilde{K}}. \tag{C.15}$$

Notice that these smooth pasting conditions are redundant due to (C.12). □

The next proposition characterize the stopping policy

Proposition C.2. *Let $w(k)$ and $\{\underline{k}, k^*, \overline{k}\}$ satisfy the following system of differential equation*

$$\tilde{\rho} w(k) = e^{\alpha k} - (\psi + \mu) w'(k) + \frac{\sigma^2}{2} w''(k) + \lambda \mathbb{E}_\xi \left[\max\left\{w(k^*) - \xi - (e^{k^*} - e^k), w(k)\right\}\right] \tag{C.16}$$
$$w(\underline{k}) - e^{\underline{k}} = w(k^*) - \kappa - e^{k^*} \tag{C.17}$$
$$w(\overline{k}) - e^{\overline{k}} = w(k^*) - \kappa - e^{k^*}, \tag{C.18}$$
$$w'(\underline{k}) = e^{\underline{k}}\ ;\ w'(\overline{k}) = e^{\overline{k}}\ ;\ w'(k^*) = e^{k^*}. \tag{C.19}$$

Then given the initial $log(K_0/E_0) = k_0$ the optimal stopping policy is given by

$$\tau(k_0) = \inf_t \{t \geq 0 : k_t \notin [\underline{k}, \overline{k}] \ o\ N_t^{k_t} - N_0^{k_0} = 1\}, \tag{C.20}$$

where $dk_t = -(\delta + \mu)dt + \sigma dB_t$ with initial condition k_0 and $N_t^{k_t}$ is a Poisson process with arrival rate $\lambda(k) = \lambda G(w(k^*) - w(k) - (e^{k^*} - e^k))$

Proof. We depart from the solution of the optimal policy given by (C.6) to (C.9). Doing a guess and verify $v\left(\tilde{K}\right) = w(\log(\tilde{K}))$, it is easy to see that w satisfies (C.16) to (C.19). Therefore the stopping policy is given by

$$\tau(k_0) = \inf_t \{t \geq 0 : k_t \notin [\underline{k}, \overline{k}] \ o\ (N_t - N_0 = 1 \ and\ \xi \leq w(k^*) - w(k) - (e^{k^*} - e^k))\}, \tag{C.21}$$

or equivalently, (C.20). □

D Extensions

For the following extensions, we maintain the assumption that the moments of the adjustment size can be written as:

$$g_m(x) = \mathbb{E}^{\hat{x}+x,S^{-x}}\left[(\hat{x}-\Delta x)^m\right] - \mathbb{E}^{\hat{x},S^{-x}}\left[(\hat{x}-\Delta x+x)^m\right] \tag{D.1}$$

In the proofs, we skip the steps that are identical to the main proofs in the Appendix.

D.1 Characterization of the CIR for higher moments of the distribution

Proposition D.1. *Assume the uncontrolled state follows $d\tilde{x}_t = \sigma dB_t$, and the processes $\left(\left\{\int_0^t s x_s^m \, dB_s\right\}_t, \tau\right)$ are well-defined stopping processes for all m,*

- *To a first order, the transitional dynamics are given by*

$$\mathcal{A}_m(\delta) = \delta \times (\Gamma_m + \Theta_m - \mathcal{M}_m[x]\Theta_0) + o(\delta^2) \tag{D.2}$$

  where the intensive and the extensive margins relate to ergodic moments as follows:

$$\Gamma_m = m\mathcal{M}_{m-1,1}[x,a]. \tag{D.3}$$

$$\Theta_m = \sum_{j=0}^{\infty} \theta_{m,j}\mathcal{M}_j[x] \quad \textit{with} \quad \theta_{m,j} \equiv \frac{2}{\sigma^2(m+1)} \sum_{k\geq j}^{\infty} \frac{\hat{x}^{k-j}}{k!j!}\left[\frac{d^{k+1}g_{m+2}(x)/(m+2)}{dx^{k+1}} - \frac{d^k g_{m+1}(x)}{dx^k}\right]\Bigg|_{x=0}. \tag{D.4}$$

  - *If $\tau|S_t \sim \tau|S_t^{-x}$, then $g_m(x) = 0$ with $\theta(m,i) = 0 \;\; \forall m, i$.*

- *The reset state and the volatility are given by (35) and (36) evaluated at $\nu = 0$. The ergodic moments are given by*

$$\mathcal{M}_m[x] = \frac{2}{(m+1)(m+2)} \frac{\mathbb{E}\left[(\hat{x}-\Delta x)^{m+2} - \hat{x}^{m+2}\right]}{\mathbb{E}\left[\Delta x^2\right]}, \tag{D.5}$$

$$\mathcal{M}_{m,1}[x,a] = \frac{2}{(m+1)(m+2)}\left[\frac{\mathbb{E}^{\hat{S}}\left[(\hat{x}-\Delta x)^{m+2}\tau\right]}{\mathbb{E}^{\hat{S}}\left[\Delta x^2\right]} - \frac{\mathcal{M}_{m+2}[x]}{\sigma^2}\right], \tag{D.6}$$

D.2 Characterization of the CIR with drift

Proposition D.2. *Assume the uncontrolled process follows $d\tilde{x}_t = \nu dt + \sigma dB_t$.*

- ***Aggregation:*** *To a first order, the CIR is given by*

$$\mathcal{A}_m(\delta) = \delta \times (\mathcal{Z}_m - \mathcal{M}_m[x]\Theta_0) + o(\delta^2) \tag{D.7}$$

  where the intensive and extensive margin are given by

$$\mathcal{Z}_m = \Theta_m + \Gamma_m - \frac{\sigma^2 m}{2\nu}\mathcal{Z}_{m-1} \tag{D.8}$$

$$\Gamma_m = \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \varphi_m^\Gamma(S_t)dt\right]}{\mathbb{E}^{\hat{S}}[\tau]} \quad ; \quad \varphi_m^\Gamma(S_t) = \frac{1}{\nu}\left(\mathbb{E}^S\left[x_\tau^m\right] - x_t^m\right) \tag{D.9}$$

$$\Theta_m = \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \varphi_m^\Theta(S_t)dt\right]}{\mathbb{E}^{\hat{S}}[\tau]} \quad ; \quad \varphi_m^\Theta(S_t) = \frac{1}{\nu}\left[\frac{\partial \mathbb{E}^S\left[x_\tau^{m+1}/(m+1)\right]}{\partial x} - \mathbb{E}^S\left[x_\tau^m\right]\right] \tag{D.10}$$

- ***Representation:***

$$\Gamma_m = m\mathcal{M}_{m-1,1}[x,a] + \frac{\mathbb{1}_{\{m\geq 2\}}\sigma^2 m(m-1)}{2\nu}\mathcal{M}_{m-2,1}[x,a] \tag{D.11}$$

$$\Theta_m = \sum_{j=0}^{\infty}\theta_{m,j}\mathcal{M}_j[x] \quad \textit{with} \quad \theta_{m,j} = \sum_{k\geq j}^{\infty}\frac{\hat{x}^{k-j}}{\nu k!j!}\left[\frac{d^{k+1}g_{m+1}(x)}{dx^{k+1}}/m+1 - \frac{d^k g_m(x)}{dx^k}\right]\Bigg|_{x=0}. \tag{D.12}$$

- ***Observability:*** *The reset state $\hat{x}$ and structural parameters (ν, σ) are recovered as*

$$\hat{x} = \mathbb{E}[\Delta x]\left(\frac{1-\mathbb{CV}^2[\tau]}{2}\right) + \frac{Cov[\tau,\Delta x]}{\mathbb{E}[\tau]}, \qquad \nu = -\frac{\mathbb{E}[\Delta x]}{\mathbb{E}[\tau]}, \qquad \sigma^2 = \frac{\mathbb{E}[\Delta x^2]}{\mathbb{E}[\tau]} + 2\nu\hat{x} \tag{D.13}$$

and the ergodic moments are recovered as:

$$\mathcal{M}_m[x] = \frac{\hat{x}^{m+1} - \mathbb{E}[(\hat{x} - \Delta x)^{m+1}]}{\mathbb{E}[\Delta x](m+1)} - \frac{\sigma^2 m}{2\nu}\mathcal{M}_{m-1}[x], \tag{D.14}$$

$$\mathcal{M}_{m,1}[x,a] = \frac{\mathbb{E}\left[\tau/\mathbb{E}[\tau]\,(\hat{x} - \Delta x)^{m+1}\right] - \mathcal{M}_{m+1}[x]}{\nu(m+1)} - \frac{\sigma^2 m}{2\nu}\mathcal{M}_{m-1,1}[x,a] \tag{D.15}$$

with initial conditions $\mathcal{M}_1[x] = 0$ *and* $\mathcal{M}_{0,1}[x,a] = \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]}$.

Proof. We start the proof from equation (B.21) in the Appendix, as all previous steps are identical.

Aggregation. The first order approximation to the CIR yields $\mathcal{A}'_m(0)$ equal to:

$$\underbrace{\int \mathbb{E}^S\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right]dF(S)}_{\mathcal{B}_m} - \mathcal{M}_m[x]\underbrace{\int \frac{\partial \mathbb{E}^S[\tau]}{\partial x}dF(S)}_{\Theta_0} + \underbrace{\int\left(\frac{\partial}{\partial x}\mathbb{E}^S\left[\int_0^\tau x_t^m dt\right] - \mathbb{E}^S\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right]\right)dF(S)}_{\mathcal{C}_m}. \tag{D.16}$$

To characterize the first term $\mathcal{B}_m \equiv \int \mathbb{E}^S\left[\int_0^\tau m x_t^{m-1}dt\right]dF(S)$, we apply Itō's lemma to x_t^m, integrate with respect to initial condition S, use the OST, divide by the drift, and rearrange:

$$\mathbb{E}^S\left[\int_0^\tau m x_t^{m-1}dt\right] = \underbrace{\frac{\mathbb{E}^S\left[x_\tau^m\right] - x_t^m}{\nu}}_{\varphi_m^\Gamma(S_t)} - \frac{\sigma^2 m}{2\nu}\mathbb{E}^S\left[\int_0^\tau (m-1)x_t^{m-2}dt\right].$$

Integrating both sides across all initial conditions, using the definition of Γ_m in (D.9), and recognizing $\mathcal{B}_m$ and $\mathcal{B}_{m-1}$ we get:

$$\mathcal{B}_m = \Gamma_m - \frac{\sigma^2 m}{2\nu}\mathcal{B}_{m-1}, \qquad \Gamma_0 = 0 \tag{D.17}$$

To characterize $\mathcal{C}_m$, we use the previous expressions to compute its two terms separately:

$$\frac{\partial \mathbb{E}^S\left[\int_0^\tau x_t^m dt\right]}{\partial x} = \frac{1}{\nu}\left(\frac{\partial \mathbb{E}^S\left[x_\tau^{m+1}\right]/(m+1)}{\partial x} - x_t^m\right) - \frac{\sigma^2 m}{2\nu}\frac{\partial \mathbb{E}^S\left[\int_0^\tau x_t^{m-1}dt\right]}{\partial x} \tag{D.18}$$

$$\mathbb{E}^S\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right] = \frac{1}{\nu}\left(\mathbb{E}^S\left[x_\tau^m\right] - x_t^m\right) - \frac{\sigma^2 m}{2\nu}\mathbb{E}^S\left[\int_0^\tau \frac{\partial x_t^{m-1}}{\partial x}dt\right] \tag{D.19}$$

Subtracting the two previous equations,

$$\frac{\partial \mathbb{E}^S\left[\int_0^\tau x_t^m dt\right]}{\partial x} - \mathbb{E}^S\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right] = \underbrace{\frac{1}{\nu}\left(\frac{\partial \mathbb{E}^S\left[x_\tau^{m+1}\right]/(m+1)}{\partial x} - \mathbb{E}^S\left[x_\tau^m\right]\right)}_{\varphi_m^\Theta(S_t)} - \frac{\sigma^2 m}{2\nu}\left\{\frac{\partial \mathbb{E}^S\left[\int_0^\tau x_t^{m-1}dt\right]}{\partial x} - \mathbb{E}^S\left[\int_0^\tau \frac{\partial x_t^{m-1}}{\partial x}dt\right]\right\}$$

Integrating with the ergodic distribution and using the definition of Θ_m in (D.10) and recognizing $\mathcal{C}_m$ and $\mathcal{C}_{m-1}$ we get:

$$\mathcal{C}_m = \Theta_m - \frac{\sigma^2 m}{2\nu}\mathcal{C}_{m-1}, \qquad \mathcal{C}_{-1} = 0. \tag{D.20}$$

Define $\mathcal{Z}_m \equiv \mathcal{B}_m + \mathcal{C}_m$, which implies $\mathcal{Z}_m = \Gamma_m + \Theta_m - \frac{\sigma^2 m}{2\nu}\mathcal{Z}_{m-1}$. Combine the results in (D.16), (D.17) and (D.20) to obtain (D.7): $\mathcal{A}'_m(0) = (\mathcal{Z}_m - \mathcal{M}_m[x]\Theta_0)$.

Lastly, we corroborate that the expression $\int \frac{\partial \mathbb{E}^S[\tau]}{\partial x}dF(S)$ is equal to Θ_0. By the OST, we have $\mathbb{E}^S[x_\tau] - x = \nu\mathbb{E}^S[\tau]$. Thus $\frac{\partial \mathbb{E}^S[\tau]}{\partial x} = \frac{1}{\nu}\left[\frac{\partial \mathbb{E}^S[x_\tau]}{\partial x} - 1\right]$. Substituting and using Auxiliary Theorem 2 we recover the expression for Θ_0 in the definition of Θ_m:

$$\Theta_0 \equiv \int \frac{\partial \mathbb{E}^S[\tau]}{\partial x}dF(S) = \int \frac{1}{\nu}\left[\frac{\partial \mathbb{E}^S[x_\tau]}{\partial x} - 1\right]dF(S)$$

Representation for the intensive margin: The proof for Γ_m are easy extended to

$$\Gamma_m = \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \left[\mathbb{E}^{S_t}\left[x_\tau^m\right] - x_t^m\right]dt\right]}{\nu\mathbb{E}^{\hat{S}}[\tau]} = \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \left[x_\tau^m - x_t^m\right]dt\right]}{\nu\mathbb{E}^{\hat{S}}[\tau]}$$

With similar steps as the case with no drift we have that

$$\frac{\mathbb{E}^{\hat{S}}\left[x_\tau^m \tau\right]}{\mathbb{E}^{\hat{S}}\left[\tau\right]} - \mathcal{M}_m[x] = \mathbb{1}_{\{m\geq 1\}}\nu m \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{m-1} t dt\right]}{\mathbb{E}^{\hat{S}}[\tau]} + \frac{\mathbb{1}_{\{m\geq 2\}}\sigma^2 m(m-1)}{2}\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{m-2} t dt\right]}{\mathbb{E}^{\hat{S}}[\tau]} \tag{D.21}$$

Thus

$$\Gamma_m = \mathbb{1}_{\{m\geq 1\}} m \mathcal{M}_{m-1,1}[x,a] + \frac{\mathbb{1}_{\{m\geq 2\}}\sigma^2 m(m-1)}{2\nu}\mathcal{M}_{m-2,1}[x,a] \tag{D.22}$$

Representation for the extensive margin: The proofs for Θ_m are easily to extend to

$$\Theta_m = \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \left(\frac{\partial \mathbb{E}^S[x_\tau^{m+1}/(m+1)]}{\partial x} - \mathbb{E}^S[x_\tau^m]\right) dt\right]}{\nu \mathbb{E}^{\hat{S}}\left[\tau\right]} \tag{D.23}$$

$$= \frac{1}{\nu}\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \left[\left[\frac{dg_{m+1}(y-\hat{x})}{dy}/(m+1) + \mathbb{E}^{\hat{x},S^{-x}}\left[(y-\hat{x}+x_\tau)^{m+1}\right]\right] - \left[g_m(y-\hat{x}) + \mathbb{E}^{\hat{x},S^{-x}}\left[(y-\hat{x}+x_\tau)^m\right]\right]\right] dt\right]}{\mathbb{E}^{\hat{S}}[\tau]}$$

$$= \frac{1}{\nu}\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \left[\frac{dg_{m+1}(y-\hat{x})}{dy}/(m+1) - g_m(y-\hat{x})\right] dt\right]}{\mathbb{E}^{\hat{S}}[\tau]}$$

$$= \frac{1}{\nu}\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \left[\sum_{j=0}^\infty \frac{d^j}{j!dx^j}\left[\frac{dg_{m+1}(x)}{dx}/(m+1) - g_m(x)\right]_{x=0}(y-\hat{x})^j\right] dt\right]}{\mathbb{E}^{\hat{S}}[\tau]}$$

$$= \frac{1}{\nu}\sum_{j=0}^\infty\sum_{z=0}^j \frac{\left[\frac{d^{j+1}g_{m+1}(0)}{dx^{j+1}}/m+1-\frac{d^j g_m(0)}{dx^j}\right]\hat{x}^z}{j!}\begin{pmatrix} j \\ z\end{pmatrix}\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{j-z}dt\right]}{\mathbb{E}^{\hat{S}}[\tau]}$$

$$= \frac{1}{\nu}\sum_{j=0}^\infty\sum_{z=0}^j \frac{\left[\frac{d^{j+1}g_{m+1}(0)}{dx^{j+1}}/m+1-\frac{d^j g_m(0)}{dx^j}\right]\hat{x}^z}{j!}\begin{pmatrix} j \\ z\end{pmatrix}\mathcal{M}_{j-z}[x]$$

$$= \sum_{j=0}^\infty\sum_{z=0}^j \underbrace{\frac{1}{\nu}\frac{\left[\frac{d^{j+1}g_{m+1}(0)}{dx^{j+1}}/m+1-\frac{d^j g_m(0)}{dx^j}\right]\hat{x}^z}{z!(j-z)!}}_{=\mathcal{H}_{m,j,z}}\mathcal{M}_{j-z}[x]$$

$$= \sum_{h=0}^\infty \theta_{m,h}\mathcal{M}_h[x],\ \textit{with}\ \theta_{m,h} = \sum_{k\geq h}^\infty \mathcal{H}_{m,k,k-h}. \tag{D.24}$$

Observation: First, we characterize the structural parameters of the stochastic process.

- It is easy to show that $\frac{\mathbb{E}^{\hat{S}}[\Delta x]}{\mathbb{E}^{\hat{S}}[\tau]} = -\nu$.
- For characterizing σ define $Y_t = x_t - \nu t$ with initial condition $Y_0 = \hat{x}$. With similar steps as the process without drift we have that

$$\sigma^2 = \frac{\mathbb{E}^{\hat{S}}\left[\Delta Y_\tau^2\right]}{\mathbb{E}^{\hat{S}}[\tau]} = \frac{\mathbb{E}^{\hat{S}}\left[(x_\tau - \nu\tau - x + x_0)^2\right]}{\mathbb{E}^{\hat{S}}[\tau]} = \frac{\mathbb{E}^{\hat{S}}\left[(\nu\tau + \Delta x)^2\right]}{\mathbb{E}^{\hat{S}}[\tau]} \tag{D.25}$$

  or equivalently

$$\sigma^2 = \frac{\mathbb{E}^{\hat{S}}[\Delta x^2]}{\mathbb{E}^{\hat{S}}[\tau]} + 2\nu\frac{\mathbb{E}^{\hat{S}}[\Delta x\tau]}{\mathbb{E}^{\hat{S}}[\tau]} + \nu^2\frac{\mathbb{E}^{\hat{S}}[\tau^2]}{\mathbb{E}^{\hat{S}}[\tau]} = \frac{\mathbb{E}^{\hat{S}}[\Delta x^2]}{\mathbb{E}[\tau]} - 2\frac{\mathbb{E}[\Delta x]\mathbb{E}[\Delta x\tau]}{\mathbb{E}[\tau]^2} + \frac{\mathbb{E}[\Delta x]^2\mathbb{E}[\tau^2]}{\mathbb{E}[\tau]^3}$$

  Applying the formula we get below we have the result.
- For the reset state $\hat{x}$, we apply Itō's lemma to x_t^2 to obtain $d(x_t^2) = 2x_t dx_t + (dx_t)^2 = \left(2\nu x_t + \sigma^2\right)dt + 2\sigma x_t dB_t$. Using the OST $\mathbb{E}^{\hat{S}}[\int_0^\tau x_s dB_s] = 0$. Moreover, given that $\mathbb{E}^{\hat{S}}[\int_0^\tau x_s ds] = \mathcal{M}_1[x]\mathbb{E}^{\hat{S}}[\tau] = 0$, we have that

$$\mathbb{E}^{\hat{S}}[x_\tau^2] = \hat{x}^2 + \sigma^2\mathbb{E}^{\hat{S}}[\tau] \tag{D.26}$$

  Completing squares

$$\mathbb{E}^{\hat{S}}[x_\tau^2] = \mathbb{E}^{\hat{S}}[(\hat{x} - (\hat{x} - x_\tau))^2] = \mathbb{E}^{\hat{S}}[\Delta x^2] - 2\hat{x}\mathbb{E}^{\hat{S}}[\Delta x] + (\hat{x})^2$$

Therefore

$$\hat{x} = \frac{1}{2\mathbb{E}^{\hat{S}}[\Delta x]}\left[\mathbb{E}^{\hat{S}}[\Delta x^2] - \sigma^2\mathbb{E}^{\hat{S}}[\tau]\right] \tag{D.27}$$

$$= \frac{1}{2\mathbb{E}^{\hat{S}}[\Delta x]}\left[\mathbb{E}^{\hat{S}}[\Delta x^2] - \left(\mathbb{E}^{\hat{S}}[\Delta x^2] + 2\frac{\mathbb{E}^{\hat{S}}[\Delta x]\mathbb{E}^{\hat{S}}[\Delta x\tau]}{\mathbb{E}^{\hat{S}}[\tau]} + \frac{\mathbb{E}^{\hat{S}}[\Delta x]^2\mathbb{E}^{\hat{S}}[\tau^2]}{\mathbb{E}^{\hat{S}}[\tau]^2}\right)\right] \tag{D.28}$$

$$= \frac{\mathbb{E}^{\hat{S}}[\Delta x\tau]}{\mathbb{E}^{\hat{S}}[\tau]} - \frac{\mathbb{E}^{\hat{S}}[\Delta x]\mathbb{E}^{\hat{S}}[\tau^2]}{2\mathbb{E}^{\hat{S}}[\tau]^2} \tag{D.29}$$

- For observability of ergodic moments of x, apply Itō's lemma to x^{m+1} and get $dx_t^{m+1} = (m+1)x_t^m\nu dt + (m+1)x_t^m\sigma dB_t + \frac{\sigma^2}{2}m(m+1)x_t^{m-1}dt$. Integrating from 0 to τ, using the OST to eliminate martingales, and rearranging:

$$\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^m dt\right] = \frac{1}{\nu(m+1)}\left(\mathbb{E}^{\hat{S}}[x_\tau^{m+1}] - \hat{x}^{m+1}\right) - \frac{\sigma^2}{2\nu}m\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{m-1}dt\right] \tag{D.30}$$

Substituting the equivalences $\mathcal{M}_m[x] = \mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^m dt\right]/\mathbb{E}^{\hat{S}}[\tau]$ and $\mathbb{E}^{\hat{S}}[\Delta x] = -\nu\mathbb{E}^{\hat{S}}[\tau]$ yields:

$$\mathcal{M}_m[x] = \frac{\hat{x}^{m+1} - \mathbb{E}^{\hat{S}}[(\hat{x}-\Delta x)^{m+1}]}{\mathbb{E}^{\hat{S}}[\Delta x](m+1)} - \frac{\sigma^2 m}{2\nu}\mathcal{M}_{m-1}[x], \qquad \mathcal{M}_1[x] = 0 \tag{D.31}$$

- For observability of ergodic moments of $x^m a$, where a stand for the duration of the last action, we use Itō's lemma and the OST on $x_t^{m+1}t$:

$$\mathbb{E}^{\hat{S}}\left[\tau\left(\hat{x}-\Delta x\right)^{m+1}\right] = \mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{m+1}dt\right] + (m+1)\nu\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^m t dt\right] + \frac{\sigma^2 m(m+1)}{2}\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{m-1}t dt\right] \tag{D.32}$$

and therefore

$$\mathcal{M}_{m,1}[x,a] = \frac{\mathbb{E}^{\hat{S}}\left[\tau\left(\hat{x}-\Delta x\right)^{m+1}\right]}{\nu(m+1)\mathbb{E}^{\hat{S}}[\tau]} - \frac{\mathcal{M}_{m+1}[x]}{\nu(m+1)} - \frac{\sigma^2 m}{2\nu}\mathcal{M}_{m-1,1}[x,a] \tag{D.33}$$

with initial condition $\mathcal{M}_{0,1}[x,a] = \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]}$.

□

D.3 Characterization of the CIR for Different Initial Conditions

For this proof we focus in the case $m = I = 1$.

Proposition D.3. *Assume a function $\mathcal{G}(x,\delta)$ s.t.*

1. $\mathcal{G}(x,0) = x$
2. $\exists z > 0$ *s.t.* $\forall \epsilon \in (-z,z)$ *the function* $\mathcal{G}(\cdot,\epsilon)$ *is bijective.*
3. $\frac{\partial \mathcal{G}(\mathcal{G}^{-1}(y,0),0)}{\partial \delta} = -\sum_{i=0}^{I} \frac{\mathcal{G}_i y^i}{i!}$ *with* $\sum_i \mathcal{G}_i^2 = 1$.

Then, up to first order, the CIR is given by:

- ***Aggregation:***

$$\mathcal{A}_1^{\mathcal{G}}(\delta) = \delta \times \sum_{i=0}^{1} \frac{\mathcal{G}_i}{i!} \left(\Gamma_{1,i} + \Theta_{1,i}\right) + o(\delta^2) \tag{D.34}$$

$$\Gamma_{1,i} = \frac{\mathbb{E}\left[\int_0^\tau \varphi_{1,i}^{\Gamma}(S_t)dt\right]}{\mathbb{E}[\tau]} \quad ; \quad \varphi_{1,i}^{\Gamma}(S) = \frac{2}{\sigma^2}\left[\frac{i x^{i-1}\mathbb{1}_{\{i\geq 2\}}}{\sigma^2}\frac{\mathbb{E}^S\left[x_\tau^3\right] - x^3}{3} + x^i \frac{\mathbb{E}^S\left[x_\tau^{m+1}\right] - x^2}{\sigma^2}\right] \tag{D.35}$$

$$\Theta_{1,i} = \frac{\mathbb{E}\left[\int_0^\tau \varphi_{1,i}^{\Theta}(S_t)dt\right]}{\mathbb{E}[\tau]} \quad ; \quad \varphi_{1,i}^{\Theta}(S) = \frac{x^i}{\sigma^2}\left[\frac{\partial \mathbb{E}^S\left[x_\tau^{m+2}/(m+2)\right]}{\partial x} - \frac{\partial \mathbb{E}^S\left[x_\tau^{m+1}\right]}{\partial x}\right] \tag{D.36}$$

- ***Representation:***

$$\Gamma_{m,i} = (i+1)\mathcal{M}_{i,1}[x,a] \tag{D.37}$$

$$\Theta_{1,i} = \sum_{j=0}^{\infty} \theta_{1,j}\mathcal{M}_{j+i}[x] \tag{D.38}$$

with $\theta_{m,j}$ defined in proposition 6.

Observation: *Without change.*

Proof. Let us explain the assumptions on the function $\mathcal{G}$. First, if there is no change in the distribution ($\delta = 0$), then the initial distribution is equal to the steady state distribution $\mathcal{G}(x,0) = x$. Second, $\mathcal{G}(\cdot,\delta)$ is bijective in a small domain around δ. The third assumption is that $\frac{\partial(\mathcal{G}^{-1}(y,0),0)}{\partial\delta}$ is differentiable for all orders, thus $\frac{\partial \mathcal{G}(\mathcal{G}^{-1}(y,0),0)}{\partial\delta} = -\sum_i \frac{\mathcal{G}_i y^i}{i!}$. Finally, since the perturbation can be re-scaled by the size of the shocks, we normalize the coefficients of the Taylor approximation such that their squares sum up to one: $\sum \mathcal{G}_i^2 = 1$. We focus on the case $m = 1$.

Most steps used in the proofs for the baseline case hold with only minor changes.

Aggregation: The main different comes at the moment of doing the Taylor approximation, since

$$\mathcal{A}_1^{\mathcal{G}}(\delta) = \int_x \left[\int_{S^{-x}} v^1(x,S^{-x})dF(S^{-x}|x)\right] dF(\mathcal{G}^{-1}(x,\delta)) \tag{D.39}$$

$$= \int_x \left[\int_{S^{-x}} v^1(x,S^{-x})dF(S^{-x}|x)\right] f(\mathcal{G}^{-1}(x,\delta))dx \tag{D.40}$$

$$= \delta \int_x v^1(x) f'(\mathcal{G}^{-1}(x,0))\frac{\partial \mathcal{G}^{-1}(x,0)}{\partial\delta}dx + o(\delta^2) \tag{D.41}$$

Notice that if $y = \mathcal{G}(x,\delta)$, then $\frac{dy}{d\delta}\big|_{\delta=0} = \frac{\partial\mathcal{G}(x,0)}{\partial\delta} = \frac{\partial\mathcal{G}(\mathcal{G}^{-1}(y,0),0)}{\partial\delta}$ and by assumption

$$\mathcal{A}_1^{\mathcal{G}}(\delta) = -\delta \sum_i \frac{\mathcal{G}_i}{i!}\int \left[x^i v^m(x)\right] f'(\mathcal{G}^{-1}(x,0))dx \tag{D.42}$$

$$= -\delta \sum_i \frac{\mathcal{G}_i}{i!}\int \left[x^i v^1(x)\right] f'(\mathcal{G}^{-1}(x,0))dx \tag{D.43}$$

$$= \delta \sum_i \frac{\mathcal{G}_i}{i!}\int \frac{\partial}{\partial x}\left[x^i v^1(x)\right] dF(S) + o(\delta) \tag{D.44}$$

$$= \delta \sum_i \frac{\mathcal{G}_i}{i!}\left[\Gamma_{1,i} + \Theta_{1,i}\right] + o(\delta) \tag{D.45}$$

where we define

$$\begin{aligned}\Gamma_{1,i} &\equiv \int \left(i x^{i-1} \mathbb{E}^S \left[\int_0^\tau x_t dt \right] + x^i \mathbb{E}^S \left[\int_0^\tau 1 dt \right] \right) dF(S) \\ \Theta_{1,i} &\equiv \int x^i \left(\frac{\partial}{\partial x} \mathbb{E}^S \left[\int_0^\tau x_t dt \right] - \mathbb{E}^S \left[\int_0^\tau 1 dt \right] \right) dF(S)\end{aligned}$$

With similar steps as in the main proof (a combination of Ito's lemma and the OST), we have that

$$\Gamma_{m,i} \equiv \frac{\mathbb{E}^{\hat{S}} \left[\int_0^\tau \varphi^{\Gamma}_{m,i}(S_t) dt \right]}{\mathbb{E}^{\hat{S}} [\tau]} \qquad \Theta_{m,i} \equiv \frac{\mathbb{E}^{\hat{S}} \left[\int_0^\tau \varphi^{\Theta}_{m,i}(S_t) dt \right]}{\mathbb{E}^{\hat{S}} [\tau]} \tag{D.46}$$

where

$$\begin{aligned}\varphi^{\Gamma}_{m,i}(S) &= \frac{2}{\sigma^2} \left(\frac{i x^{i-1} \mathbb{1}_{\{i \geq 1\}}}{3\sigma^2} \left[\mathbb{E}^S \left[x_\tau^3 \right] - x^3 \right] + \frac{x^i}{\sigma^2} \left[\mathbb{E}^S \left[x_\tau^2 \right] - x^2 \right] \right) \\ \varphi^{\Theta}_{m,i}(S) &= \frac{x^i}{\sigma^2} \left(\frac{\partial \mathbb{E}^S \left[x_\tau^3 / 3 \right]}{\partial x} - \mathbb{E}^S \left[x_\tau^2 \right] \right)\end{aligned}$$

Representation for the intensive margin: Repeating the steps as in the main proof, it is easy to show that

$$\Gamma_{1,i} = \Gamma^1_{1,i} + \Gamma^2_{1,i} \tag{D.47}$$

$$\Gamma^1_{1,i} = \frac{\mathbb{1}_{\{i \geq 1\}}}{3\sigma^2} \frac{\mathbb{E}^{\hat{S}} \left[\int_0^\tau i x_t^{i-1} \left[x_\tau^3 - x_t^3 \right] dt \right]}{\mathbb{E}^{\hat{S}} [\tau]} \tag{D.48}$$

$$\Gamma^2_{1,i} = \frac{\mathbb{E}^{\hat{S}} \left[\int_0^\tau x_t^i \left[x_\tau^2 - x_t^2 \right] dt \right]}{\sigma^2 \mathbb{E}^{\hat{S}} [\tau]} \tag{D.49}$$

Notice that in the case $i = 0$ we have that

$$\Gamma_{1,0} = \Gamma^1_{1,0} + \Gamma^2_{1,0} = \Gamma^2_{1,0} = \mathcal{M}_1[a]. \tag{D.50}$$

Next, we characterize the case with $i = 1$. For $\Gamma^1_{1,1}$, we can use proposition (1) were we have shown that

$$\frac{\mathbb{E}^{\hat{S}} \left[\int_0^\tau \left[x_\tau^3 - x_t^3 \right] dt \right]}{\sigma^2 3 \mathbb{E}^{\hat{S}} [\tau]} 2 = 2\mathcal{M}_{1,1}[x, a], \tag{D.51}$$

Thus, $\Gamma^1_{1,1} = \mathcal{M}_{1,1}[x, a]$. Let us characterize the term $\Gamma^2_{1,i}$. Using the occupancy measure

$$\Gamma^2_{1,1} = \frac{\mathbb{E}^{\hat{S}} \left[\int_0^\tau x_t \left[x_\tau^2 - x_t^2 \right] dt \right]}{\sigma^2 \mathbb{E}^{\hat{S}} [\tau]} = \frac{\mathbb{E}^{\hat{S}} \left[x_\tau^2 \int_0^\tau x_t dt \right]}{\sigma^2 \mathbb{E}^{\hat{S}} [\tau]} - \frac{\mathbb{E}^{\hat{S}} \left[\int_0^\tau x_t^3 dt \right]}{\sigma^2 \mathbb{E}^{\hat{S}} [\tau]} = \frac{\mathbb{E}^{\hat{S}} \left[x_\tau^2 \int_0^\tau x_t dt \right]}{\sigma^2 \mathbb{E}^{\hat{S}} [\tau]} - \frac{\mathcal{M}_3[x]}{\sigma^2} \tag{D.52}$$

To characterize the first term of the previous equation, we apply Ito's Lemma to $x_t^2 \int_0^t x_s ds$, and we have

$$d(x_t^2 \int_0^t x_s ds) = x_t^3 dt + 2x_t \int_0^t x_s ds dB_t + \frac{2\sigma^2}{2} \int_0^t x_s ds dt \tag{D.53}$$

Using the OTS and properties of the Ito's integral, we have that

$$\mathbb{E}^{\hat{S}} \left[x_\tau^2 \int_0^\tau x_t dt \right] = \mathbb{E}^{\hat{S}} \left[\int_0^\tau x_t^3 dt \right] + \sigma^2 \mathbb{E}^{\hat{S}} \left[\int_0^\tau \int_0^t x_s ds dt \right]. \tag{D.54}$$

Using Fubini's for the Reimman integral

$$\mathbb{E}^{\hat{S}} \left[\int_0^\tau \int_0^t x_s ds dt \right] = \mathbb{E}^{\hat{S}} \left[\int_0^\tau x_s \int_{\tau-2}^\tau dt ds \right] = \mathbb{E}^{\hat{S}} \left[\int_0^\tau x_s s ds \right], \tag{D.55}$$

and thus we have that

$$\Gamma^2_{1,1} = \mathcal{M}_{1,1}[x, a] \tag{D.56}$$

Representation for the extensive margin: For $\Theta_{m,i}$ we have that

$$\begin{aligned}
\Theta_{1,i} &= \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \varphi_m^\Theta(S_t)dt\right]}{\mathbb{E}^{\hat{S}}[\tau]} = \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^i \left[\frac{dg_3(x_t-\hat{x})}{dy}/(3) - g_2(x_t-\hat{x})\right] dt\right]}{\sigma^2\mathbb{E}^{\hat{S}}[\tau]} \\
&= \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^i \left[\sum_{j=0}^\infty \frac{d^j}{j!dx^j}\left[\frac{dg_3(x)}{dx}/(3) - g_2(x)\right]_{x=0}(x_t-\hat{x})^j\right] dt\right]}{\sigma^2\mathbb{E}^{\hat{S}}[\tau]} \\
&= \sum_{j=0}^\infty \sum_{z=0}^j \frac{\left[\frac{d^{j+1}g_3(0)}{dx^{j+1}}/3 - \frac{d^{j+1}g_2(0)}{dx^{j+1}}\right]\hat{x}^z}{j!}\begin{pmatrix} j \\ z \end{pmatrix}\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{j+i-z}dt\right]}{\sigma^2\mathbb{E}^{\hat{S}}[\tau]} \\
&= \sum_{j=0}^\infty \sum_{z=0}^j \frac{\left[\frac{d^{j+1}g_3(0)}{dx^{j+1}}/3 - \frac{d^{j}g_2(0)}{dx^{j}}\right]\hat{x}^z}{\sigma^2 j!}\begin{pmatrix} j \\ z \end{pmatrix}\mathcal{M}_{j+i-z}[x] \\
&= \sum_{j=0}^\infty \theta_{m,j}\mathcal{M}_{j+i}[x],\ \textit{with}\ \theta_{m,j} = \sum_{k\geq j}^\infty \frac{\left[\frac{d^{k+1}g_3(x)}{dx^{k+1}}/3 - \frac{d^{k}g_2(x)}{dx^{k}}\right]\Big|_{x=0}\hat{x}^{k-j}}{\sigma^2 k!j!}.
\end{aligned} \tag{D.57}$$

□

D.4 Characterization of the CIR for mean-reverting process

Proposition D.4. *Assume that the uncontrolled state follows a mean-reverting Brownian motion with $\rho < 0$:*

$$d\tilde{x}_t = \rho \tilde{x}_t dt + \sigma dB_t, \qquad B_t \sim Wiener.$$

- ***Aggregation:*** *To a first order, the transitional dynamics are given by*

$$\mathcal{A}_m(\delta) = \delta \times \left(\mathcal{Z}_m - \mathcal{M}_m[x] \left[\Theta_0 - \frac{2\rho}{\sigma^2} \mathcal{C}_2 \right] \right) + o(\delta^2) \tag{D.58}$$

where the intensive Γ_m and the extensive Θ_m margin components are:

$$\begin{aligned}
\mathcal{Z}_m &= \Gamma_m + \Theta_m - \frac{2\rho}{\sigma^2(m+1)} \mathcal{Z}_{m+2} \\
\mathcal{C}_2 &= \int \frac{\partial}{\partial x} \mathbb{E}^S \left[\int_0^\tau x_t^2 dt \right] - \mathbb{E}^S \left[\int_0^\tau 2x_t dt \right] dF(S) \\
\Gamma_m &\equiv \frac{\mathbb{E}^{\hat{S}} \left[\int_0^\tau \varphi_m^\Gamma(S_t) dt \right]}{\mathbb{E}^{\hat{S}}[\tau]} \quad with \quad \varphi_m^\Gamma(S) \equiv \frac{2}{\sigma^2(m+1)} \left(\mathbb{E}^S \left[x_\tau^{m+1} \right] - x^{m+1} \right) \\
\Theta_m &\equiv \frac{\mathbb{E}^{\hat{S}} \left[\int_0^\tau \varphi_m^\Theta(S_t) dt \right]}{\mathbb{E}^{\hat{S}}[\tau]} \quad with \quad \varphi_m^\Theta(S) \equiv \frac{2}{\sigma^2(m+1)} \left(\frac{\partial \mathbb{E}^S[x_\tau^{m+2}]/(m+2)}{\partial x} - \mathbb{E}^S \left[x_\tau^{m+1} \right] \right)
\end{aligned}$$

- ***Representation:***

$$\begin{aligned}
\Gamma_m &= \frac{2}{\sigma^2(m+1)} \mathcal{M}_{m+1}[x] + m \mathcal{M}_{m-1,1}[x,a] \\
\Theta_m &= \sum_{h=0}^{\infty} \theta_{m,h} \mathcal{M}_h[x], \quad \theta_{m,h} = \sum_{k \geq h}^{\infty} \frac{(-1)^{k-h}}{\rho} \frac{\hat{x}^{k-h}}{(k-h)!h!} \left[\frac{d^{j+1} g_{m+2}(0)}{dx^{j+1}} / (m+2) - \frac{d^{j+1} g_{m+1}(0)}{dx^{j+1}} \right]
\end{aligned}$$

- ***Observation:*** *The reset state and structural parameters are recovered with a system of equations:*

$$\hat{x} = \frac{\mathbb{E}^{\hat{S}}[e^{-\rho\tau} \Delta x]}{\mathbb{E}^{\hat{S}}[e^{-\rho\tau}] - 1} \tag{D.59}$$

$$\frac{\sigma^2}{\rho} = 2 \frac{\hat{x}^2 - \mathbb{E}^{\hat{S}} \left[e^{-2\rho\tau} (\hat{x} - \Delta x)^2 \right]}{\mathbb{E}^{\hat{S}} \left[e^{-2\rho\tau} \right] - 1} \tag{D.60}$$

$$\mathbb{E}^{\hat{S}} \left[erf \left((\hat{x} - \Delta x) \sqrt{\frac{\rho}{\sigma^2}} \right) \right] = erf \left(\hat{x} \sqrt{\frac{\rho}{\sigma^2}} \right) \tag{D.61}$$

where $erf(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ is the Gauss error function; and the ergodic moments are recovered as

$$\mathcal{M}_m[x] = \frac{\mathbb{E}[(\hat{x} - \Delta x)^m - \hat{x}^m]}{\rho m \mathbb{E}[\tau]} - \frac{\sigma^2(m-1)}{2\rho} \mathcal{M}_{m-2}[x] \tag{D.62}$$

$$\mathcal{M}_{m,1}[x,a] = \frac{\mathbb{E}[\tau(\hat{x} - \Delta x)^m]}{\mathbb{E}[\tau]} \frac{\sigma^2(m-1)}{2\rho} \mathcal{M}_{m-2,1}[x,a] - \frac{\mathcal{M}_m[x]}{m\rho} \tag{D.63}$$

with $\mathcal{M}_0[x] = 1$, $\mathcal{M}_1[x] = 0$, $\mathcal{M}_{1,1}[x,a] = 0, \mathcal{M}_{0,1}[x,a] = \mathcal{M}_1[a]$.

Proof. **Aggregation** We start the proof from equation (B.21) in the Appendix, as all previous steps are identical. The first order approximation to the CIR yields $\mathcal{A}'_m(0)$ equal to:

$$\underbrace{\int \mathbb{E}^S \left[\int_0^\tau \frac{\partial x_t^m}{\partial x} dt \right] dF(S)}_{\mathcal{B}_m} - \mathcal{M}_m[x] \underbrace{\int \frac{\partial \mathbb{E}^S[\tau]}{\partial x} dF(S)}_{\Theta_0} + \underbrace{\int \left(\frac{\partial}{\partial x} \mathbb{E}^S \left[\int_0^\tau x_t^m dt \right] - \mathbb{E}^S \left[\int_0^\tau \frac{\partial x_t^m}{\partial x} dt \right] \right) dF(S)}_{\mathcal{C}_m}. \tag{D.64}$$

To characterize the first term $\mathcal{B}_m \equiv \int \mathbb{E}^S \left[\int_0^\tau m x_t^{m-1} dt \right] dF(S)$, we apply Itō's Lemma to x_t^{m+1} to get $dx_t^{m+1} = \rho(m+1) x_t^{m+1} dt + \frac{\sigma^2}{2} m(m+1) x_t^{m-1} dt + \sigma(m+1) x_t^m dB_t$. Integrating both sides from 0 to τ and taking expectations with initial condition S, and applying the OST to eliminate martingales with zero initial condition, we get:

$$\frac{\mathbb{E}^S \left[x_\tau^{m+1} \right] - x^{m+1}}{m+1} = \rho \mathbb{E}^S \left[\int_0^\tau x_t^{m+1} dt \right] + \frac{\sigma^2}{2} \mathbb{E}^S \left[\int_0^\tau m x_t^{m-1} dt \right]$$

Solving for $\mathbb{E}^S\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right]$ and multiplying/dividing by $m+2$:

$$\mathbb{E}^S\left[\int_0^\tau m x_t^{m-1}dt\right] = \underbrace{\frac{2}{\sigma^2}\frac{\mathbb{E}^S\left[x_\tau^{m+1}\right]-x^{m+1}}{m+1}}_{\varphi_m^\Gamma(S)} - \frac{2\rho}{\sigma^2(m+2)}\mathbb{E}^S\left[\int_0^\tau (m+2)x_t^{m+1}dt\right] \tag{D.65}$$

Integrating both sides across with the initial distribution, and defining $\Gamma_m \equiv \int \frac{2}{\sigma^2}\frac{\mathbb{E}^S\left[x_\tau^{m+1}\right]-x^{m+1}}{m+1}dF(S)$, we get a recursive formula for $\mathcal{B}_m$:

$$\mathcal{B}_m = \Gamma_m - \frac{2\rho}{\sigma^2(m+2)}\mathcal{B}_{m+2} \tag{D.66}$$

To characterize the term $\mathcal{C}_m$, we focus separately on each of its terms. If we apply Itō's Lemma to x_t^{m+2} we get $dx_t^{m+2} = \rho(m+2)x_t^{m+2}dt + \frac{\sigma^2}{2}(m+2)(m+1)x_t^m dt + \sigma(m+2)x_t^{m+2}dB_t$, and we can express it as

$$\mathbb{E}^S[x_\tau^{m+2} - x^{m+2}] = \rho(m+2)\mathbb{E}^S\left[\int_0^\tau x_t^{m+2}dt\right] + \frac{\sigma^2}{2}(m+2)(m+1)\mathbb{E}^S\left[\int_0^\tau x_t^m dt\right]$$

Solving for $\mathbb{E}^S\left[\int_0^\tau x_t^m dt\right]$ and taking the derivative with respect to initial condition x:

$$\frac{\partial}{\partial x}\mathbb{E}^S\left[\int_0^\tau x_t^m dt\right] = \frac{2}{\sigma^2(m+1)}\left(\frac{\partial \mathbb{E}^S[x_\tau^{m+2}]/(m+2)}{\partial x} - x^{m+1}\right) - \frac{2\rho}{\sigma^2(m+1)}\frac{\partial}{\partial x}\mathbb{E}^S\left[\int_0^\tau x_t^{m+2}dt\right] \tag{D.67}$$

Subtract (D.67) minus (D.65):

$$\begin{aligned}
& \frac{\partial}{\partial x}\mathbb{E}^S\left[\int_0^\tau x_t^m dt\right] - \mathbb{E}^S\left[\int_0^\tau m x_t^{m-1}dt\right] \\
= \; & \frac{2}{\sigma^2(m+1)}\left(\frac{\partial \mathbb{E}^S[x_\tau^{m+2}]/(m+2)}{\partial x} - x^{m+1}\right) - \frac{2\rho}{\sigma^2(m+1)}\frac{\partial}{\partial x}\mathbb{E}^S\left[\int_0^\tau x_t^{m+2}dt\right] \\
- \; & \frac{2}{\sigma^2}\frac{\mathbb{E}^S\left[x_\tau^{m+1}\right]-x^{m+1}}{m+1} + \frac{2\rho}{\sigma^2(m+2)}\mathbb{E}^S\left[\int_0^\tau (m+2)x_t^{m+1}dt\right] \\
= \; & \underbrace{\frac{2}{\sigma^2(m+1)}\left(\frac{\partial \mathbb{E}^S[x_\tau^{m+2}]/(m+2)}{\partial x} - \mathbb{E}^S\left[x_\tau^{m+1}\right]\right)}_{\varphi_m^\Theta(S_t)} - \frac{2\rho}{\sigma^2(m+1)}\left\{\frac{\partial}{\partial x}\mathbb{E}^S\left[\int_0^\tau x_t^{m+2}dt\right] - \mathbb{E}^S\left[\int_0^\tau (m+2)x_t^{m+1}dt\right]\right\} \\
- \; & \frac{2\rho}{\sigma^2(m+2)(m+1)}\mathbb{E}^S\left[\int_0^\tau (m+2)x_t^{m+1}dt\right]
\end{aligned}$$

Integrating with the ergodic distribution, we obtain:

$$\mathcal{C}_m = \Theta_m - \frac{2\rho}{\sigma^2(m+1)}\mathcal{C}_{m+2} - \frac{2\rho}{\sigma^2(m+1)(m+2)}\mathcal{B}_{m+2} \tag{D.68}$$

where $\Theta_m \equiv \int \frac{2}{\sigma^2(m+1)}\left(\frac{\partial \mathbb{E}^S[x_\tau^{m+2}]/(m+2)}{\partial x} - \mathbb{E}^S\left[x_\tau^{m+1}\right]\right)dF(S)$. Finally, let $\mathcal{Z}_m \equiv \mathcal{B}_m + \mathcal{C}_m$ which equals

$$\begin{aligned}
\mathcal{Z}_m &\equiv \mathcal{B}_m + \mathcal{C}_m \\
&= \Gamma_m + \Theta_m - \frac{2\rho}{\sigma^2}\left[\frac{\mathcal{B}_{m+2}}{(m+2)} + \frac{\mathcal{C}_{m+2}}{(m+1)} + \frac{\mathcal{B}_{m+2}}{(m+1)(m+2)}\right] \\
&= \Gamma_m + \Theta_m - \frac{2\rho}{\sigma^2(m+1)}\mathcal{Z}_{m+2}
\end{aligned}$$

Lastly, we find the expression for $\int \frac{\partial \mathbb{E}^S[\tau]}{\partial x}dF(S)$. By Ito's Lemma applied to x_t^2 and the OST, we have that

$$\mathbb{E}^S[x_\tau^2] - x^2 = 2\rho\mathbb{E}^S\left[\int_0^\tau x_t^2 dt\right] + \sigma^2\mathbb{E}^S[\tau]$$

Solving for $\mathbb{E}^S[\tau]$ and taking derivative with respect to the initial condition yields:

$$\frac{\partial \mathbb{E}^S[\tau]}{\partial x} = \frac{2}{\sigma^2}\left(\frac{\partial \mathbb{E}^S[x_\tau^2/2]}{\partial x} - x - \rho\frac{\partial}{\partial x}\mathbb{E}^S\left[\int_0^\tau x_t^2 dt\right]\right)$$

By the OST, we also have that $\mathbb{E}^S[x_\tau] - x = \rho\mathbb{E}^S\left[\int_0^\tau x_t dt\right]$, which can be substituted back

$$\frac{\partial \mathbb{E}^S[\tau]}{\partial x} = \frac{2}{\sigma^2}\left(\frac{\partial \mathbb{E}^S[x_\tau^2/2]}{\partial x} - \mathbb{E}^S[x_\tau] - \rho\left\{\frac{\partial}{\partial x}\mathbb{E}^S\left[\int_0^\tau x_t^2 dt\right] - \mathbb{E}^S\left[\int_0^\tau x_t dt\right]\right\}\right)$$

Integrating with the ergodic distribution, we get

$$\int \frac{\partial \mathbb{E}^S[\tau]}{\partial x} dF(S) = \Theta_0 - \frac{2\rho}{\sigma^2}\mathcal{C}_2$$

since the last two terms can be manipulated as $\int \frac{\partial}{\partial x}\mathbb{E}^S\left[\int_0^\tau x_t^2 dt\right] - \mathbb{E}^S\left[\int_0^\tau 2x_t dt\right] + \mathbb{E}^S\left[\int_0^\tau x_t dt\right] dF(S) = \mathcal{C}_2 + 0 = \mathcal{C}_2$.

Representation for the intensive margin. Start from the definition of Γ_m and $\varphi_m^\Gamma(S)$ and repeat the steps as in previous proofs to reach:

$$\frac{\sigma^2(m+1)}{2}\mathbb{E}^{\hat{S}}[\tau]\Gamma_m = \mathbb{E}^{\hat{S}}\left[x_\tau^{m+1}\int_0^\tau dt\right] - \mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{m+1} dt\right] \tag{D.69}$$

We now characterize $\mathbb{E}^{\hat{S}}\left[x_\tau^{m+1}\int_0^\tau dt\right]$. Applying Ito's lemma followed by the OST to $Y_t^{m+1} \equiv x_t^{m+1}\int_0^t ds$

$$\begin{aligned} dY_t^{m+1} &= x_t^{m+1}dt + (m+1)\left[\rho x_t^{m+1} + \sigma x_t^m\right]\int_0^t ds dB_t + \frac{m(m+1)\sigma^2}{2}x_t^{m-1}\int_0^t ds dt \\ \mathbb{E}^{\hat{S}}\left[Y_\tau^{m+1}\right] &= \underbrace{\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{m+1}dt\right]}_{\mathcal{M}_{m+1}[x]\mathbb{E}^{\hat{S}}[\tau]} + \frac{m(m+1)\sigma^2}{2}\underbrace{\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{m-1}tdt\right]}_{\mathcal{M}_{m-1,1}[x,a]\mathbb{E}^{\hat{S}}[\tau]} \end{aligned}$$

Substituting back into (D.69) and rearranging:

$$\frac{\sigma^2(m+1)}{2}\mathbb{E}^{\hat{S}}[\tau]\Gamma_m = \mathcal{M}_{m+1}[x]\mathbb{E}^{\hat{S}}[\tau] + \frac{m(m+1)\sigma^2}{2}\mathcal{M}_{m-1,1}[x,a]\mathbb{E}^{\hat{S}}[\tau]$$

implying:

$$\Gamma_m = \frac{2}{\sigma^2(m+1)}\mathcal{M}_{m+1}[x] + m\mathcal{M}_{m-1,1}[x,a] \tag{D.70}$$

Representation for the extensive margin Start from the definition of $g_m(x)$ in (D.1) evaluated at $x = y - \hat{x}$

$$g_m(y-\hat{x}) = \mathbb{E}^{y,S^{-x}}\left[x_\tau^m\right] - \mathbb{E}^{\hat{x},S^{-x}}\left[(x_\tau + y - \hat{x})^m\right] \tag{D.71}$$

and find expressions for the following objects:

$$\begin{aligned} \mathbb{E}^{y,S^{-x}}[x_\tau^{m+1}] &= g_{m+1}(y-\hat{x}) + \mathbb{E}^{\hat{x},S^{-x}}\left[(x_\tau + y - \hat{x})^{m+1}\right] \\ \frac{\partial \mathbb{E}^{y,S^{-x}}[x_\tau^{m+2}/(m+2)]}{\partial x} &= \frac{dg_{m+2}(y-\hat{x})}{dy}/(m+2) + \mathbb{E}^{\hat{x},S^{-x}}\left[(x_\tau + y - \hat{x})^{m+1}\right] \end{aligned}$$

Recover $\varphi_m^\Theta(y, S^{-x})$ by subtracting the two previous expressions and simplifying

$$\rho\varphi_m^\Theta(y, S^{-x}) = \frac{dg_{m+2}(y-\hat{x})}{dy}/(m+2) - g_{m+1}(y-\hat{x})$$

An infinite Taylor approximation around 0 yields:

$$\rho\varphi_m^\Theta(y, S^{-x}) = \sum_{j=0}^\infty \frac{d^j}{j!dx^j}\left[\frac{dg_{m+2}(x)}{dx}/(m+2) - g_{m+1}(x)\right]_{x=0}(y-\hat{x})^j = \sum_{j=0}^\infty \frac{1}{j!}\left[\frac{d^{j+1}g_{m+2}(0)}{dx^{j+1}}/(m+2) - \frac{d^j g_{m+1}(0)}{dx^j}\right](y-\hat{x})^j$$

Opening the binomial term as $(y-\hat{x})^j = \sum_{z=0}^j (-1)^z \frac{j!}{z!(j-z)!}\frac{\hat{x}^z}{j!}y^{j-z}$ and substituting back

$$\varphi_m^\Theta(y, S^{-x}) = \frac{1}{\rho}\sum_{j=0}^\infty\sum_{z=0}^j (-1)^z \frac{\hat{x}^z}{z!(j-z)!}\left[\frac{d^{j+1}g_{m+2}(0)}{dx^{j+1}}/(m+2) - \frac{d^j g_{m+1}(0)}{dx^j}\right]y^{j-z}$$

Finally, we find Θ_m by integrating over all initial conditions y with the occupancy measure (note that the integral only

affects the term y^{j-z}):

$$\begin{aligned}
\Theta_m &= \sum_{j=0}^{\infty}\sum_{z=0}^{j} \underbrace{\frac{1}{\rho}(-1)^z \frac{\hat{x}^z}{z!(j-z)!}\left[\frac{d^{j+1}g_{m+2}(0)}{dx^{j+1}}/(m+2) - \frac{d^{j+1}g_{m+1}(0)}{dx^{j+1}}\right]}_{\mathcal{H}_{m,j,z}} \underbrace{\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{j-z}dt\right]}{\mathbb{E}^{\hat{S}}[\tau]}}_{\mathcal{M}_{j-z}[x]} \\
&= \sum_{j=0}^{\infty}\sum_{z=0}^{j} \mathcal{H}_{m,j,z}\mathcal{M}_{j-z}[x] \\
&= \sum_{h=0}^{\infty} \theta_{m,h}\mathcal{M}_h[x], \qquad where \qquad \theta_{m,h} = \sum_{k\geq h}^{\infty}\mathcal{H}_{m,k,k-h}
\end{aligned}$$

Observation:

- To characterize the three structural parameters $(\hat{x}, \rho, \sigma)$ we need three equations that are independent of the initial conditions. Apply Itō's Lemma to $Y_t = e^{-\rho t}x_t$ and obtain $dY_t = \sigma e^{\rho t}dB_t$. Integrating from 0 to τ, taking expectations with initial condition $\hat{S}$, and using the OST we obtain $\mathbb{E}^{\hat{S}}[e^{-\rho\tau}(\hat{x}-\Delta x)] = \hat{x}$. Thus, we have that

$$\hat{x} = \frac{\mathbb{E}^{\hat{S}}[e^{-\rho\tau}\Delta x]}{\mathbb{E}^{\hat{S}}[e^{-\rho\tau}]-1} \tag{D.72}$$

  Apply Itō's Lemma to $Y_t = e^{-2\rho t}x_t^2$ and obtain $dY_t = 2\sigma x_t e^{-2\rho t}dB_t + \sigma^2 e^{-2\rho t}$. Following the same steps we obtain $\mathbb{E}^{\hat{S}}[e^{-2\rho\tau}x_\tau^2] = \hat{x}^2 + \sigma^2\mathbb{E}^{\hat{S}}\left[\int_0^\tau e^{-2\rho t}dt\right] = \hat{x}^2 + \frac{\sigma^2}{2\rho}\mathbb{E}^{\hat{S}}\left[\left(1-e^{-2\rho\tau}\right)\right]$. Thus

$$\frac{\sigma^2}{\rho} = 2\frac{\hat{x}^2 - \mathbb{E}^{\hat{S}}\left[e^{-2\rho\tau}(\hat{x}-\Delta x)^2\right]}{\mathbb{E}^{\hat{S}}\left[e^{-2\rho\tau}\right]-1} \tag{D.73}$$

  Define the error function $erf(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$. Apply Ito's Lemma to $Y_t = erf\left(x_t\sqrt{\frac{\rho}{\sigma^2}}\right)$ to obtain With the same steps that include the OST, we find the third equation for the system:

$$\mathbb{E}^{\hat{S}}\left[erf\left((\hat{x}-\Delta x)\sqrt{\frac{\rho}{\sigma^2}}\right)\right] = erf\left(\hat{x}\sqrt{\frac{\rho}{\sigma^2}}\right) \tag{D.74}$$

- To characterize the ergodic moment $\mathcal{M}_m[x]$, we apply Itō's Lemma to $Y_t = x_t^m$ and obtain $dY_t = \rho m x_t^m dt + m\sigma x_t^{m-1}dB_t + \frac{\sigma^2 m(m-1)}{2}x_t^{m-2}dt$. Integrating from 0 to τ and taking expectations with initial condition $\hat{S}$,

$$\mathbb{E}[x_\tau^m] - \hat{x}^m = \rho m\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^m dt\right] + \frac{\sigma^2 m(m-1)}{2}\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{m-2}dt\right] + m\sigma\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{m-1}dB_t\right]$$

  In the first two terms we substitute the relationship $\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^m dt\right] = \mathbb{E}[\tau]\mathcal{M}_m[x]$; the last term is equal to zero by the OST. Rearranging:

$$\mathcal{M}_m[x] = \frac{\mathbb{E}[(\hat{x}-\Delta x)^m - \hat{x}^m]}{\rho m\mathbb{E}[\tau]} - \frac{\sigma^2(m-1)}{2\rho}\mathcal{M}_{m-2}[x].$$

- To characterize the covariance with age $\mathcal{M}_{m,1}[x,a]$, we apply Itō's theorem to $Y_t = x_t^m t$ and obtain: $dY_t = x_t^m dt + \rho m t x_t^m dt + \frac{\sigma^2 m(m-1)}{2}t x_t^{m-2}dt + \sigma m t x_t^{m-1}dB_t$. Integrating from 0 to τ and taking expectations with initial condition $\hat{S}$,

$$\mathbb{E}[\tau x_\tau^m] = \mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^m dt\right] + \rho m\mathbb{E}^{\hat{S}}\left[\int_0^\tau t x_t^m dt\right] + \frac{\sigma^2 m(m-1)}{2}\mathbb{E}^{\hat{S}}\left[\int_0^\tau t x_t^{m-2}dt\right] + \sigma m\mathbb{E}^{\hat{S}}\left[\int_0^\tau t x_t^{m-1}dB_t\right]$$

  In the first three terms we substitute the relationships $\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^m dt\right] = \mathbb{E}[\tau]\mathcal{M}_m[x]$ and $\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^k t^l dt\right] = \mathbb{E}[\tau]\mathcal{M}_{k,l}[x,a]$ the last term is equal to zero by the OST:

$$\mathbb{E}[\tau x_\tau^m] = \mathbb{E}[\tau]\mathcal{M}_m[x] + \rho m\mathbb{E}[\tau]\mathcal{M}_{m,1}[x,a] + \frac{\sigma^2 m(m-1)}{2}\mathbb{E}[\tau]\mathcal{M}_{m-2,1}[x,a]$$

  Rearranging and solving for $\mathcal{M}_{m,1}[x,a]$ we obtain a recursive representation for the moment:

$$\mathcal{M}_{m,1}[x,a] = \frac{\mathbb{E}[\tau(\hat{x}-\Delta x)^m]}{\mathbb{E}[\tau]}\frac{\sigma^2(m-1)}{2\rho}\mathcal{M}_{m-2,1}[x,a] - \frac{\mathcal{M}_m[x]}{m\rho}$$

□

E Additional proofs

E.1 Relationship between kurtosis and age in drift-less time-dependent models

Lemma 1. *If the uncontrolled state follows $d\tilde{x}_t = \sigma dB_t$, and τ is independent of x, then the following equalities hold:*

$$\frac{\mathcal{M}_2[x]}{\sigma^2} = \mathcal{M}_1[a] \tag{E.75}$$

$$\mathbb{K}ur\left[\Delta x\right] = 3\left(1+\mathbb{CV}^2[\tau]\right). \tag{E.76}$$

Proof. First, let us show that (E.75). By Auxiliary Theorem 2, we have that $\frac{\mathcal{M}_2[x]}{\sigma^2} = \frac{\mathbb{E}[\int_0^\tau x_t^2 dt]}{\mathbb{E}[\tau]}$ and we have that

$$\frac{\mathcal{M}_2[x]}{\sigma^2} = \frac{\mathbb{E}[\int_0^\tau B_t^2 dt]}{\mathbb{E}[\tau]} = \frac{\int_0^\infty \mathbb{E}\left[B_t^2 I(t\le\tau)\right]dt}{\mathbb{E}[\tau]} = \frac{\int_0^\infty \mathbb{E}\left[B_t^2\right]\mathbb{E}\left[I(t\le\tau)\right]dt}{\mathbb{E}[\tau]} = \mathbb{E}\left[\int_0^\tau tdt\right] = \mathcal{M}_1[a], \tag{E.77}$$

where in the last step we used independence of the stopping time. Let us show (E.76). Start from the equivalence $-\Delta x = \sigma B_\tau$, raise to the 4-th power and take expectations to obtain $\mathbb{E}[\Delta x^4] = \sigma^4\mathbb{E}[B_\tau^4]$. By the independence assumption of the stopping time and the normal distribution of the Brownian motions, $\mathbb{E}[B_\tau^4] = 3\mathbb{E}[\tau^2]$. Now using this result, we express the kurtosis of Δx as follows:

$$\mathbb{K}ur\left[\Delta x\right] = \frac{\mathbb{E}[\Delta x^4]}{\mathbb{E}[\Delta x^2]^2} = \frac{3\sigma^4\mathbb{E}[\tau^2]}{\sigma^4\mathbb{E}[\tau]^2} = \frac{3(\mathbb{V}ar[\tau]+\mathbb{E}[\tau]^2)}{\mathbb{E}[\tau]^2} = 3(1+\mathbb{CV}^2[\tau]) \tag{E.78}$$

□

E.2 Proof of Example 1 with no Drift

This subsection of the appendix proofs the theorem in example 1 in the case of no drift since it is less involve—then, we generalized in the case with drift. We use the following identities shown in subsection in the Online Appendix.

$$\tilde{\lambda} = \frac{\lambda}{\sigma^2} \tag{E.79}$$

$$\xi_1 = -\sqrt{2\tilde{\lambda}} \quad ; \quad \xi_2 = \sqrt{2\tilde{\lambda}} \tag{E.80}$$

$$v_2(\hat{x}) = \frac{-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_2^2(\underline{x}) - \underline{\alpha}_2\kappa_2^2(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_2^2(\overline{x}) - \overline{\alpha}_1\kappa_2^2(\underline{x})\right] + \kappa_2^2(\hat{x})}{\lambda} \tag{E.81}$$

$$v_1(x) = \frac{-e^{\xi_1 x}\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right] - e^{\xi_2 x}\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(x)\right] + \kappa_1^1(x)}{\lambda} \tag{E.82}$$

$$\overline{\alpha}_1 = \frac{e^{-\sqrt{2\tilde{\lambda}}\overline{x}}}{e^{\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})} - e^{-\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})}} \quad ; \quad \overline{\alpha}_2 = \frac{e^{\sqrt{2\tilde{\lambda}}\overline{x}}}{e^{\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})} - e^{-\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})}} \tag{E.83}$$

$$\underline{\alpha}_1 = \frac{e^{-\sqrt{2\tilde{\lambda}}\underline{x}}}{e^{\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})} - e^{-\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})}} \quad ; \quad \underline{\alpha}_2 = \frac{e^{\sqrt{2\tilde{\lambda}}\underline{x}}}{e^{\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})} - e^{-\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})}} \tag{E.84}$$

$$\kappa_1^1(x) = x, \quad ; \quad \kappa_2^2(x) = \frac{1}{\tilde{\lambda}} + x^2 \tag{E.85}$$

$$\mathbb{E}^{\hat{x}}\left[\int_0^\tau e^{\xi_1 x_t}dt\right] = \frac{-e^{\xi_1\hat{x}}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_1\hat{x}}\hat{x}}{-\sigma^2\xi_1} \tag{E.86}$$

$$\mathbb{E}^{\hat{x}}\left[\int_0^\tau e^{\xi_2 x_t}dt\right] = \frac{-e^{\xi_1\hat{x}}(e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_2\hat{x}}\hat{x}}{-\sigma^2\xi_2} \tag{E.87}$$

$$\hat{x} = e^{-\sqrt{2\tilde{\lambda}}\hat{x}}\left[\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}\right] + e^{\sqrt{2\tilde{\lambda}}\hat{x}}\left[\underline{\alpha}_1\overline{x} - \overline{\alpha}_1 x\right] \tag{E.88}$$

Proposition E.5. *Assume $\nu = 0$ and $H(\xi) = 1$ for all $\xi \in [0,\kappa]$ in the model presented in Section 2. Then the CIR is given by*

$$\mathcal{A}_1(\delta) = \delta\frac{\mathcal{M}_2[x]}{\sigma^2} + o(\delta^2) \tag{E.89}$$

Proof. Define the following objects

$$LHS = \left.\frac{d\mathcal{A}_1(\delta)}{d\delta}\right|_{\delta=0} \tag{E.90}$$

$$RHS = \frac{\mathcal{M}_2[x]}{\sigma^2} \tag{E.91}$$

We need to show that $LHS = RHS$. We divide the proof in 4 steps.

Step 1: Let us operate with the LHS and the RHS with the occupancy measure. The LHS and the RHS are given by

$$LHS = \frac{d\mathcal{A}_1(\delta)}{d\delta}\bigg|_{\delta=0} = \int_{\underline{x}}^{\overline{x}} v_1'(x)f(x) = \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t)dt\right]}{\mathbb{E}^{\hat{x}}[\tau]} \tag{E.92}$$

$$RHS = \frac{\mathcal{M}_2[x]}{\sigma^2} = \frac{v_2(\hat{x})}{\sigma^2\mathbb{E}^{\hat{x}}[\tau]}. \tag{E.93}$$

Therefore $LHS = RHS$ iff $\frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t)dt\right]}{\mathbb{E}^{\hat{x}}[\tau]} = \frac{v_2(\hat{x})}{\sigma^2\mathbb{E}^{\hat{x}}[\tau]}$, or equivalently $\lambda\mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t)dt\right]\lambda = \lambda\frac{v_2(\hat{x})}{\sigma^2}$ (i.e. $\lambda\mathbb{E}^{\hat{x}}[\tau]\,LHS = \lambda\mathbb{E}^{\hat{x}}[\tau]\,RHS$).

Step 2: Now, let us simply $\lambda\mathbb{E}^{\hat{x}}[\tau]\,LHS = \lambda\mathbb{E}^{\hat{x}}[\tau]\,RHS$. Defining Now we will show that the left hand side is equal to the right hand side. Define

$$\mathcal{T}_1 = \frac{-e^{\sqrt{2\tilde{\lambda}}\hat{x}}(\overline{\alpha}_2 - \underline{\alpha}_2) - e^{-\sqrt{2\tilde{\lambda}}\hat{x}}(\underline{\alpha}_1 - \overline{\alpha}_1) + 1}{\lambda}, \tag{E.94}$$

then we have that

$$\begin{aligned}
&= \lambda\mathbb{E}^{\hat{x}}[\tau]\,RHS \\
&= \frac{\lambda v_2(\hat{x})}{\sigma^2} \\
&= \frac{\frac{-e^{-\sqrt{2\tilde{\lambda}}\hat{x}}(\overline{\alpha}_2-\underline{\alpha}_2)-e^{\sqrt{2\tilde{\lambda}}\hat{x}}(\underline{\alpha}_1-\overline{\alpha}_1)+1}{\tilde{\lambda}} - e^{-\sqrt{2\tilde{\lambda}}\hat{x}}(\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2) - e^{\sqrt{2\tilde{\lambda}}\hat{x}}(\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2) + \hat{x}^2}{\sigma^2} \\
&= \mathcal{T}_1 + \frac{-e^{-\sqrt{2\tilde{\lambda}}\hat{x}}(\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2) - e^{\sqrt{2\tilde{\lambda}}\hat{x}}(\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2) + \hat{x}^2}{\sigma^2}
\end{aligned} \tag{E.95}$$

$$\begin{aligned}
&= \lambda\mathbb{E}^{\hat{x}}[\tau]\,LHS \\
&= \lambda\mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t)dt\right] \\
&= -\xi_1\mathbb{E}^{\hat{x}}\left[\int_0^\tau e^{\xi_1 x_t}dt\right](\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) - \xi_2\mathbb{E}^{\hat{x}}\left[\int_0^\tau e^{\xi_2 x_t}dt\right](\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x}) + \mathbb{E}^{\hat{x}}[\tau] \\
&= \frac{\left(-e^{-\sqrt{2\tilde{\lambda}}\hat{x}}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\sqrt{2\tilde{\lambda}}\hat{x}}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{-\sqrt{2\tilde{\lambda}}\hat{x}}\hat{x}\right)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x})}{\sigma^2}\cdots \\
&\cdots + \frac{\left(-e^{-\sqrt{2\tilde{\lambda}}\hat{x}}(e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\sqrt{2\tilde{\lambda}}\hat{x}}\left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\sqrt{2\tilde{\lambda}}\hat{x}}\hat{x}\right)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})}{\sigma^2} + \mathcal{T}_1.
\end{aligned} \tag{E.96}$$

Simplifying the σ^2 and $\mathcal{T}_1$ from both sides, we have

$$\begin{aligned}
&= \lambda\mathbb{E}^{\hat{x}}[\tau]\,RHS \\
&= -e^{-\sqrt{2\tilde{\lambda}}\hat{x}}(\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2) - e^{\sqrt{2\tilde{\lambda}}\hat{x}}(\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2) + \hat{x}^2
\end{aligned} \tag{E.97}$$

$$\begin{aligned}
&= \lambda\mathbb{E}^{\hat{x}}[\tau]\,LHS \\
&= \left(-e^{-\sqrt{2\tilde{\lambda}}\hat{x}}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\sqrt{2\tilde{\lambda}}\hat{x}}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{-\sqrt{2\tilde{\lambda}}\hat{x}}\hat{x}\right)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x})\cdots \\
&\cdots + \left(-e^{-\sqrt{2\tilde{\lambda}}\hat{x}}(e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\sqrt{2\tilde{\lambda}}\hat{x}}\left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\sqrt{2\tilde{\lambda}}\hat{x}}\hat{x}\right)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x}).
\end{aligned} \tag{E.98}$$

Therefore $LHS = RHS$ iff equation (E.97) is equal to equation (E.98).

Step 3: Using the definition of $\hat{x} = e^{-\sqrt{2\tilde{\lambda}}\hat{x}}\left[\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}\right] + e^{\sqrt{2\tilde{\lambda}}\hat{x}}\left[\underline{\alpha}_1\overline{x} - \overline{\alpha}_1 x\right]$ we can operate in the LHS and the RHS in equations (E.97) and (E.98).

$$
\begin{aligned}
&= \lambda\mathbb{E}^{\hat{x}}\left[\tau\right] RHS,\\
&= -e^{-\sqrt{2\tilde{\lambda}}\hat{x}}(\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2) - e^{\sqrt{2\tilde{\lambda}}\hat{x}}(\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2) + \hat{x}^2
\end{aligned}
\tag{E.99}
$$

$$
\begin{aligned}
&= \lambda\mathbb{E}^{\hat{x}}\left[\tau\right] LHS,\\
&= -e^{-\sqrt{2\tilde{\lambda}}\hat{x}}\left((e^{-\sqrt{2\tilde{\lambda}}\underline{x}}\underline{x}\overline{\alpha}_2 - e^{-\sqrt{2\tilde{\lambda}}\overline{x}}\overline{x}\underline{\alpha}_2)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + (e^{\sqrt{2\tilde{\lambda}}\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\sqrt{2\tilde{\lambda}}\overline{x}}\overline{x}\underline{\alpha}_2)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right)\dots\\
&\dots - e^{\sqrt{2\tilde{\lambda}}\hat{x}}\left(\left(e^{-\sqrt{2\tilde{\lambda}}\overline{x}}\overline{x}\underline{\alpha}_1 - e^{-\sqrt{2\tilde{\lambda}}\underline{x}}\underline{x}\overline{\alpha}_1\right)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + \left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right)\dots\\
&\dots + \hat{x}\left[e^{-\sqrt{2\tilde{\lambda}}\hat{x}}(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + e^{\sqrt{2\tilde{\lambda}}\hat{x}}(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right]\\
&= -e^{-\sqrt{2\tilde{\lambda}}\hat{x}}\left((e^{-\sqrt{2\tilde{\lambda}}\underline{x}}\underline{x}\overline{\alpha}_2 - e^{-\sqrt{2\tilde{\lambda}}\overline{x}}\overline{x}\underline{\alpha}_2)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + (e^{\sqrt{2\tilde{\lambda}}\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\sqrt{2\tilde{\lambda}}\overline{x}}\overline{x}\underline{\alpha}_2)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right)\dots\\
&\dots - e^{\sqrt{2\tilde{\lambda}}\hat{x}}\left(\left(e^{-\sqrt{2\tilde{\lambda}}\overline{x}}\overline{x}\underline{\alpha}_1 - e^{-\sqrt{2\tilde{\lambda}}\underline{x}}\underline{x}\overline{\alpha}_1\right)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + \left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right) + \hat{x}^2,
\end{aligned}
\tag{E.100}
$$

canceling $\hat{x}^2$ from both sizes and multiplying and dividing the LHS by $\mathcal{T}_2 = e^{\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})} - e^{-\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})}$ we have that

$$
\begin{aligned}
&= \lambda\mathbb{E}^{\hat{x}}\left[\tau\right] RHS,\\
&= -e^{-\sqrt{2\tilde{\lambda}}\hat{x}}(\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2) - e^{\sqrt{2\tilde{\lambda}}\hat{x}}(\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2)
\end{aligned}
\tag{E.101}
$$

$$
\begin{aligned}
&= \lambda\mathbb{E}^{\hat{x}}\left[\tau\right] LHS,\\
&= -e^{-\sqrt{2\tilde{\lambda}}\hat{x}}\mathcal{T}_2\left((\underline{\alpha}_1\underline{x}\overline{\alpha}_2 - \overline{\alpha}_1\overline{x}\underline{\alpha}_2)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + (\underline{\alpha}_2\underline{x}\overline{\alpha}_2 - \overline{\alpha}_2\overline{x}\underline{\alpha}_2)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right)\dots\\
&\dots - e^{\sqrt{2\tilde{\lambda}}\hat{x}}\mathcal{T}_2\left((\overline{\alpha}_1\overline{x}\underline{\alpha}_1 - \underline{\alpha}_1\underline{x}\overline{\alpha}_1)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + (\overline{\alpha}_2\overline{x}\underline{\alpha}_1 - \underline{\alpha}_2\underline{x}\overline{\alpha}_1)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right).
\end{aligned}
\tag{E.102}
$$

Therefore $LHS = RHS$ iff equation (E.101) is equal to equation (E.102).

Step 4: Now we show that (E.101) is equal to equation (E.102). By definition of $\overline{\alpha}_2$, $\underline{\alpha}_1$, $\overline{\alpha}_1$ and $\underline{\alpha}_2$, we have that

$$
\overline{\alpha}_2\underline{\alpha}_1 - \overline{\alpha}_1\underline{\alpha}_2 = \frac{e^{\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})} - e^{-\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})}}{(e^{\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})} - e^{-\sqrt{2\tilde{\lambda}}(\overline{x}-\underline{x})})^2} = \mathcal{T}_2^{-1}.
\tag{E.103}
$$

Using this result we have that

$$
\begin{aligned}
&= (\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2)\\
&= \mathcal{T}_2\left(\underline{x}^2\overline{\alpha}_2\left(\underline{\alpha}_1\overline{\alpha}_2 - \overline{\alpha}_1\underline{\alpha}_2\right) - \overline{x}^2\underline{\alpha}_2\left(\underline{\alpha}_1\overline{\alpha}_2 - \overline{\alpha}_1\underline{\alpha}_2\right)\right)\\
&= \mathcal{T}_2\left(\underline{x}\overline{x}\left(-\underline{\alpha}_1\overline{\alpha}_2\underline{\alpha}_2 - \overline{\alpha}_1\underline{\alpha}_2\overline{\alpha}_2\right) + \underline{\alpha}_1\underline{x}^2\overline{\alpha}_2^2 + \overline{\alpha}_1\overline{x}^2\underline{\alpha}_2^2\right)\dots\\
&\dots + \mathcal{T}_2\left(\underline{x}\overline{x}\left(\underline{\alpha}_1\overline{\alpha}_2\underline{\alpha}_2 + \overline{\alpha}_1\underline{\alpha}_2\overline{\alpha}_2\right) - \overline{\alpha}_1\underline{x}^2\overline{\alpha}_2\underline{\alpha}_2 - \underline{\alpha}_1\overline{x}^2\underline{\alpha}_2\overline{\alpha}_2\right)\\
&= \mathcal{T}_2\left((\underline{\alpha}_1\underline{x}\overline{\alpha}_2 - \overline{\alpha}_1\overline{x}\underline{\alpha}_2)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + (\underline{\alpha}_2\underline{x}\overline{\alpha}_2 - \overline{\alpha}_2\overline{x}\underline{\alpha}_2)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right)
\end{aligned}
\tag{E.104}
$$

and also

$$
\begin{aligned}
&= (\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2)\\
&= \mathcal{T}_2\left(\underline{\alpha}_1\overline{x}^2\left(\underline{\alpha}_1\overline{\alpha}_2 - \overline{\alpha}_1\underline{\alpha}_2\right) - \overline{\alpha}_1\underline{x}^2\left(\underline{\alpha}_1\overline{\alpha}_2 - \overline{\alpha}_1\underline{\alpha}_2\right)\right)\\
&= \mathcal{T}_2\left(\underline{x}\overline{x}\left(\overline{\alpha}_2\overline{\alpha}_1\underline{\alpha}_1 + \underline{\alpha}_2\underline{\alpha}_1\overline{\alpha}_1\right) - \underline{\alpha}_2\underline{x}^2\overline{\alpha}_1\underline{\alpha}_1 - \overline{\alpha}_2\overline{x}^2\underline{\alpha}_1\overline{\alpha}_1\right)\dots\\
&\dots + \mathcal{T}_2\left(\underline{x}\overline{x}\left(-\overline{\alpha}_2\overline{\alpha}_1\underline{\alpha}_1 - \underline{\alpha}_2\underline{\alpha}_1\overline{\alpha}_1\right) + \overline{\alpha}_1^2\underline{x}^2\underline{\alpha}_2 + \underline{\alpha}_1^2\overline{x}^2\overline{\alpha}_2\right)\\
&= \mathcal{T}_2\left((\overline{\alpha}_1\overline{x}\underline{\alpha}_1 - \underline{\alpha}_1\underline{x}\overline{\alpha}_1)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + (\overline{\alpha}_2\overline{x}\underline{\alpha}_1 - \underline{\alpha}_2\underline{x}\overline{\alpha}_1)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right).
\end{aligned}
\tag{E.105}
$$

Combining from equations (E.101) to (E.105) we have that

$$0 = LHS - RHS \iff \tag{E.106}$$

$$0 = -e^{-\sqrt{2\tilde{\lambda}}\hat{x}}\left(\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2 - (\mathcal{T}_2\left((\underline{\alpha}_1\underline{x}\overline{\alpha}_2 - \overline{\alpha}_1\overline{x}\underline{\alpha}_2)(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}) + (\underline{\alpha}_2\underline{x}\overline{\alpha}_2 - \overline{\alpha}_2\overline{x}\underline{\alpha}_2)(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x})\right))\right)\ldots$$
$$- e^{\sqrt{2\tilde{\lambda}}\hat{x}}\left(\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2 - (\mathcal{T}_2\left((\overline{\alpha}_1\overline{x}\underline{\alpha}_1 - \underline{\alpha}_1\underline{x}\overline{\alpha}_1)\left(\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}\right) + (\overline{\alpha}_2\overline{x}\underline{\alpha}_1 - \underline{\alpha}_2\underline{x}\overline{\alpha}_1)\left(\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x}\right)\right))\right) \iff \tag{E.107}$$

$$0 = -e^{-\sqrt{2\tilde{\lambda}}\hat{x}}\left(\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2 - \left(\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2\right)\right)\ldots$$
$$- e^{\sqrt{2\tilde{\lambda}}\hat{x}}\left(\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2 - \left(\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2\right)\right) \iff \tag{E.108}$$

$$0 = 0 \tag{E.109}$$

Thus, we have shown the result. □

E.3 Proof for Example 1

This subsection in the Online appendix proofs the theorem in example 1. We use the following identities shown in section F.1 in the Online Appendix.

$$\tilde{\nu} = -\frac{\psi + \mu}{\sigma^2} \quad , \quad \nu = -(\psi + \mu) \tag{E.110}$$

$$\tilde{\lambda} = \frac{\lambda}{\sigma^2}\ ;\ \tilde{\lambda}(\varphi) = \frac{\lambda - \varphi}{\sigma^2} \tag{E.111}$$

$$\xi_1 = -\tilde{\nu} - \sqrt{\tilde{\nu}^2 + 2\tilde{\lambda}}\ ;\ \xi_1(\varphi) = -\tilde{\nu} - \sqrt{\tilde{\nu}^2 + 2\tilde{\lambda}(\varphi)} \tag{E.112}$$

$$\xi_2 = -\tilde{\nu} + \sqrt{\tilde{\nu}^2 + 2\tilde{\lambda}}\ ;\ \xi_2(\varphi) = -\tilde{\nu} + \sqrt{\tilde{\nu}^2 + 2\tilde{\lambda}(\varphi)} \tag{E.113}$$

$$\overline{\alpha}_1 = \frac{e^{\xi_1\overline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}}\ ;\ \overline{\alpha}_1(\varphi) = \frac{e^{\xi_1(\varphi)\overline{x}}}{e^{\xi_1(\varphi)\underline{x}+\xi_2(\varphi)\overline{x}} - e^{\xi_2(\varphi)\underline{x}+\xi_1(\varphi)\overline{x}}} \tag{E.114}$$

$$\overline{\alpha}_2 = \frac{e^{\xi_2\overline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}}\ ;\ \overline{\alpha}_2(\varphi) = \frac{e^{\xi_2(\varphi)\overline{x}}}{e^{\xi_1(\varphi)\underline{x}+\xi_2(\varphi)\overline{x}} - e^{\xi_2(\varphi)\underline{x}+\xi_1(\varphi)\overline{x}}} \tag{E.115}$$

$$\underline{\alpha}_1 = \frac{e^{\xi_1\underline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}}\ ;\ \underline{\alpha}_1(\varphi) = \frac{e^{\xi_1(\varphi)\underline{x}}}{e^{\xi_1(\varphi)\underline{x}+\xi_2(\varphi)\overline{x}} - e^{\xi_2(\varphi)\underline{x}+\xi_1(\varphi)\overline{x}}} \tag{E.116}$$

$$\underline{\alpha}_2 = \frac{e^{\xi_2\underline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}}\ ;\ \underline{\alpha}_2(\varphi) = \frac{e^{\xi_2(\varphi)\underline{x}}}{e^{\xi_1(\varphi)\underline{x}+\xi_2(\varphi)\overline{x}} - e^{\xi_2(\varphi)\underline{x}+\xi_1(\varphi)\overline{x}}} \tag{E.117}$$

$$\kappa_j^m(x) = \sum_{i=0}^{j}(x)^m\,\frac{m!}{i!}\left[\frac{\xi_1 + \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1 - \xi_2}(\xi_1 x)^{i-m} + \frac{\xi_2 + \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2 - \xi_1}(\xi_2 x)^{i-m}\right] \tag{E.118}$$

$$\kappa_1^1(x) = \frac{\tilde{\nu}}{\tilde{\lambda}} + x \tag{E.119}$$

$$\kappa_1^1(x,\varphi) = \frac{\tilde{\nu}}{\tilde{\lambda}(\varphi)} + x \tag{E.120}$$

$$\kappa_2^2(x) = 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + \frac{1}{2\tilde{\lambda}}\right] + 2\frac{\tilde{\nu}}{\tilde{\lambda}}x + x^2 \tag{E.121}$$

$$v_2(\hat{x}) = \frac{-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_2^2(\underline{x}) - \underline{\alpha}_2\kappa_2^2(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_2^2(\overline{x}) - \overline{\alpha}_1\kappa_2^2(\underline{x})\right] + \kappa_2^2(\hat{x})}{\lambda} \tag{E.122}$$

$$v_1(x) = \frac{-e^{\xi_1 x}\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right] - e^{\xi_2 x}\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(x)\right] + \kappa_1^1(x)}{\lambda} \tag{E.123}$$

$$\hat{x} = -\frac{\tilde{\nu}}{\tilde{\lambda}} + e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right] + e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(\underline{x})\right] \tag{E.124}$$

$$\mathbb{E}^{\hat{x}}\left[\int_0^\tau e^{\xi_1 x_t}dt\right] = -\frac{-e^{\xi_1\hat{x}}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_1\hat{x}}\hat{x}}{\sigma^2(\tilde{\nu} + \xi_1)} \tag{E.125}$$

$$\mathbb{E}^{\hat{x}}\left[\int_0^\tau e^{\xi_2 x_t}dt\right] = -\frac{-e^{\xi_1\hat{x}}(e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_2\hat{x}}\hat{x}}{\sigma^2(\tilde{\nu} + \xi_2)} \tag{E.126}$$

$$h_1(\varphi) = \frac{-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\kappa_1^1(\underline{x},\varphi) - \underline{\alpha}_2(\varphi)\kappa_1^1(\overline{x},\varphi)\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\kappa_1^1(\overline{x},\varphi) - \overline{\alpha}_1(\varphi)\kappa_1^1(\underline{x},\varphi)\right] + \overline{\kappa}_1^1(\hat{x})}{\lambda - \varphi} \tag{E.127}$$

Proposition E.6. *Assume $H(\xi) = 1$ for all $\xi \in [0, \overline{\kappa}]$ in the model presented in Section 2. Then the CIR is given by*

$$\mathcal{A}_1(\delta) = \delta \frac{\mathcal{M}_2[x] - \nu \mathcal{M}_{1,1}[x,a]}{\sigma^2} \tag{E.128}$$

Proof. We proceed with the same steps as in the case with no drift. Define the following object

$$LHS = \left. \frac{d\mathcal{A}_1(\delta)}{d\delta} \right|_{\delta=0} \tag{E.129}$$

$$RHS = \frac{\mathcal{M}_2[x] - \nu \mathcal{M}_{1,1}[x,a]}{\sigma^2}. \tag{E.130}$$

For showing that the $LHS = RHS$, we divide the proof in 5 steps.

Step 1: Let us operate with the LHS and the RHS with the occupancy measure and the moment generating function for age. Using the occupancy measure and the moment generating function for age, we have that

$$\mathcal{M}_{1,1}[x,a] = \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t t dt\right]}{\mathbb{E}^{\hat{x}}[\tau]} = \frac{\left. \frac{\partial \mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t e^{\varphi t} dt\right]}{\partial \varphi} \right|_{\varphi=0}}{\mathbb{E}^{\hat{x}}[\tau]} = \frac{h_1'(0)}{\mathbb{E}^{\hat{x}}[\tau]}. \tag{E.131}$$

The LHS and the RHS are given by

$$\begin{aligned} LHS &= \left. \frac{d\mathcal{A}_1(\delta)}{d\delta} \right|_{\delta=0} \\ &= \int_{\underline{x}}^{\overline{x}} v_1'(x) f(x) \\ &= \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t) dt\right]}{\mathbb{E}^{\hat{x}}[\tau]} \end{aligned} \tag{E.132}$$

$$\begin{aligned} RHS &= \frac{\mathcal{M}_2[x]}{\sigma^2} \\ &= \frac{v_2(\hat{x}) - \nu h_1'(0)}{\sigma^2 \mathbb{E}^{\hat{x}}[\tau]}. \end{aligned} \tag{E.133}$$

Therefore $LHS = RHS$ iff $\frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t) dt\right]}{\mathbb{E}^{\hat{x}}[\tau]} = \frac{v_2(\hat{x}) - \nu h_1'(0)}{\sigma^2 \mathbb{E}^{\hat{x}}[\tau]}$, or equivalently $\sigma^2 \lambda \mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t) dt\right] = \lambda v_2(\hat{x}) - \lambda \nu h_1'(0)$.

Step 2: This steps writes

$$\sigma^2 \lambda \mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t) dt\right] = \mathcal{K}_1 - \tilde{\nu} \mathcal{K}_2, \tag{E.134}$$

where

$$\begin{aligned} \mathcal{K}_1 = &\left[-e^{\xi_1 \hat{x}}(e^{\xi_1 \underline{x}} \underline{x} \overline{\alpha}_2 - e^{\xi_1 \overline{x}} \overline{x} \underline{\alpha}_2) - e^{\xi_2 \hat{x}}\left(e^{\xi_1 \overline{x}} \overline{x} \underline{\alpha}_1 - e^{\xi_1 \underline{x}} \underline{x} \overline{\alpha}_1\right) + e^{\xi_1 \hat{x}} \hat{x}\right]\left[\overline{\alpha}_2 \kappa_1^1(\underline{x}) - \underline{\alpha}_2 \kappa_1^1(\overline{x})\right] \dots \\ &\dots + \left[-e^{\xi_1 \hat{x}}(e^{\xi_2 \underline{x}} \underline{x} \overline{\alpha}_2 - e^{\xi_2 \overline{x}} \overline{x} \underline{\alpha}_2) - e^{\xi_2 \hat{x}}\left(e^{\xi_2 \overline{x}} \overline{x} \underline{\alpha}_1 - e^{\xi_2 \underline{x}} \underline{x} \overline{\alpha}_1\right) + e^{\xi_2 \hat{x}} \hat{x}\right]\left[\underline{\alpha}_1 \kappa_1^1(\overline{x}) - \overline{\alpha}_1 \kappa_1^1(x)\right] + \sigma^2 \mathbb{E}^{\hat{x}}[\tau] \end{aligned} \tag{E.135}$$

$$\begin{aligned} \mathcal{K}_2 = &\frac{-e^{\xi_1 \hat{x}}(e^{\xi_1 \underline{x}} \underline{x} \overline{\alpha}_2 - e^{\xi_1 \overline{x}} \overline{x} \underline{\alpha}_2) - e^{\xi_2 \hat{x}}\left(e^{\xi_1 \overline{x}} \overline{x} \underline{\alpha}_1 - e^{\xi_1 \underline{x}} \underline{x} \overline{\alpha}_1\right) + e^{\xi_1 \hat{x}} \hat{x}}{\tilde{\nu} + \xi_1}\left[\overline{\alpha}_2 \kappa_1^1(\underline{x}) - \underline{\alpha}_2 \kappa_1^1(\overline{x})\right] \dots \\ &\dots + \frac{-e^{\xi_1 \hat{x}}(e^{\xi_2 \underline{x}} \underline{x} \overline{\alpha}_2 - e^{\xi_2 \overline{x}} \overline{x} \underline{\alpha}_2) - e^{\xi_2 \hat{x}}\left(e^{\xi_2 \overline{x}} \overline{x} \underline{\alpha}_1 - e^{\xi_2 \underline{x}} \underline{x} \overline{\alpha}_1\right) + e^{\xi_2 \hat{x}} \hat{x}}{\tilde{\nu} + \xi_2}\left[\underline{\alpha}_1 \kappa_1^1(\overline{x}) - \overline{\alpha}_1 \kappa_1^1(x)\right]. \end{aligned} \tag{E.136}$$

Operating over $\sigma^2\lambda\mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t)dt\right]$ we have that

$$
\begin{aligned}
&= \sigma^2\lambda\mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t)dt\right] \\
&= -\xi_1\sigma^2\mathbb{E}^{\hat{x}}\left[\int_0^\tau e^{\xi_1 x_t}dt\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right] - \xi_2\sigma^2\mathbb{E}^{\hat{x}}\left[\int_0^\tau e^{\xi_2 x_t}dt\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(x)\right] + \sigma^2\mathbb{E}^{\hat{x}}[\tau] \\
&= \frac{\xi_1}{(\tilde{\nu}+\xi_1)}\left[-e^{\xi_1\hat{x}}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_1\hat{x}}\hat{x}\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right]\dots \\
&\dots + \frac{\xi_2}{(\tilde{\nu}+\xi_2)}\left[-e^{\xi_1\hat{x}}(e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_2\hat{x}}\hat{x}\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(x)\right] + \sigma^2\mathbb{E}^{\hat{x}}[\tau] \\
&= [1 + \frac{\xi_1}{(\tilde{\nu}+\xi_1)} - 1]\left[-e^{\xi_1\hat{x}}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_1\hat{x}}\hat{x}\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right]\dots \\
&\dots + [1 + \frac{\xi_2}{(\tilde{\nu}+\xi_2)} - 1]\left[-e^{\xi_1\hat{x}}(e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_2\hat{x}}\hat{x}\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(x)\right] + \sigma^2\mathbb{E}^{\hat{x}}[\tau] \\
&= \left[-e^{\xi_1\hat{x}}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_1\hat{x}}\hat{x}\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right]\dots \\
&\dots + \left[-e^{\xi_1\hat{x}}(e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_2\hat{x}}\hat{x}\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(x)\right] + \sigma^2\mathbb{E}^{\hat{x}}[\tau]\dots \\
&\dots - \frac{\tilde{\nu}}{\tilde{\nu}+\xi_1}\left[-e^{\xi_1\hat{x}}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_1\hat{x}}\hat{x}\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right]\dots \\
&\dots - \frac{\tilde{\nu}}{\tilde{\nu}+\xi_2}\left[-e^{\xi_1\hat{x}}(e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_2\hat{x}}\hat{x}\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(x)\right] \qquad \text{(E.137)} \\
&= \mathcal{K}_1 - \tilde{\nu}\mathcal{K}_2 \qquad \text{(E.138)}
\end{aligned}
$$

where $\mathcal{K}_1$ and $\mathcal{K}_2$ are defined in equations (E.135) and (E.136).

Step 3: This step characterizes $\mathcal{K}_1$ equal to

$$
\mathcal{K}_1 = v_2(\hat{x})\lambda - \left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\mathbb{E}^{\hat{x}}[\tau]\lambda. \qquad \text{(E.139)}
$$

Define $\mathcal{T}_2 = \overline{\alpha}_2\underline{\alpha}_1 - \overline{\alpha}_1\underline{\alpha}_2$. Using the definition of $\kappa_1^1(x)$, $\hat{x}$, and operating over $\mathcal{K}_1$ we have that

$$
\begin{aligned}
\mathcal{K}_1 &= \left[-e^{\xi_1\hat{x}}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_1\hat{x}}\hat{x}\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right]\dots \\
&\quad\dots + \left[-e^{\xi_1\hat{x}}(e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_2\hat{x}}\hat{x}\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(x)\right] + \sigma^2\mathbb{E}^{\hat{x}}[\tau] \\
&\quad = \hat{x}^2 + \hat{x}\frac{\tilde{\nu}}{\tilde{\lambda}} - \frac{\tilde{\nu}}{\tilde{\lambda}}e^{\xi_1\hat{x}}\mathcal{T}_2\left[\underline{\alpha}_1\underline{x}\overline{\alpha}_2 - \overline{\alpha}_1\overline{x}\underline{\alpha}_2\right]\left[\overline{\alpha}_2 - \underline{\alpha}_2\right] + \left(\underline{\alpha}_2\underline{x}\overline{\alpha}_2 - \overline{\alpha}_2\overline{x}\underline{\alpha}_2\right)\left[\underline{\alpha}_1 - \overline{\alpha}_1\right]]\dots \\
&\quad\dots - \frac{\tilde{\nu}}{\tilde{\lambda}}e^{\xi_2\hat{x}}\mathcal{T}_2\left[(\overline{\alpha}_1\overline{x}\underline{\alpha}_1 - \underline{\alpha}_1\underline{x}\overline{\alpha}_1)\left[\overline{\alpha}_2 - \underline{\alpha}_2\right] + (\overline{\alpha}_2\overline{x}\underline{\alpha}_1 - \underline{\alpha}_2\underline{x}\overline{\alpha}_1)\left[\underline{\alpha}_1 - \overline{\alpha}_1\right]\right]\dots \\
&\quad\dots - e^{\xi_1\hat{x}}\mathcal{T}_2\left[\underline{\alpha}_1\underline{x}\overline{\alpha}_2 - \overline{\alpha}_1\overline{x}\underline{\alpha}_2\right]\left[\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}\right] + \left(\underline{\alpha}_2\underline{x}\overline{\alpha}_2 - \overline{\alpha}_2\overline{x}\underline{\alpha}_2\right)\left[\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x}\right]]\dots \\
&\quad\dots - e^{\xi_2\hat{x}}\mathcal{T}_2\left[(\overline{\alpha}_1\overline{x}\underline{\alpha}_1 - \underline{\alpha}_1\underline{x}\overline{\alpha}_1)\left[\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}\right] + (\overline{\alpha}_2\overline{x}\underline{\alpha}_1 - \underline{\alpha}_2\underline{x}\overline{\alpha}_1)\left[\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x}\right]\right] + \sigma^2\mathbb{E}^{\hat{x}}[\tau] \\
&\quad = \hat{x}^2 + \hat{x}\frac{\tilde{\nu}}{\tilde{\lambda}} - \frac{\tilde{\nu}}{\tilde{\lambda}}e^{\xi_1\hat{x}}\mathcal{T}_2\left[\underline{x}\overline{\alpha}_2(\underline{\alpha}_1\overline{\alpha}_2 - \underline{\alpha}_2\overline{\alpha}_1) - \underline{x}\overline{\alpha}_2(\overline{\alpha}_2\overline{\alpha}_1 - \overline{\alpha}_1\underline{\alpha}_2)\right]\dots \\
&\quad\dots - \frac{\tilde{\nu}}{\tilde{\lambda}}e^{\xi_2\hat{x}}\mathcal{T}_2\left[\overline{x}\underline{\alpha}_1(\underline{\alpha}_1\overline{\alpha}_2 - \underline{\alpha}_2\overline{\alpha}_1) - \overline{x}\underline{\alpha}_1(\overline{\alpha}_2\overline{\alpha}_1 - \overline{\alpha}_1\underline{\alpha}_2)\right] - e^{\xi_1\hat{x}}\mathcal{T}_2\left(\underline{x}^2\overline{\alpha}_2\left(\underline{\alpha}_1\overline{\alpha}_2 - \overline{\alpha}_1\underline{\alpha}_2\right) - \overline{x}^2\underline{\alpha}_2\left(\underline{\alpha}_1\overline{\alpha}_2 - \overline{\alpha}_1\underline{\alpha}_2\right)\right)\dots \\
&\quad\dots - e^{\xi_2\hat{x}}\mathcal{T}_2\left(\underline{\alpha}_1\overline{x}^2\left(\underline{\alpha}_1\overline{\alpha}_2 - \overline{\alpha}_1\underline{\alpha}_2\right) - \overline{\alpha}_1\underline{x}^2\left(\underline{\alpha}_1\overline{\alpha}_2 - \overline{\alpha}_1\underline{\alpha}_2\right)\right) + \sigma^2\mathbb{E}^{\hat{x}}[\tau] \\
&\quad = \hat{x}^2 + \hat{x}\frac{\tilde{\nu}}{\tilde{\lambda}} - \frac{\tilde{\nu}}{\tilde{\lambda}}\left[e^{\xi_1\hat{x}}\left[\underline{x}\overline{\alpha}_2 - \overline{x}\underline{\alpha}_2\right] + e^{\xi_2\hat{x}}\left[\overline{x}\underline{\alpha}_1 - \underline{x}\overline{\alpha}_1\right]\right] + \left[-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2\right]\right]\dots \\
&\quad\dots + \sigma^2\frac{-e^{\sqrt{2\tilde{\lambda}}\hat{x}}(\overline{\alpha}_2 - \underline{\alpha}_2) - e^{-\sqrt{2\tilde{\lambda}}\hat{x}}(\underline{\alpha}_1 - \overline{\alpha}_1) + 1}{\lambda}. \qquad \text{(E.140)}
\end{aligned}
$$

Next we operate over the previous equations to show (E.139). We use the that the mean state is zero $\mathcal{M}_1[x] = 0$, thus

$$
v_1(\hat{x}) \equiv -e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_1\kappa_1^1(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(\underline{x})\right] + \kappa_1^1(\hat{x}) = 0. \qquad \text{(E.141)}
$$

Using this result we have that

$$
\begin{aligned}
\mathcal{K}_1 &= \hat{x}^2 + \hat{x}\frac{\tilde{\nu}}{\tilde{\lambda}} - \frac{\tilde{\nu}}{\tilde{\lambda}}\left[e^{\xi_1\hat{x}}\left[\underline{x}\overline{\alpha}_2 - \overline{x}\underline{\alpha}_2\right] + e^{\xi_2\hat{x}}\left[\overline{x}\underline{\alpha}_1 - \underline{x}\overline{\alpha}_1\right]\right] + \left[-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\underline{x}^2 - \underline{\alpha}_2\overline{x}^2\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\overline{x}^2 - \overline{\alpha}_1\underline{x}^2\right]\right] + \dots \\
&\quad + \sigma^2\frac{-e^{\xi_1\hat{x}}(\overline{\alpha}_2 - \underline{\alpha}_2) - e^{\xi_2\hat{x}}(\underline{\alpha}_1 - \overline{\alpha}_1) + 1}{\lambda}\dots \\
&= \hat{x}^2 + \hat{x}\frac{\tilde{\nu}}{\tilde{\lambda}}2 + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + \frac{1}{2\tilde{\lambda}}\right] - 2\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 - \hat{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\dots \\
&\quad \dots - \left[e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\left[\underline{x}^2 + \underline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}2 + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + \frac{1}{2\tilde{\lambda}}\right]\right] - \underline{\alpha}_2\left[\overline{x}^2 + \overline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}2 + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + \frac{1}{2\tilde{\lambda}}\right]\right]\right]\right]\dots \\
&\quad \dots - \left[e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\left[\overline{x}^2 + \overline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}2 + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + \frac{1}{2\tilde{\lambda}}\right]\right] - \overline{\alpha}_1\left[\underline{x}^2 + \underline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}2 + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + \frac{1}{2\tilde{\lambda}}\right]\right]\right]\right]\dots \\
&\quad \dots + \left[e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\left[\underline{x}\frac{\tilde{\nu}}{\tilde{\lambda}} + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\right]\right] - \underline{\alpha}_2\left[\overline{x}\frac{\tilde{\nu}}{\tilde{\lambda}} + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\right]\right]\right]\right]\dots \\
&\quad \dots + \left[e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\left[\overline{x}\frac{\tilde{\nu}}{\tilde{\lambda}} + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\right]\right] - \overline{\alpha}_1\left[\underline{x}\frac{\tilde{\nu}}{\tilde{\lambda}} + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\right]\right]\right]\right] \\
&= \kappa_2^2(\hat{x}) - e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_2^2(\underline{x}) - \underline{\alpha}_2\kappa_2^2(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_2^2(\overline{x}) - \overline{\alpha}_1\kappa_2^2(\underline{x})\right] - 2\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 - \hat{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\dots \\
&\quad \dots + \left[e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\left[\underline{x}\frac{\tilde{\nu}}{\tilde{\lambda}} + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\right]\right] - \underline{\alpha}_2\left[\overline{x}\frac{\tilde{\nu}}{\tilde{\lambda}} + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\right]\right]\right]\right]\dots \\
&\quad \dots + \left[e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\left[\overline{x}\frac{\tilde{\nu}}{\tilde{\lambda}} + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\right]\right] - \overline{\alpha}_1\left[\underline{x}\frac{\tilde{\nu}}{\tilde{\lambda}} + 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\right]\right]\right]\right] \\
&= \lambda v_2(\hat{x}) - 2\frac{\tilde{\nu}}{\tilde{\lambda}}\left(-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_1\kappa_1^1(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(\underline{x})\right] + \kappa_1^1(\hat{x})\right) + \hat{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\dots \\
&\quad \dots + \left[e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\left[-\underline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\right] - \underline{\alpha}_2\left[-\overline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\right]\right]\right]\dots \\
&\quad \dots + \left[e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\left[-\overline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\right] - \overline{\alpha}_1\left[-\underline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\right]\right]\right] \\
&= \lambda v_2(\hat{x}) - 2\frac{\tilde{\nu}}{\tilde{\lambda}}v_1(\hat{x}) + \hat{x}\frac{\tilde{\nu}}{\tilde{\lambda}} + \left[e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\left[-\underline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\right] - \underline{\alpha}_2\left[-\overline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\right]\right]\right] + \left[e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\left[-\overline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\right] - \overline{\alpha}_1\left[-\underline{x}\frac{\tilde{\nu}}{\tilde{\lambda}}\right]\right]\right]\dots \\
&= \lambda v_2(\hat{x}) + \frac{\tilde{\nu}}{\tilde{\lambda}}\left(-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_1\kappa_1^1(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(\underline{x})\right] + \kappa_1^1(\hat{x})\right)\dots \\
&\quad \dots + \frac{\tilde{\nu}}{\tilde{\lambda}}\left(-\frac{\tilde{\nu}}{\tilde{\lambda}}(\kappa_0^0(\hat{x}) - e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_0^0(\underline{x}) - \underline{\alpha}_1\kappa_0^0(\overline{x})\right]) - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_0^0(\overline{x}) - \overline{\alpha}_1\kappa_0^0(\underline{x})\right]\right)\dots \\
&= \lambda v_2(\hat{x}) - \left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2\mathbb{E}^{\hat{x}}[\tau]\lambda.
\end{aligned}
\tag{E.142}
$$

Therefore, we have the result (E.139).

Step 4: This step writes $\mathcal{K}_2$ equal to

$$
\mathcal{K}_2 = \lambda\sigma^2 h_1'(0) - \sigma^2\frac{\nu}{\lambda}\mathbb{E}^{\hat{x}}[\tau] + \mathcal{C} \tag{E.143}
$$

$$
\begin{aligned}
\mathcal{C} &= \sigma^2 e^{\xi_1\hat{x}}\left[\frac{d\overline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi) - \frac{d\underline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi)\right]_{\varphi=0} + \sigma^2 e^{\xi_2\hat{x}}\left[\frac{d\underline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi) - \frac{d\overline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi)\right]_{\varphi=0}\dots \\
&\quad + \frac{\mathcal{T}_2}{\tilde{\nu} - \xi_1}\left[-e^{\xi_1\hat{x}}(\underline{\alpha}_1\underline{x}\overline{\alpha}_2 - \overline{\alpha}_1\underline{\alpha}_2\overline{x}) - e^{\xi_2\hat{x}}\left(\overline{\alpha}_1\overline{x}\underline{\alpha}_1 - \underline{\alpha}_1\underline{x}\overline{\alpha}_1\right)\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right]\dots \\
&\quad + \frac{\mathcal{T}_2}{\tilde{\nu} - \xi_2}\left[-e^{\xi_1\hat{x}}(\underline{\alpha}_2\underline{x}\overline{\alpha}_2 - \overline{\alpha}_2\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(\overline{\alpha}_2\overline{x}\underline{\alpha}_1 - \underline{\alpha}_2\underline{x}\overline{\alpha}_1\right)\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(x)\right],
\end{aligned}
\tag{E.144}
$$

and in the next step we show that (E.144) is equal to zero.

First, we characterize $h_1'(0)$. Given

$$h_1(\varphi) = \frac{-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\kappa_1^1(\underline{x},\varphi) - \underline{\alpha}_2(\varphi)\kappa_1^1(\overline{x},\varphi)\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\kappa_1^1(\overline{x},\varphi) - \overline{\alpha}_1(\varphi)\kappa_1^1(\underline{x},\varphi)\right] + \overline{\kappa}_1^1(\hat{x},\varphi)}{\lambda - \varphi}, \tag{E.145}$$

and that

$$\frac{de^{\xi_1(\varphi)\hat{x}}}{d\varphi} = -\frac{e^{\xi_1(\varphi)\hat{x}}\hat{x}}{\sigma^2(\tilde{\nu} + \xi_1(\varphi))} \tag{E.146}$$

$$\frac{de^{\xi_2(\varphi)\hat{x}}}{d\varphi} = -\frac{e^{\xi_2(\varphi)\hat{x}}\hat{x}}{\sigma^2(\tilde{\nu} + \xi_2(\varphi))} \tag{E.147}$$

$$\frac{d\kappa_1^1(x,\varphi)}{d\varphi} = \frac{\nu}{(\lambda - \varphi)^2} \tag{E.148}$$

we have that

$$\begin{aligned}
h_1'(\varphi) = {} & \frac{-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\kappa_1^1(\underline{x},\varphi) - \underline{\alpha}_2(\varphi)\kappa_1^1(\overline{x},\varphi)\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\kappa_1^1(\overline{x},\varphi) - \overline{\alpha}_1(\varphi)\kappa_1^1(\underline{x},\varphi)\right] + \overline{\kappa}_1^1(\hat{x})}{(\lambda - \varphi)^2} \cdots \\
\cdots - {} & \frac{-e^{\xi_1(\varphi)\hat{x}}\hat{x}\frac{1}{\sigma^2(\tilde{\nu}+\xi_1(\varphi))}\left[\overline{\alpha}_2(\varphi)\kappa_1^1(\underline{x},\varphi) - \underline{\alpha}_2(\varphi)\kappa_1^1(\overline{x},\varphi)\right] - e^{\xi_2(\varphi)\hat{x}}\hat{x}\frac{1}{\sigma^2(\tilde{\nu}+\xi_2(\varphi))}\left[\underline{\alpha}_1(\varphi)\kappa_1^1(\overline{x},\varphi) - \overline{\alpha}_1(\varphi)\kappa_1^1(\underline{x},\varphi)\right]}{(\lambda - \varphi)} \cdots \\
\cdots + {} & \frac{-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\frac{d\kappa_1^1(\underline{x},\varphi)}{d\varphi} - \underline{\alpha}_2(\varphi)\frac{d\kappa_1^1(\overline{x},\varphi)}{d\varphi}\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\frac{d\kappa_1^1(\overline{x},\varphi)}{d\varphi} - \overline{\alpha}_1(\varphi)\frac{d\kappa_1^1(\underline{x},\varphi)}{d\varphi}\right] + \frac{d\overline{\kappa}_1^1(\hat{x})}{d\varphi}}{\lambda - \varphi} \cdots \\
\cdots + {} & \frac{-e^{\xi_1(\varphi)\hat{x}}\left[\frac{d\overline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi) - \frac{d\underline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi)\right] - e^{\xi_2\hat{x}}\left[\frac{d\underline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi) - \frac{d\overline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi)\right]}{\lambda - \varphi} \\
= {} & \frac{-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\kappa_1^1(\underline{x},\varphi) - \underline{\alpha}_2(\varphi)\kappa_1^1(\overline{x},\varphi)\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\kappa_1^1(\overline{x},\varphi) - \overline{\alpha}_1(\varphi)\kappa_1^1(\underline{x},\varphi)\right] + \overline{\kappa}_1^1(\hat{x})}{(\lambda - \varphi)^2} \cdots \\
\cdots - {} & \frac{-e^{\xi_1(\varphi)\hat{x}}\hat{x}\frac{1}{\sigma^2(\tilde{\nu}+\xi_1(\varphi))}\left[\overline{\alpha}_2(\varphi)\kappa_1^1(\underline{x},\varphi) - \underline{\alpha}_2(\varphi)\kappa_1^1(\overline{x},\varphi)\right] - e^{\xi_2\hat{x}}\hat{x}\frac{1}{\sigma^2(\tilde{\nu}+\xi_2(\varphi))}\left[\underline{\alpha}_1(\varphi)\kappa_1^1(\overline{x},\varphi) - \overline{\alpha}_1(\varphi)\kappa_1^1(\underline{x},\varphi)\right]}{(\lambda - \varphi)} \cdots \\
\cdots + {} & \frac{\nu}{(\lambda - \varphi)^2}\frac{-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi) - \underline{\alpha}_2(\varphi)\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi) - \overline{\alpha}_1(\varphi)\right] + 1}{\lambda - \varphi} \cdots \\
\cdots - {} & \frac{e^{\xi_1(\varphi)\hat{x}}\left[\frac{d\overline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi) - \frac{d\underline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi)\right]}{\lambda - \varphi} - \frac{e^{\xi_2\hat{x}}\left[\frac{d\underline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi) - \frac{d\overline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi)\right]}{\lambda - \varphi}.
\end{aligned} \tag{E.149}$$

Evaluating equations (E.149) at zero

$$\begin{aligned}
& = h_1'(0) \\
& = \frac{v_1(\hat{x})}{\lambda} - \hat{x}\left(\frac{\frac{-e^{\xi_1\hat{x}}}{\sigma^2(\tilde{\nu}+\xi_1)}\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right] - \frac{e^{\xi_2\hat{x}}\hat{x}}{\sigma^2(\tilde{\nu}+\xi_2)}\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(\underline{x})\right]}{\lambda}\right) \cdots \\
& \cdots - \frac{\nu}{\lambda^2}\frac{-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2 - \underline{\alpha}_2\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1 - \overline{\alpha}_1\right] + 1}{\lambda} \cdots \\
& \cdots + \left.\frac{e^{\xi_1(\varphi)\hat{x}}\left[\frac{d\overline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi) - \frac{d\underline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi)\right]}{\lambda - \varphi}\right|_{\varphi=0} - \left.\frac{e^{\xi_2\hat{x}}\left[\frac{d\underline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi) - \frac{d\overline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi)\right]}{\lambda - \varphi}\right|_{\varphi=0} \\
& = -\hat{x}\left(\frac{\frac{-e^{\xi_1\hat{x}}}{\sigma^2(\tilde{\nu}+\xi_1)}\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right] - \frac{e^{\xi_2\hat{x}}\hat{x}}{\sigma^2(\tilde{\nu}+\xi_2)}\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(\underline{x})\right]}{\lambda}\right) + \frac{\nu}{(\lambda)^2}\mathbb{E}^{\hat{x}}[\tau] \cdots \\
& \cdots - \left.\frac{e^{\xi_1(\varphi)\hat{x}}\left[\frac{d\overline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi) - \frac{d\underline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi)\right]}{\lambda - \varphi}\right|_{\varphi=0} - \left.\frac{e^{\xi_2\hat{x}}\left[\frac{d\underline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi) - \frac{d\overline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi)\right]}{\lambda - \varphi}\right|_{\varphi=0}.
\end{aligned} \tag{E.150}$$

Therefore we have

$$
\begin{aligned}
&= h_1'(0)\lambda\sigma^2 \\
&= -\hat{x}\left[-\frac{e^{\xi_1\hat{x}}}{(\tilde{\nu}+\xi_1)}\left[\overline{\alpha}_2\kappa_1^1(\underline{x})-\underline{\alpha}_2\kappa_1^1(\overline{x})\right]-\frac{e^{\xi_2\hat{x}}\hat{x}}{(\tilde{\nu}+\xi_2)}\left[\underline{\alpha}_1\kappa_1^1(\overline{x})-\overline{\alpha}_1\kappa_1^1(\underline{x})\right]\right]+\frac{\nu}{\lambda}\sigma^2\mathbb{E}^{\hat{x}}[\tau]\ldots \\
&-\sigma^2\ e^{\xi_1(\varphi)\hat{x}}\left[\frac{d\overline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi)-\frac{d\underline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi)\right]\Bigg|_{\varphi=0}-\sigma^2\ e^{\xi_2\hat{x}}\left[\frac{d\underline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi)-\frac{d\overline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi)\right]\Bigg|_{\varphi=0}\ldots \\
&\cdots-\frac{\mathcal{T}_2}{\tilde{\nu}+\xi_1}\left[-e^{\xi_1\hat{x}}(\underline{\alpha}_1\underline{x}\overline{\alpha}_2-\overline{\alpha}_1\underline{\alpha}_2\overline{x})-e^{\xi_2\hat{x}}\left(\overline{\alpha}_1\overline{x}\underline{\alpha}_1-\underline{\alpha}_1\underline{x}\overline{\alpha}_1\right)\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x})-\underline{\alpha}_2\kappa_1^1(\overline{x})\right]\ldots \\
&\cdots-\frac{\mathcal{T}_2}{\tilde{\nu}+\xi_2}\left[-e^{\xi_1\hat{x}}(\underline{\alpha}_2\underline{x}\overline{\alpha}_2-\overline{\alpha}_2\overline{x}\underline{\alpha}_2)-e^{\xi_2\hat{x}}\left(\overline{\alpha}_2\overline{x}\underline{\alpha}_1-\underline{\alpha}_2\underline{x}\overline{\alpha}_1\right)\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x})-\overline{\alpha}_1\kappa_1^1(x)\right]\ldots \\
&\cdots+\frac{\mathcal{T}_2}{\tilde{\nu}+\xi_1}\left[-e^{\xi_1\hat{x}}(\underline{\alpha}_1\underline{x}\overline{\alpha}_2-\overline{\alpha}_1\underline{\alpha}_2\overline{x})-e^{\xi_2\hat{x}}\left(\overline{\alpha}_1\overline{x}\underline{\alpha}_1-\underline{\alpha}_1\underline{x}\overline{\alpha}_1\right)\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x})-\underline{\alpha}_2\kappa_1^1(\overline{x})\right]\ldots \\
&\cdots+\frac{\mathcal{T}_2}{\tilde{\nu}\xi_2}\left[-e^{\xi_1\hat{x}}(\underline{\alpha}_2\underline{x}\overline{\alpha}_2-\overline{\alpha}_2\overline{x}\underline{\alpha}_2)-e^{\xi_2\hat{x}}\left(\overline{\alpha}_2\overline{x}\underline{\alpha}_1-\underline{\alpha}_2\underline{x}\overline{\alpha}_1\right)\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x})-\overline{\alpha}_1\kappa_1^1(x)\right] \\
&= -\hat{x}\left[-\frac{e^{\xi_1\hat{x}}}{\tilde{\nu}+\xi_1}\left[\overline{\alpha}_2\kappa_1^1(\underline{x})-\underline{\alpha}_2\kappa_1^1(\overline{x})\right]-\frac{e^{\xi_2\hat{x}}\hat{x}}{\tilde{\nu}+\xi_2}\left[\underline{\alpha}_1\kappa_1^1(\overline{x})-\overline{\alpha}_1\kappa_1^1(\underline{x})\right]\right]+\frac{\nu}{\lambda}\sigma^2\mathbb{E}^{\hat{x}}[\tau]-\mathcal{C}\ldots \\
&\cdots+\frac{\mathcal{T}_2}{\tilde{\nu}+\xi_1}\left[-e^{\xi_1\hat{x}}(\underline{\alpha}_1\underline{x}\overline{\alpha}_2-\overline{\alpha}_1\underline{\alpha}_2\overline{x})-e^{\xi_2\hat{x}}\left(\overline{\alpha}_1\overline{x}\underline{\alpha}_1-\underline{\alpha}_1\underline{x}\overline{\alpha}_1\right)\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x})-\underline{\alpha}_2\kappa_1^1(\overline{x})\right]\ldots \\
&\cdots+\frac{\mathcal{T}_2}{\tilde{\nu}+\xi_2}\left[-e^{\xi_1\hat{x}}(\underline{\alpha}_2\underline{x}\overline{\alpha}_2-\overline{\alpha}_2\overline{x}\underline{\alpha}_2)-e^{\xi_2\hat{x}}\left(\overline{\alpha}_2\overline{x}\underline{\alpha}_1-\underline{\alpha}_2\underline{x}\overline{\alpha}_1\right)\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x})-\overline{\alpha}_1\kappa_1^1(x)\right]
\end{aligned}
\tag{E.151}
$$

Using the definition of $\mathcal{K}_2$ in equation (E.136) and equation (E.151), we have equation (E.143). Using equations (E.134), (E.143) and (E.143) , we have that

$$
\begin{aligned}
&= \sigma^2\lambda\mathbb{E}^{\hat{x}}\left[\int_0^\tau v_1'(x_t)dt\right] && \text{(E.152)}\\
&= \mathcal{K}_1-\tilde{\nu}\mathcal{K}_2 && \text{(E.153)}\\
&= (v_2(\hat{x})\lambda-\left(\frac{\tilde{\nu}}{\overline{\lambda}}\right)^2\mathbb{E}^{\hat{x}}[\tau]\lambda)-\tilde{\nu}(\lambda\sigma^2h_1'(0)-\sigma^2\frac{\nu}{\lambda}\mathbb{E}^{\hat{x}}[\tau]+\mathcal{C}) && \text{(E.154)}\\
&= v_2(\hat{x})\lambda-\nu h_1'(0)-\tilde{\nu}\mathcal{C} && \text{(E.155)}
\end{aligned}
$$

Step 5: This step shows that $\mathcal{C}=0$, where $\mathcal{C}$ is defined as

$$
\begin{aligned}
\mathcal{C} = {}&\sigma^2e^{\xi_1\hat{x}}\left[\frac{d\overline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi)-\frac{d\underline{\alpha}_2(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi)\right]_{\varphi=0}+\sigma^2e^{\xi_2\hat{x}}\left[\frac{d\underline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\overline{x},\varphi)-\frac{d\overline{\alpha}_1(\varphi)}{d\varphi}\kappa_1^1(\underline{x},\varphi)\right]_{\varphi=0}\ldots \\
&\cdots-\frac{\mathcal{T}_2}{\tilde{\nu}+\xi_1}\left[-e^{\xi_1\hat{x}}(\underline{\alpha}_1\underline{x}\overline{\alpha}_2-\overline{\alpha}_1\underline{\alpha}_2\overline{x})-e^{\xi_2\hat{x}}\left(\overline{\alpha}_1\overline{x}\underline{\alpha}_1-\underline{\alpha}_1\underline{x}\overline{\alpha}_1\right)\right]\left[\overline{\alpha}_2\kappa_1^1(\underline{x})-\underline{\alpha}_2\kappa_1^1(\overline{x})\right]\ldots \\
&\cdots-\frac{\mathcal{T}_2}{\tilde{\nu}+\xi_2}\left[-e^{\xi_1\hat{x}}(\underline{\alpha}_2\underline{x}\overline{\alpha}_2-\overline{\alpha}_2\overline{x}\underline{\alpha}_2)-e^{\xi_2\hat{x}}\left(\overline{\alpha}_2\overline{x}\underline{\alpha}_1-\underline{\alpha}_2\underline{x}\overline{\alpha}_1\right)\right]\left[\underline{\alpha}_1\kappa_1^1(\overline{x})-\overline{\alpha}_1\kappa_1^1(x)\right].
\end{aligned}
\tag{E.156}
$$

First, let us the derivative of the $\underline{\alpha}_i$ and $\overline{\alpha}_i$

$$
\sigma^2\frac{d\overline{\alpha}_2(\varphi)}{d\varphi}=\overline{\alpha}_2(\varphi)\mathcal{T}_2\left[\frac{\underline{\alpha}_1(\varphi)\overline{\alpha}_2(\varphi)\underline{x}-\overline{\alpha}_1(\varphi)\underline{\alpha}_2(\varphi)\overline{x}}{\tilde{\nu}+\xi_1}-\frac{\underline{\alpha}_1(\varphi)\overline{\alpha}_2(\varphi)\overline{x}-\overline{\alpha}_1(\varphi)\underline{\alpha}_2(\varphi)\underline{x}}{\tilde{\nu}+\xi_2}\right]-\frac{\overline{\alpha}_2(\varphi)\overline{x}}{\tilde{\nu}+\xi_2} \tag{E.157}
$$

$$
\sigma^2\frac{d\overline{\alpha}_1(\varphi)}{d\varphi}=\overline{\alpha}_1(\varphi)\mathcal{T}_2\left[\frac{\underline{\alpha}_1(\varphi)\overline{\alpha}_2(\varphi)\underline{x}-\overline{\alpha}_1(\varphi)\underline{\alpha}_2(\varphi)\overline{x}}{\tilde{\nu}+\xi_1}-\frac{\underline{\alpha}_1(\varphi)\overline{\alpha}_2(\varphi)\overline{x}-\overline{\alpha}_1(\varphi)\underline{\alpha}_2(\varphi)\underline{x}}{\tilde{\nu}+\xi_2}\right]-\frac{\overline{\alpha}_1(\varphi)\overline{x}}{\tilde{\nu}+\xi_1} \tag{E.158}
$$

$$
\sigma^2\frac{d\underline{\alpha}_2(\varphi)}{d\varphi}=\underline{\alpha}_2(\varphi)\mathcal{T}_2\left[\frac{\underline{\alpha}_1(\varphi)\overline{\alpha}_2(\varphi)\underline{x}-\overline{\alpha}_1(\varphi)\underline{\alpha}_2(\varphi)\overline{x}}{\tilde{\nu}+\xi_1}-\frac{\underline{\alpha}_1(\varphi)\overline{\alpha}_2(\varphi)\overline{x}-\overline{\alpha}_1(\varphi)\underline{\alpha}_2(\varphi)\underline{x}}{\tilde{\nu}+\xi_2}\right]-\frac{\underline{\alpha}_2(\varphi)\underline{x}}{\tilde{\nu}+\xi_2} \tag{E.159}
$$

$$
\sigma^2\frac{d\underline{\alpha}_1(\varphi)}{d\varphi}=\underline{\alpha}_1(\varphi)\mathcal{T}_2\left[\frac{\underline{\alpha}_1(\varphi)\overline{\alpha}_2(\varphi)\underline{x}-\overline{\alpha}_1(\varphi)\underline{\alpha}_2(\varphi)\overline{x}}{\tilde{\nu}+\xi_1}-\frac{\underline{\alpha}_1(\varphi)\overline{\alpha}_2(\varphi)\overline{x}-\overline{\alpha}_1(\varphi)\underline{\alpha}_2(\varphi)\underline{x}}{\tilde{\nu}+\xi_2}\right]-\frac{\underline{\alpha}_1(\varphi)\underline{x}}{\tilde{\nu}+\xi_1}. \tag{E.160}
$$

Using the previous derivatives in the first part of equation (E.156) we have that

$$
\begin{aligned}
&= \sigma^2 e^{\xi_1 \hat{x}} \left[\frac{d\overline{\alpha}_2(\varphi)}{d\varphi} \kappa_1^1(\underline{x}, \varphi) - \frac{d\underline{\alpha}_2(\varphi)}{d\varphi} \kappa_1^1(\overline{x}, \varphi) \right]_{\varphi=0} + \sigma^2 e^{\xi_2 \hat{x}} \left[\frac{d\underline{\alpha}_1(\varphi)}{d\varphi} \kappa_1^1(\overline{x}, \varphi) - \frac{d\overline{\alpha}_1(\varphi)}{d\varphi} \kappa_1^1(\underline{x}, \varphi) \right]_{\varphi=0} \\
&= \frac{\mathcal{T}_2}{\tilde{\nu} + \xi_1} e^{\xi_1 \hat{x}} \left[(\underline{\alpha}_1 \overline{\alpha}_2 \underline{x} - \overline{\alpha}_1 \underline{\alpha}_2 \overline{x}) \overline{\alpha}_2 \kappa_1^1(\underline{x}) - (\underline{\alpha}_1 \overline{\alpha}_2 \underline{x} - \overline{\alpha}_1 \underline{\alpha}_2 \overline{x}) \underline{\alpha}_2 \kappa_1^1(\overline{x}) \right] \dots \\
&\dots + \frac{\mathcal{T}_2}{\tilde{\nu} + \xi_1} e^{\xi_2 \hat{x}} \left[(\underline{\alpha}_1 \overline{\alpha}_2 \underline{\alpha}_1 \underline{x} - \overline{\alpha}_1 \underline{\alpha}_2 \overline{x} \underline{\alpha}_1 - \underline{\alpha}_1 \underline{x}/\mathcal{T}_2) \kappa_1^1(\overline{x}) - (\underline{\alpha}_1 \overline{\alpha}_2 \underline{x} \overline{\alpha}_1 - \overline{\alpha}_1 \underline{\alpha}_2 \overline{x} \overline{\alpha}_1 - \overline{\alpha}_1 \overline{x}/\mathcal{T}_2) \kappa_1^1(\underline{x}) \right] \dots \\
&\dots + \frac{\mathcal{T}_2}{\tilde{\nu} + \xi_2} e^{\xi_1 \hat{x}} \left[(\underline{\alpha}_1 \overline{\alpha}_2 \overline{x} - \overline{\alpha}_1 \underline{\alpha}_2 \underline{x} + \overline{\alpha}_2 \overline{x}/\mathcal{T}_2) \kappa_1^1(\underline{x}) - (\underline{\alpha}_1 \overline{\alpha}_2 \overline{x} - \overline{\alpha}_1 \underline{\alpha}_2 \underline{x} + \underline{\alpha}_2 \underline{x}/\mathcal{T}_2) \kappa_1^1(\overline{x}) \right] \dots \\
&\dots + \frac{\mathcal{T}_2}{\tilde{\nu} + \xi_2} \left[e^{\xi_2 \hat{x}} \left[(\overline{\alpha}_2 \overline{x} \underline{\alpha}_1 - \underline{\alpha}_2 \underline{x} \overline{\alpha}_1) \underline{\alpha}_1 \kappa_1^1(\overline{x}) - (\overline{\alpha}_2 \overline{x} \underline{\alpha}_1 - \underline{\alpha}_2 \underline{x} \overline{\alpha}_1) \overline{\alpha}_1 \kappa_1^1(\underline{x}) \right] \right] \dots \\
&= \frac{\mathcal{T}_2}{\tilde{\nu} + \xi_1} e^{\xi_1 \hat{x}} (\underline{\alpha}_1 \overline{\alpha}_2 \underline{x} - \overline{\alpha}_1 \underline{\alpha}_2 \overline{x}) \left[\overline{\alpha}_2 \kappa_1^1(\underline{x}) - \underline{\alpha}_2 \kappa_1^1(\overline{x}) \right] - \frac{\mathcal{T}_2}{\tilde{\nu} + \xi_1} e^{\xi_2 \hat{x}} (\overline{\alpha}_1 \overline{x} \underline{\alpha}_1 - \underline{\alpha}_1 \underline{x} \overline{\alpha}_1) \left[\overline{\alpha}_2 \kappa_1^1(\underline{x}) - \underline{\alpha}_2 \kappa_1^1(\overline{x}) \right] \dots \\
&\dots + \frac{\mathcal{T}_2}{\tilde{\nu} + \xi_2} e^{\xi_1 \hat{x}} (\underline{\alpha}_2 \underline{x} \overline{\alpha}_2 - \overline{\alpha}_2 \overline{x} \underline{\alpha}_2) \left[\underline{\alpha}_1 \kappa_1^1(\overline{x}) - \overline{\alpha}_1 \kappa_1^1(\underline{x}) \right] - \frac{\mathcal{T}_2}{\tilde{\nu} + \xi_2} e^{\xi_2 \hat{x}} (\overline{\alpha}_2 \overline{x} \underline{\alpha}_1 - \underline{\alpha}_2 \underline{x} \overline{\alpha}_1) \left[\underline{\alpha}_1 \kappa_1^1(\overline{x}) - \overline{\alpha}_1 \kappa_1^1(\underline{x}) \right]. \qquad \text{(E.161)}
\end{aligned}
$$

Using the previous equation we have the equality. □

E.4 Proof of example 2

Proposition E.7. *Assume a random Ss model with $\lambda = 0$ and $\sigma^2 = 0$, then*

$$\Theta_1 = \theta_{1,0} \mathcal{M}_0[x] = -T/2. \qquad \text{(E.162)}$$

Proof. Assume the Ss model with lower Ss band $\bar{x}$ and upper Ss band $\hat{x}$. Define the following objects:

$$g_m(x) = \mathbb{E}^{\hat{x}+x} \left[(\hat{x} - \Delta x)^m \right] - \mathbb{E}^{\hat{x}} \left[(\hat{x} - \Delta x + x)^m \right] \qquad \text{(E.163)}$$

and

$$\theta_{1,j} \equiv \frac{1}{\nu} \sum_{k \geq j}^{\infty} \frac{\hat{x}^{k-j}}{k!j!} \left[\frac{d^{k+1} g_2(0)/2}{dx^{k+1}} - \frac{d^k g_1(0)}{dx^k} \right], \qquad \text{(E.164)}$$

then we need to show that

$$\Theta_1 = \sum_{j=0}^{\infty} \theta_{1,j} \mathcal{M}_[x] = -\frac{T}{2}. \qquad \text{(E.165)}$$

where T is the time between adjustments. Applying the definition of $g_m(x)$, we have that

$$g_m(x) = \bar{x}(T)^m - (\bar{x}(T) + x)^m, \qquad \text{(E.166)}$$

Note that $\nu T + \hat{x} = \bar{x}$. Then, by Proposition 5, we have that $\hat{x} = \frac{\mathbb{E}[\Delta x]}{2} = \frac{-\nu T}{2}$ and $\bar{x}(T) = \frac{\nu T}{2}$. Therefore,

$$\frac{d^k g_m(x)}{dx^k} = -\frac{m!}{(m-k)!} \left(\left(\frac{\nu T}{2} \right) + x \right)^{m-k} \mathbb{1}_{\{m-k \geq 0\}} \quad \forall k \geq 1 \qquad \text{(E.167)}$$

Thus, for all $k > 1$ we have that

$$\frac{d^1 g_2(0)/2}{dx^1} - \frac{d^0 g_1(0)}{dx^0} = -2 \left(\frac{\nu T}{2} \right) / 2 - \left[\left(\frac{\nu T}{2} \right) - \left(\frac{-\nu T}{2} \right) \right] = \frac{-\nu T}{2} \qquad \text{(E.168)}$$

$$\frac{d^2 g_2(0)/2}{dx^2} - \frac{d^1 g_1(0)}{dx^1} = -2/2 - (-1) = 0 \qquad \text{(E.169)}$$

$$\frac{d^{k+1} g_2(0)/2}{dx^{k+1}} - \frac{d^k g_1(0)}{dx^k} = 0 - 0 = 0 \qquad \text{(E.170)}$$

$$\text{(E.171)}$$

Thus,

$$\theta_{1,0} = \frac{\hat{x}^0 ((\frac{-\nu T}{2}))}{\nu} + \frac{1}{\nu} \sum_{k \geq 1}^{\infty} \frac{\hat{x}^{k-j}}{k!j!} 0 = -T/2, \qquad \textit{and} \qquad \theta_{1,j} = \frac{1}{\nu} \sum_{k \geq 1}^{\infty} \frac{\hat{x}^{k-j}}{k!j!} 0 = 0. \ \forall j > 0, \qquad \text{(E.172)}$$

Since $\mathcal{M}_0[x] = 1$, we obtain (E.165). □

F Application of the three properties

This section solves numerically and analytically two models: the random Ss model presented in 2 with $H(\xi) = 1$ for all $\xi \in [0, \overline{k}]$ and the Calvo-Taylor model. We use these model to obtain intuitions of the theory develop in this paper.

F.1 Lumpy Investment in the Random Ss Model

Before we compute all the endogenous objects in this model, we define a set of function and parameters

$$\tilde{\nu} = -\frac{\psi + \mu}{\sigma^2} \ ; \ \nu = -\psi + \mu \tag{F.1}$$

$$\tilde{\lambda} = \frac{\lambda}{\sigma^2} \ ; \ \tilde{\lambda}(\varphi) = \frac{\lambda - \varphi}{\sigma^2} \tag{F.2}$$

$$\xi_1 = -\tilde{\nu} - \sqrt{\tilde{\mu}^2 + 2\tilde{\lambda}} \ ; \ \xi_1(\varphi) = -\tilde{\nu} - \sqrt{\tilde{\mu}^2 + 2\tilde{\lambda}(\varphi)} \tag{F.3}$$

$$\xi_2 = -\tilde{\nu} + \sqrt{\tilde{\mu}^2 + 2\tilde{\lambda}} \ ; \ \xi_2(\varphi) = -\tilde{\nu} + \sqrt{\tilde{\mu}^2 + 2\tilde{\lambda}(\varphi)} \tag{F.4}$$

$$\overline{\alpha}_1 = \frac{e^{\xi_1 \overline{x}}}{e^{\xi_1 \underline{x} + \xi_2 \overline{x}} - e^{\xi_2 \underline{x} + \xi_1 \overline{x}}} \ ; \ \overline{\alpha}_1(\varphi) = \frac{e^{\xi_1(\varphi) \overline{x}}}{e^{\xi_1(\varphi) \underline{x} + \xi_2(\varphi) \overline{x}} - e^{\xi_2(\varphi) \underline{x} + \xi_1(\varphi) \overline{x}}} \tag{F.5}$$

$$\overline{\alpha}_2 = \frac{e^{\xi_2 \overline{x}}}{e^{\xi_1 \underline{x} + \xi_2 \overline{x}} - e^{\xi_2 \underline{x} + \xi_1 \overline{x}}} \ ; \ \overline{\alpha}_2(\varphi) = \frac{e^{\xi_2(\varphi) \overline{x}}}{e^{\xi_1(\varphi) \underline{x} + \xi_2(\varphi) \overline{x}} - e^{\xi_2(\varphi) \underline{x} + \xi_1(\varphi) \overline{x}}} \tag{F.6}$$

$$\underline{\alpha}_1 = \frac{e^{\xi_1 \underline{x}}}{e^{\xi_1 \underline{x} + \xi_2 \overline{x}} - e^{\xi_2 \underline{x} + \xi_1 \overline{x}}} \ ; \ \underline{\alpha}_1(\varphi) = \frac{e^{\xi_1(\varphi) \underline{x}}}{e^{\xi_1(\varphi) \underline{x} + \xi_2(\varphi) \overline{x}} - e^{\xi_2(\varphi) \underline{x} + \xi_1(\varphi) \overline{x}}} \tag{F.7}$$

$$\underline{\alpha}_2 = \frac{e^{\xi_2 \underline{x}}}{e^{\xi_1 \underline{x} + \xi_2 \overline{x}} - e^{\xi_2 \underline{x} + \xi_1 \overline{x}}} \ ; \ \underline{\alpha}_2(\varphi) = \frac{e^{\xi_2(\varphi) \underline{x}}}{e^{\xi_1(\varphi) \underline{x} + \xi_2(\varphi) \overline{x}} - e^{\xi_2(\varphi) \underline{x} + \xi_1(\varphi) \overline{x}}} \tag{F.8}$$

$$\kappa_j^m(x) = \sum_{i=0}^{j} (x)^m \frac{m!}{i!} \left[\frac{\xi_1 - \xi_1 \xi_2 \frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1 - \xi_2} (\xi_1 x)^{i-m} + \frac{\xi_2 - \xi_1 \xi_2 \frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2 - \xi_1} (\xi_2 x)^{i-m} \right] \tag{F.9}$$

$$\overline{\kappa}_j^m(x) = \sum_{i=0}^{j} (x)^m \frac{m!}{i!} \left[\frac{\xi_1 - \xi_1 \xi_2 \frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1 - \xi_2} (-\xi_1 x)^{i-m} + \frac{\xi_2 - \xi_1 \xi_2 \frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2 - \xi_1} (-\xi_2 x)^{i-m} \right] \tag{F.10}$$

$$\kappa_j^m(x, \varphi) = \sum_{i=0}^{j} (x)^m \frac{m!}{i!} \left[\frac{\xi_1(\varphi) + \xi_1(\varphi)\xi_2(\varphi) \frac{\tilde{\mu}}{\tilde{\lambda}(\varphi)}}{\xi_1(\varphi) - \xi_2(\varphi)} (\xi_1(\varphi) x)^{i-m} + \frac{\xi_2(\varphi) + \xi_1(\varphi)\xi_2(\varphi) \frac{\tilde{\mu}}{\tilde{\lambda}(\varphi)}}{\xi_2(\varphi) - \xi_1(\varphi)} (\xi_2(\varphi) x)^{i-m} \right]. \tag{F.11}$$

With these objects we can characterize the policy function and the cross-sectional distribution of investment rates.

Policy. Following Section C in the online appendix, the policy function for the Ss bands and the reset state before the normalization solve the HJB, value matching, and smooth pasting conditions defined in 2. Let us use the notation $(\underline{k}, \hat{k}, \overline{k})$ to denote the lower Ss band, the reset state and the upper Ss band before the re-normalization, respectively. Then $(\underline{k}, \hat{k}, \overline{k})$ are the solution to

$$\tilde{\rho} v(k) = e^{\alpha k} + \lambda \left(v(\hat{k}) - (e^{\hat{k}} - e^{k}) \right) + \nu v'(k) + \frac{\sigma^2}{2} v''(k) \qquad \forall k \in \left(\underline{k}, \overline{k}\right) \tag{F.12}$$

$$v(\underline{k}) - e^{\underline{k}} = v(\overline{k}) - e^{\overline{k}} = v\left(\hat{k}\right) - \kappa - e^{\hat{k}} \tag{F.13}$$

$$v'(\underline{k}) = e^{\underline{k}}, \qquad v'(\overline{k}) = e^{\overline{k}}, \qquad v'\left(\hat{k}\right) = e^{\hat{k}}. \tag{F.14}$$

Proposition F.1. *In the random Ss model the firm's policies are given by*

$$\tau = \inf\{t \geq 0 : k_t \notin [\underline{k}, \overline{k}] \text{ or } N_t \geq 1\} \tag{F.15}$$

where the Ss bands and the reset state with $\underline{k} < \hat{k} < \overline{k}$ are defined implicitly as the solution to the system

$$v'(\underline{k}|\underline{k}, \hat{k}, \overline{k}) = e^{\underline{k}} \quad v'(\hat{k}|\underline{k}, \hat{k}, \overline{k}) = e^{\hat{k}} \quad v'(\overline{k}|\underline{k}, \hat{k}, \overline{k}) = e^{\overline{k}} \tag{F.16}$$

where $v(x|\underline{k}, \hat{k}, \overline{k})$ is given by

$$v(k|\underline{k}, \hat{k}, \overline{k}) = A(\lambda_2) e^{\lambda_1 k} + A(\lambda_1) e^{\lambda_2 k} + C(\alpha) e^{\alpha k} + C(1) \lambda e^{k} + E \tag{F.17}$$

with roots

$$\lambda_1 = -\tilde{\nu} - \sqrt{\tilde{\nu}^2 + \frac{2\tilde{\rho}}{\sigma^2}} \quad ; \quad \lambda_2 = -\tilde{\nu} + \sqrt{\tilde{\nu}^2 + \frac{2\tilde{\rho}}{\sigma^2}} \tag{F.18}$$

and coefficients computed as:

$$C(\alpha) = \left(\tilde{\rho} - \alpha\tilde{\nu} - \frac{\sigma^2}{2}\alpha^2\right)^{-1} \tag{F.19}$$

$$B(k) = -\theta - \left(e^{\hat{k}} - e^{k}\right) + C(\alpha)\left(e^{\alpha\hat{k}} - e^{\alpha k}\right) + \lambda C(1)\left(e^{\hat{k}} - e^{k}\right) \tag{F.20}$$

$$A(\lambda) = \frac{B(\underline{k})\left(e^{\lambda\overline{k}} - e^{\lambda\hat{k}}\right) - B(\overline{k})\left(e^{\lambda\underline{k}} - e^{\lambda\hat{k}}\right)}{D} \tag{F.21}$$

$$E = \frac{\lambda}{\tilde{\rho} - \lambda}\left(A_1 e^{\lambda_1\hat{k}} + A_2 e^{\lambda_2\hat{k}} + C(\alpha)e^{\alpha\hat{k}} + C(1)\lambda\left(e^{\hat{k}} - e^{\hat{k}}\right)\right) \tag{F.22}$$

$$D = \left(e^{-\lambda_1\underline{k}} - e^{-\lambda_1\hat{k}}\right)\left(e^{-\lambda_2\overline{k}} - e^{-\lambda_2\hat{k}}\right) - \left(e^{-\lambda_2\underline{k}} - e^{-\lambda_2\hat{k}}\right)\left(e^{-\lambda_1\overline{k}} - e^{-\lambda_1\hat{k}}\right) \tag{F.23}$$

Proof. The homogenous solution of (F.13) is given by

$$v^h(k) = A_1 e^{\lambda_1 k} + A_2 e^{\lambda_2 k}, \tag{F.24}$$

where the roots λ_1 and λ_2 are given by (F.18).

Using the method of undetermined coefficients to find the non-homogenous solution, we have that

$$v(k) = A_1 e^{\lambda_1 k} + A_2 e^{\lambda_2 k} + C(\alpha)e^{\alpha k} + C(1)\lambda e^{k} + E \tag{F.25}$$

$$C(\alpha) = \left(\tilde{\rho} - \alpha\tilde{\nu} - \frac{\sigma^2}{2}\alpha^2\right)^{-1} \tag{F.26}$$

$$E = \frac{\lambda}{\tilde{\rho} - \lambda}\left(A_1 e^{\lambda_1\hat{k}} + A_2 e^{\lambda_2\hat{k}} + C(\alpha)e^{\alpha\hat{k}} + C(1)e^{\hat{k}} - e^{\hat{k}}\right). \tag{F.27}$$

The value matching conditions in (F.13) imply

$$A_1\left(e^{\lambda_1\underline{k}} - e^{\lambda_1\hat{k}}\right) + A_2\left(e^{\lambda_2\underline{k}} - e^{\lambda_2\hat{k}}\right) = -\kappa - \left(e^{\hat{k}} - e^{\underline{k}}\right) + C(\alpha)\left(e^{\alpha\hat{k}} - e^{\alpha\underline{k}}\right) + \lambda C(1)\left(e^{\hat{k}} - e^{\underline{k}}\right) =: B(\underline{k}) \tag{F.28}$$

$$A_1\left(e^{\lambda_1\overline{k}} - e^{\lambda_1\hat{k}}\right) + A_2\left(e^{\lambda_2\overline{k}} - e^{\lambda_2\hat{k}}\right) = -\kappa - \left(e^{\hat{k}} - e^{\overline{k}}\right) + C(\alpha)\left(e^{\alpha\hat{k}} - e^{\alpha\overline{k}}\right) + \lambda C(1)\left(e^{\hat{k}} - e^{\overline{k}}\right) =: B(\overline{k}), \tag{F.29}$$

and using Cramer's rule to solve this system we have that

$$A_1 = \frac{B(\underline{k})\left(e^{\lambda_2\overline{k}} - e^{\lambda_2\hat{k}}\right) - B(\overline{k})\left(e^{\lambda_2\underline{k}} - e^{\lambda_2\hat{k}}\right)}{D} \tag{F.30}$$

$$A_2 = -\frac{B(\underline{k})\left(e^{\lambda_1\overline{k}} - e^{\lambda_1\hat{k}}\right) + B(\overline{k})\left(e^{\lambda_1\underline{k}} - e^{\lambda_1\hat{k}}\right)}{D} \tag{F.31}$$

with the determinant

$$D = \left(e^{-\lambda_1\underline{k}} - e^{-\lambda_1\hat{k}}\right)\left(e^{-\lambda_2\overline{k}} - e^{-\lambda_2\hat{k}}\right) - \left(e^{-\lambda_1\overline{k}} - e^{-\lambda_1\hat{k}}\right)\left(e^{-\lambda_2\underline{k}} - e^{-\lambda_2\hat{k}}\right). \tag{F.32}$$

□

Re-normalization We normalize the policy function in both models to work with centralized moments. We redefine the reset state $\hat{x}$ and the Ss bands $\underline{x}, \overline{x}$ in the following way

$$(\underline{x}, \hat{x}, \overline{x}) = ((\underline{k} - \mathcal{M}_1[k], \hat{k} - \mathcal{M}_1[k], \overline{k} - \mathcal{M}_1[k])), \tag{F.33}$$

where $\mathcal{M}_1[k]$ is the average capital-gap under the firms' policies $(\underline{k}, \hat{k}, \overline{k})$. It is easy to check that under the re-normalized policy $\mathcal{M}_1[x] = 0$.

Inputs for the verification of observability and representation. In order to verify all the main proposition in the main text, first we compute several object:

$$\mathbb{E}^{\hat{x}}[\Delta x^m]\ ;\ \mathbb{E}^{\hat{x}}[x_\tau^m]\ ;\ \mathbb{E}^{\hat{x}}[e^{\varphi\tau}\Delta x^m]\ ;\ \mathcal{M}_m[x]\ ;\ \mathcal{M}_{m,1}[x,a]\ ;\ f(x)\ ;\ \mathcal{M}[e^{\lambda_i x}] \tag{F.34}$$

We divide the computation of each object in a separate propositions just to order the presentation.

Proposition F.2. *The following relations hold*

$$\mathbb{E}^{\hat{x}}[x_\tau^m] = -e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_{m-1}^m(\underline{x}) - \underline{\alpha}_2\kappa_{m-1}^m(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_{m-1}^m(\overline{x}) - \overline{\alpha}_1\kappa_{m-1}^m(\underline{x})\right] + \kappa_m^m(\hat{x}) \tag{F.35}$$

$$\mathbb{E}^{\hat{x}}[\Delta x^m] = -e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\overline{\kappa}_{m-1}^m(\hat{x}-\underline{x}) - \underline{\alpha}_2\overline{\kappa}_{m-1}^m(\hat{x}-\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\overline{\kappa}_{m-1}^m(\hat{x}-\overline{x}) - \overline{\alpha}_1\overline{\kappa}_{m-1}^m(\hat{x}-\underline{x})\right] + \overline{\kappa}_0^m(0) \tag{F.36}$$

Proof. The function $v_m(x) = \mathbb{E}^x[x_\tau^m]$ satisfies the HJB

$$0 = \nu v_m'(x) + \frac{\sigma^2}{2}v_m''(x) + \lambda\left(x^m - v_m(x)\right) \tag{F.37}$$

with the border conditions $v_m(\underline{x}) = \underline{x}^m$ and $v_m(\overline{x}) = \overline{x}^m$. The homogenous solution is given by

$$v^h(x) = A_1e^{\xi_1 x} + A_2e^{\xi_2 x}, \tag{F.38}$$

where ξ_i are defined in (F.3) and (F.4).

To compute the non-homogenous solution let us use a guess and verify $v^{nh}(x) = \sum_{i=0}^m a_{i,m}x^i$. Then

$$0 = \tilde{\nu}\left(\sum_{i=1}^m a_{i,m}ix^{i-1}\right) + \frac{1}{2}\left(\sum_{i=2}^m a_{i,m}i(i-1)x^{i-2}\right) + \tilde{\lambda}\left(x^m - \sum_{i=0}^m a_{i,m}x^i\right) \tag{F.39}$$

Notice that the solution consist in the following system of equations

$$a_{m,m} = 1\ ;\ a_{m-1,m} = \frac{\tilde{\nu}ma_{m,m}}{\tilde{\lambda}}\ ;\ a_{i,m} = \frac{\tilde{\nu}(i+1)a_{i+1,m} + (i+2)(i+1)a_{i+2,m}/2}{\tilde{\lambda}}. \tag{F.40}$$

The border conditions we have that A_1 and A_2 satisfy

$$A_{1,m}e^{\xi_1\underline{x}} + A_{2,m}e^{\xi_2\underline{x}} = -\sum_{i=0}^{m-1}a_{i,m}\underline{x}^i, \tag{F.41}$$

$$A_{1,m}e^{\xi_1\overline{x}} + A_{2,m}e^{\xi_2\overline{x}} = -\sum_{i=0}^{m-1}a_{i,m}\overline{x}^i. \tag{F.42}$$

and the solution to this system of equations is given by

$$A_{1,m} = \frac{\left(-\sum_{i=0}^{m-1}a_{i,m}\underline{x}^i\right)e^{\xi_2\overline{x}} - \left(-\sum_{i=0}^{m-1}a_{i,m}\overline{x}^i\right)e^{\xi_2\underline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}} \tag{F.43}$$

$$A_{2,m} = \frac{\left(-\sum_{i=0}^{m-1}a_{i,m}\overline{x}^i\right)e^{\xi_1\underline{x}} - \left(-\sum_{i=0}^{m-1}a_{i,m}\underline{x}^i\right)e^{\xi_1\overline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}}. \tag{F.44}$$

For the next verifications it is useful to simply these expressions in several ways. Note that the coefficient are an equation in difference

$$a_{i,m} = \left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)(i+1)a_{i+1,m} + \frac{(i+2)(i+1)}{2\tilde{\lambda}}a_{i+2,m}. \tag{F.45}$$

Using the guess and verify method with $a_{i,m} = \frac{c}{i!}x^i$, we have that

$$a_{i,m} = \frac{m!}{i!}\left[c_1\xi_1^{i-m} + c_2\xi_2^{i-m}\right], \tag{F.46}$$

and using the border conditions

$$a_{i,m} = \frac{m!}{i!}\left[\frac{\xi_1 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1-\xi_2}\xi_1^{i-m} - \frac{\xi_2 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2-\xi_1}\xi_2^{i-m}\right], \tag{F.47}$$

Putting all the results togethers we have (F.35).

The function $h_m(x) = \mathbb{E}^x[(\hat{x}-x_\tau)^m]$ satisfies the HJB

$$0 = \nu h_m'(x) + \frac{\sigma^2}{2}h_m''(x) + \lambda\left((\hat{x}-x)^m - h_m(x)\right) \tag{F.48}$$

with the border conditions $v_m(\underline{x}) = (\hat{x}-\underline{x})^m$ and $v_m(\overline{x}) = (\hat{x}-\overline{x})^m$. The homogenous solution is given by

$$v^h(x) = B_1e^{\xi_1 x} + B_2e^{\xi_2 x}. \tag{F.49}$$

To compute the non-homogenous solution let us use a guess and verify $v^{nh}(x) = \sum_{i=0}^{m} b_{i,m}(\hat{x}-x)^i$. Then

$$0 = -\tilde{\nu}\left(\sum_{i=1}^{m} b_i i(\hat{x}-x)^{i-1}\right) + \frac{1}{2}\left(\sum_{i=2}^{m} b_i i(i-1)(\hat{x}-x)^{i-2}\right) + \tilde{\lambda}\left(x^m - \sum_{i=0}^{m} b_i(\hat{x}-x)^i\right). \tag{F.50}$$

Note that the solution consist in the following system of equations

$$b_{m,m} = 1 \ ; \ b_{m-1,m} = -\frac{\tilde{\nu} m b_m}{\tilde{\lambda}} \ ; \ b_{i,m} = \frac{-\tilde{\nu}(i+1)b_{i+1} + (i+2)(i+1)a_{i+2}/2}{\tilde{\lambda}}. \tag{F.51}$$

with the solution given by

$$b_{i,m} = \frac{m!}{i!}\left[\frac{\xi_1 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1-\xi_2}(-\xi_1)^{i-m} + \frac{\xi_2 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2-\xi_1}(-\xi_2)^{i-m}\right], \tag{F.52}$$

with the border conditions we have that B_1 and B_2 satisfy

$$B_{1,m}e^{\xi_1\underline{x}} + B_{2,m}e^{\xi_2\underline{x}} = -\sum_{i=0}^{m-1} b_{i,m}(\hat{x}-\underline{x})^i, \tag{F.53}$$

$$B_{1,m}e^{\xi_1\overline{x}} + B_{2,m}e^{\xi_2\overline{x}} = -\sum_{i=0}^{m-1} b_{i,m}(\hat{x}-\overline{x})^i. \tag{F.54}$$

and the solution to this system of equations is given by

$$B_{1,m} = \frac{\left(-\sum_{i=0}^{m-1} b_{i,m}(\hat{x}-\underline{x})^i\right)e^{\xi_2\overline{x}} - \left(-\sum_{i=0}^{m-1} b_{i,m}(\hat{x}-\overline{x})^i\right)e^{\xi_2\underline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}} \tag{F.55}$$

$$B_{2,m} = \frac{\left(-\sum_{i=0}^{m-1} b_{i,m}(\hat{x}-\overline{x})^i\right)e^{\xi_1\underline{x}} - \left(-\sum_{i=0}^{m-1} b_{i,m}(\hat{x}-\underline{x})^i\right)e^{\xi_1\overline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}} \tag{F.56}$$

Putting all the results togethers we have (F.36). □

Proposition F.3. *The moments of the ergodic distribution are given by*

$$\mathcal{M}_m[x] = \frac{-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_m^m(\underline{x}) - \underline{\alpha}_2\kappa_m^m(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_m^m(\overline{x}) - \overline{\alpha}_1\kappa_m^m(\underline{x})\right] + \kappa_m^m(\hat{x})}{\lambda\mathbb{E}^{\hat{x}}[\tau]} \tag{F.57}$$

Proof. Remember that $\mathcal{M}_m[x] = \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^m dt\right]}{\mathbb{E}^{\hat{x}}[\tau]}$. Define $h_m(x) = \mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^m dt\right]$. Then function $h_m(x)$ satisfies the HJB

$$0 = x^m + \nu h_m'(x) + \frac{\sigma^2}{2}h_m''(x) + \lambda\left(-h_m(x)\right) \tag{F.58}$$

with the border conditions $h_m(\underline{x}) = 0$ and $h_m(\overline{x}) = 0$. The homogenous solution is given by

$$h^h(x) = B_{1,m}e^{\xi_1 x} + B_{2,m}e^{\xi_2 x}. \tag{F.59}$$

For the non-homogenous solution we use a guess and verify $v^{nh}(x) = \sum_{i=0}^{m} b_{i,m}x^i$. Then

$$0 = \frac{x^m}{\sigma^2} + \tilde{\nu}\left(\sum_{i=1}^{m} b_{i,m} i x^{i-1}\right) + \frac{1}{2}\left(\sum_{i=2}^{m} b_{i,m} i(i-1)x^{i-2}\right) + \tilde{\lambda}\left(-\sum_{i=0}^{m} b_{i,m}x^i\right). \tag{F.60}$$

The solution consist in the following system of equations

$$b_{m,m} = \frac{1}{\lambda} \ ; \ b_{m-1,m} = \frac{\tilde{\nu} m b_{m,m}}{\tilde{\lambda}} \ ; \ b_{i,m} = \frac{\tilde{\nu}(i+1)b_{i+1,m} + (i+2)(i+1)b_{i+2,m}/2}{\tilde{\lambda}}. \tag{F.61}$$

The solution of the previous differential equation is given by

$$b_{i,m} = \frac{1}{\lambda}\frac{m!}{i!}\left[\frac{\xi_1 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1-\xi_2}\xi_1^{i-m} + \frac{\xi_2 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2-\xi_1}\xi_2^{i-m}\right], \tag{F.62}$$

With the border conditions we have that A_1 and A_2 satisfy

$$B_{1,m}e^{\xi_1\underline{x}} + B_{2,m}e^{\xi_1\underline{x}} = -\sum_{i=0}^{m} b_i\underline{x}^i, \tag{F.63}$$

$$B_{1,m}e^{\xi_1\overline{x}} + B_{2,m}e^{\xi_1\overline{x}} = -\sum_{i=0}^{m} b_i\overline{x}^i. \tag{F.64}$$

and the solution to this system of equations is given by

$$B_{1,m} = \frac{\left(-\sum_{i=0}^{m} b_{i,m}\underline{x}^i\right)e^{\xi_2\overline{x}} - \left(-\sum_{i=0}^{m} b_{i,m}\overline{x}^i\right)e^{\xi_2\underline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}} \tag{F.65}$$

$$B_{2,m} = \frac{\left(-\sum_{i=0}^{m} b_{i,m}\overline{x}^i\right)e^{\xi_1\underline{x}} - \left(-\sum_{i=0}^{m} b_{i,m}\underline{x}^i\right)e^{\xi_1\overline{x}}}{e^{\xi_1\underline{x}+\xi_2\overline{x}} - e^{\xi_2\underline{x}+\xi_1\overline{x}}} \tag{F.66}$$

Putting all the results togethers we have that

$$\mathcal{M}_m(\hat{x}) = \frac{-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_m^m(\underline{x}) - \underline{\alpha}_2\kappa_m^m(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_m^m(\overline{x}) - \overline{\alpha}_1\kappa_m^m(\underline{x})\right] + \kappa_m^m(\hat{x})}{\lambda\mathbb{E}[\tau]} \tag{F.67}$$

□

Proposition F.4. *The ergodic distribution is given*

$$f(x) = A\begin{cases} \frac{e^{\beta 1(x-\underline{x})} - e^{\beta 2(x-\underline{x})}}{e^{\beta 1(\hat{x}-\underline{x})} - e^{\beta 2(\hat{x}-\underline{x})}} & \text{if } x \in (\underline{x}, \hat{x}] \\ \frac{e^{\beta 1(x-\overline{x})} - e^{\beta 2(x-\overline{x})}}{e^{\beta 1(\hat{x}-\overline{x})} - e^{\beta 2(\hat{x}-\overline{x})}} & \text{if } x \in [\hat{x}, \overline{x}) \end{cases} \tag{F.68}$$

$$A = \left[\frac{\frac{e^{\beta 1(\hat{x}-\underline{x})}-1}{\beta 1} - \frac{e^{\beta 2(\hat{x}-\underline{x})}-1}{\beta 2}}{e^{\beta 1(\hat{x}-\underline{x})} - e^{\beta 2(\hat{x}-\underline{x})}} + \frac{\frac{1-e^{\beta 1(\hat{x}-\overline{x})}}{\beta 1} - \frac{1-e^{\beta 2(\hat{x}-\overline{x})}}{\beta 2}}{e^{\beta 1(\hat{x}-\overline{x})} - e^{\beta 2(\hat{x}-\overline{x})}}\right]^{-1} \tag{F.69}$$

Proof. The distribution of state solves the following KFE given by

$$\lambda f(x) = -\nu f'(x) + \frac{\sigma_a^2}{2}f''(x) \quad \forall x \in (\underline{x}, \overline{x}) \cap \{\hat{x}\}^c \tag{F.70}$$

with the border conditions $f(\underline{x}) = f(\overline{x}) = 0$ and $\int_{\underline{x}}^{\overline{x}} f(x)dx = 1$. The solution for $f(x)$ is given by

$$f(x) = \begin{cases} A_{11}e^{\beta_1 x} + A_{12}e^{\beta_2 x} & \text{if } x \in (\underline{x}, \hat{x}) \\ A_{21}e^{\beta_1 x} + A_{22}e^{\beta_2 x} & \text{if } x \in (\hat{x}, \overline{x}) \end{cases} \tag{F.71}$$

$$\beta_1 = \tilde{\nu} + \sqrt{\tilde{\nu}^2 + 2\tilde{\lambda}} \tag{F.72}$$

$$\beta_2 = \tilde{\nu} - \sqrt{\tilde{\nu}^2 + 2\tilde{\lambda}}. \tag{F.73}$$

The border conditions of zero mass in the boundary of the continuation region imply that

$$f(x) = \begin{cases} A_1\left(e^{\beta_1(x-\underline{x})} - e^{\beta_2(x-\underline{x})}\right) & \text{if } x \in (\underline{x}, \hat{x}) \\ A_2\left(e^{\beta_1(x-\overline{x})} - e^{\beta_2(x-\overline{x})}\right) & \text{if } x \in (\hat{x}, \overline{x}) \end{cases}. \tag{F.74}$$

Continuity at the reset state implies

$$f(x) = \begin{cases} A\frac{e^{\beta 1(x-\underline{x})} - e^{\beta 2(x-\underline{x})}}{e^{\beta 1(\hat{x}-\underline{x})} - e^{\beta 2(\hat{x}-\underline{x})}} & \text{if } x \in (\underline{x}, \hat{x}) \\ A\frac{e^{\beta 1(x-\overline{x})} - e^{\beta 2(x-\overline{x})}}{e^{\beta 1(\hat{x}-\overline{x})} - e^{\beta 2(\hat{x}-\overline{x})}} & \text{if } x \in (\hat{x}, \overline{x}) \end{cases} \tag{F.75}$$

$$\tag{F.76}$$

Using the unitary mass of firms condition we have that

$$\begin{aligned} A &= \left[\frac{\frac{e^{-\beta_1\underline{x}}}{\beta_1}\left(e^{\beta_1\hat{x}} - e^{\beta_1\underline{x}}\right) - \frac{e^{-\beta_2\underline{x}}}{\beta_2}\left(e^{\beta_2\hat{x}} - e^{\beta_2\underline{x}}\right)}{e^{\beta_1(\hat{x}-\underline{x})} - e^{\beta_2(\hat{x}-\underline{x})}} + \frac{\frac{e^{-\beta_1\overline{x}}}{\beta_1}\left(e^{\beta_1\overline{x}} - e^{\beta_1\hat{x}}\right) - \frac{e^{-\beta_2\overline{x}}}{\beta_2}\left(e^{\beta_2\overline{x}} - e^{\beta_2\hat{x}}\right)}{e^{\beta_1(\hat{x}-\overline{x})} - e^{\beta_2(\hat{x}-\overline{x})}}\right]^{-1} \\ &= \left[\frac{\frac{e^{\beta_1(\hat{x}-\underline{x})}-1}{\beta_1} - \frac{e^{\beta_2(\hat{x}-\underline{x})}-1}{\beta_2}}{e^{\beta_1(\hat{x}-\underline{x})} - e^{\beta_2(\hat{x}-\underline{x})}} + \frac{\frac{1-e^{\beta_1(\hat{x}-\overline{x})}}{\beta_1} - \frac{1-e^{\beta_2(\hat{x}-\overline{x})}}{\beta_2}}{e^{\beta_1(\hat{x}-\overline{x})} - e^{\beta_2(\hat{x}-\overline{x})}}\right]^{-1} \end{aligned} \tag{F.77}$$

□

Proposition F.5. *The following relations hold*

$$\mathbb{E}^{\hat{x}}[e^{\varphi\tau}x_\tau^m] = \frac{\lambda}{\lambda-\varphi}\Big[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\overline{\kappa}_{m-1}^m(\underline{x},\varphi)-\underline{\alpha}_2(\varphi)\overline{\kappa}_{m-1}^m(\overline{x},\varphi)\right] \tag{F.78}$$

$$\cdots - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\overline{\kappa}_{m-1}^m(\overline{x},\varphi)-\overline{\alpha}_1(\varphi)\overline{\kappa}_{m-1}^m(\underline{x},\varphi)\right]+\overline{\kappa}_m^m(\hat{x})\Big]+\dots \tag{F.79}$$

$$\cdots + \frac{\varphi}{\lambda-\varphi}\left[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\underline{x}^m-\underline{\alpha}_2(\varphi)\overline{x}^m\right]+e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\overline{x}^m-\overline{\alpha}_1(\varphi)\underline{x}^m\right]\right] \tag{F.80}$$

$$\mathbb{E}^{\hat{x}}[e^{\varphi\tau}\Delta x^m] = \frac{\lambda}{\lambda-\varphi}\Big[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\overline{\kappa}_{m-1}^m(\hat{x}-\underline{x},\varphi)-\underline{\alpha}_2(\varphi)\overline{\kappa}_{m-1}^m(\hat{x}-\overline{x},\varphi)\right]\dots$$

$$\cdots - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\overline{\kappa}_{m-1}^m(\hat{x}-\overline{x},\varphi)-\overline{\alpha}_1(\varphi)\overline{\kappa}_{m-1}^m(\hat{x}-\underline{x},\varphi)\right]+\overline{\kappa}_0^m(0)\Big]\dots$$

$$\cdots + \frac{\varphi}{\lambda-\varphi}\left[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)(\hat{x}-\underline{x})^m-\underline{\alpha}_2(\varphi)(\hat{x}-\overline{x})^m\right]-e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)(\hat{x}-\overline{x})^m-\overline{\alpha}_1(\varphi)(\hat{x}-\underline{x})^m\right]\right] \tag{F.81}$$

Proof. The function $v_m(x)=\mathbb{E}^x[e^{\varphi\tau}(\hat{x}-x_\tau)^m]$ satisfies the HJB

$$-\varphi v_m(x) = \nu v_m'(x)+\frac{\sigma^2}{2}v_m''(x)+\lambda\left((\hat{x}-x)^m-v_m(x)\right) \tag{F.82}$$

with the border conditions $v_m(\underline{x})=(\hat{x}-\underline{x})^m$ and $v_m(\overline{x})=(\hat{x}-\overline{x})^m$. The homogenous solution is given by

$$v^h(x) = B_{1,m}e^{\xi_1(\varphi)x}+B_{2,m}e^{\xi_2(\varphi)x}, \tag{F.83}$$

To compute the non-homogenous solution let us use a guess and verify $v^{nh}(x)=\sum_{i=0}^m b_{i,m}(\hat{x}-x)^i$. Then

$$0 = -\tilde{\nu}\left(\sum_{i=1}^m b_{im}i(\hat{x}-x)^{i-1}\right)+\frac{1}{2}\left(\sum_{i=2}^m b_{im}i(i-1)(\hat{x}-x)^{i-2}\right)+\tilde{\lambda}(\varphi)\left(\frac{\tilde{\lambda}}{\tilde{\lambda}(\varphi)}(\hat{x}-x)^m-\sum_{i=0}^m b_i(\hat{x}-x)^i\right) \tag{F.84}$$

Notice that the solution consist in the following system of equations

$$b_{m,m}=\frac{\tilde{\lambda}}{\tilde{\lambda}(\varphi)}\ ;\ b_{m-1,m}=-\frac{\tilde{\nu}mb_{m,m}}{\tilde{\lambda}(\varphi)}\ ;\ b_{i,m}=\frac{-\tilde{\nu}(i+1)b_{i+1,m}+(i+2)(i+1)b_{i+2,m}/2}{\tilde{\lambda}(\varphi)}. \tag{F.85}$$

with the solution given by

$$b_{i,m}(\varphi)=\frac{\tilde{\lambda}}{\tilde{\lambda}(\varphi)}\frac{m!}{i!}\left[\frac{\xi_1(\varphi)-\xi_1(\varphi)\xi_2(\varphi)\frac{\tilde{\nu}}{\tilde{\lambda}(\varphi)}}{\xi_1(\varphi)-\xi_2(\varphi)}(-\xi_1(\varphi))^{i-m}+\frac{\xi_2(\varphi)-\xi_1(\varphi)\xi_2(\varphi)\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2(\varphi)-\xi_1(\varphi)}(-\xi_2(\varphi))^{i-m}\right], \tag{F.86}$$

with the border conditions we have that B_1 and B_2 satisfy

$$B_{1,m}(\varphi)e^{\xi_1(\varphi)\underline{x}}+B_{2,m}(\varphi)e^{\xi_2(\varphi)\underline{x}} = -\sum_{i=0}^m b_{i,m}(\varphi)(\hat{x}-\underline{x})^i+(\hat{x}-\underline{x})^m, \tag{F.87}$$

$$B_{1,m}(\varphi)e^{\xi_1(\varphi)\overline{x}}+B_{2,m}(\varphi)e^{\xi_2(\varphi)\overline{x}} = -\sum_{i=0}^m b_{i,m}(\varphi)(\hat{x}-\overline{x})^i+(\hat{x}-\underline{x})^m. \tag{F.88}$$

Putting all the results togethers we have that

$$h_m(\hat{x}) = \frac{\lambda}{\lambda-\varphi}\Big[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\overline{\kappa}_{m-1}^m(\hat{x}-\underline{x},\varphi)-\underline{\alpha}_2(\varphi)\overline{\kappa}_{m-1}^m(\hat{x}-\overline{x},\varphi)\right] \tag{F.89}$$

$$\cdots - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\overline{\kappa}_{m-1}^m(\hat{x}-\overline{x},\varphi)-\overline{\alpha}_1(\varphi)\overline{\kappa}_{m-1}^m(\hat{x}-\underline{x},\varphi)\right]+\overline{\kappa}_0^m(0)\Big]+\dots \tag{F.90}$$

$$\cdots + \frac{\varphi}{\lambda-\varphi}\left[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)(\hat{x}-\underline{x})^m-\underline{\alpha}_2(\varphi)(\hat{x}-\overline{x})^m\right]+e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)(\hat{x}-\overline{x})^m-\overline{\alpha}_1(\varphi)(\hat{x}-\underline{x})^m\right]\right] \tag{F.91}$$

In a similar way, the function $k_m(x)=\mathbb{E}^x[e^{\varphi\tau}x_\tau^m]$ satisfies the HJB

$$-\varphi k_m(x) = \nu k_m'(x)+\frac{\sigma^2}{2}k_m''(x)+\lambda\left(x^m-k_m(x)\right) \tag{F.92}$$

with the border conditions $k_m(\underline{x})=\underline{x}^m$ and $k_m(\overline{x})=\overline{x}^m$. The homogenous solution is given by

$$v^h(x) = B_{1,m}e^{\xi_1(\varphi)x}+B_{2,m}e^{\xi_2(\varphi)x}. \tag{F.93}$$

To compute the non-homogenous solution let us use a guess and verify $v^{nh}(x) = \sum_{i=0}^{m} b_{i,m} x^i$. Then

$$0 = \tilde{\nu}\left(\sum_{i=1}^{m} b_{im} i x^{i-1}\right) + \frac{1}{2}\left(\sum_{i=2}^{m} b_{i,m} i(i-1)x^{i-2}\right) + \tilde{\lambda}(\varphi)\left(\frac{\tilde{\lambda}}{\tilde{\lambda}(\varphi)}x^m - \sum_{i=0}^{m} b_i x^i\right) \tag{F.94}$$

Notice that the solution consist in the following system of equations

$$b_{m,m} = \frac{\tilde{\lambda}}{\tilde{\lambda}(\varphi)} \ ; \ b_{m-1,m} = \frac{\tilde{\nu} m b_m}{\tilde{\lambda}(\varphi)} \ ; \ b_{i,m} = \frac{\tilde{\nu}(i+1)b_{i+1} + (i+2)(i+1)a_{i+2}/2}{\tilde{\lambda}(\varphi)}. \tag{F.95}$$

with the solution given by

$$b_{i,m}(\varphi) = \frac{\tilde{\lambda}}{\tilde{\lambda}(\varphi)}\frac{m!}{i!}\left[\frac{\xi_1(\varphi) - \xi_1(\varphi)\xi_2(\varphi)\frac{\tilde{\nu}}{\tilde{\lambda}(\varphi)}}{\xi_1(\varphi) - \xi_2(\varphi)}(\xi_1(\varphi))^{i-m} + \frac{\xi_2(\varphi) - \xi_1(\varphi)\xi_2(\varphi)\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2(\varphi) - \xi_1(\varphi)}(\xi_2(\varphi))^{i-m}\right], \tag{F.96}$$

with the border conditions we have that B_1 and B_2 satisfy

$$B_{1,m}(\varphi)e^{\xi_1(\varphi)\underline{x}} + B_{2,m}(\varphi)e^{\xi_2(\varphi)\underline{x}} = -\sum_{i=0}^{m} b_{i,m}(\varphi)\underline{x}^i + \underline{x}^m, \tag{F.97}$$

$$B_{1,m}(\varphi)e^{\xi_1(\varphi)\overline{x}} + B_{2,m}(\varphi)e^{\xi_2(\varphi)\overline{x}} = -\sum_{i=0}^{m} b_{i,m}(\varphi)\overline{x}^i + \underline{x}^m. \tag{F.98}$$

Putting all the results togethers we have that

$$k_m(\hat{x}) = \frac{\lambda}{\lambda - \varphi}\Big[- e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\overline{\kappa}_{m-1}^m(\underline{x},\varphi) - \underline{\alpha}_2(\varphi)\overline{\kappa}_{m-1}^m(\overline{x},\varphi)\right] \tag{F.99}$$

$$\cdots - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\overline{\kappa}_{m-1}^m(\overline{x},\varphi) - \overline{\alpha}_1(\varphi)\overline{\kappa}_{m-1}^m(\underline{x},\varphi)\right] + \overline{\kappa}_m^m(\hat{x})\Big] + \ldots \tag{F.100}$$

$$\cdots + \frac{\varphi}{\lambda - \varphi}\left[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\underline{x}^m - \underline{\alpha}_2(\varphi)\overline{x}^m\right] + e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\overline{x}^m - \overline{\alpha}_1(\varphi)\underline{x}^m\right]\right] \tag{F.101}$$

□

Proposition F.6. *The following relation holds*

$$\mathcal{M}_{m,1}[x,a] = \frac{h_m'(0)}{\mathbb{E}^{\hat{x}}[\tau]} \tag{F.102}$$

$$h_m(\varphi) = \frac{1}{\lambda - \varphi}\Big[- e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\kappa_m^m(\underline{x},\varphi) - \underline{\alpha}_2(\varphi)\kappa_m^m(\overline{x},\varphi)\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\kappa_m^m(\overline{x},\varphi) - \overline{\alpha}_1(\varphi)\kappa_m^m(\underline{x},\varphi)\right] + \overline{\kappa}_m^m(\hat{x})\Big] \tag{F.103}$$

Proof. The function $v_m(x) = \mathbb{E}^x[\int_0^\tau e^{\varphi t} x_t^m dt]$ satisfies the HJB

$$-\varphi v_m(x,\varphi) = x^m + \nu\frac{\partial_m^v(x,\varphi)}{\partial x^2} + \frac{\sigma^2}{2}\frac{\partial_m^v(x,\varphi)}{\partial x^2} + \lambda\left(-v_m(x,\varphi)\right) \tag{F.104}$$

with the border conditions $v_m(\underline{x},\varphi) = 0$ and $v_m(\overline{x},\varphi) = 0$. The homogenous solution is given by

$$v^h(x,\varphi) = B_{1,m}e^{\xi_1(\varphi)x} + B_{2,m}e^{\xi_2(\varphi)x}, \tag{F.105}$$

and the non-homogenous solution is given by $v^{nh}(x,\varphi) = \sum_{i=0}^{m} b_{i,m}x^i$ with

$$b_{i,m}(\varphi) = \frac{1}{\lambda - \varphi}\frac{m!}{i!}\left[\frac{\xi_1(\varphi) - \xi_1(\varphi)\xi_2(\varphi)\frac{\tilde{\nu}}{\tilde{\lambda}(\varphi)}}{\xi_1(\varphi) - \xi_2(\varphi)}(\xi_1(\varphi))^{i-m} + \frac{\xi_2(\varphi) - \xi_1(\varphi)\xi_2(\varphi)\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2(\varphi) - \xi_1(\varphi)}(\xi_2(\varphi))^{i-m}\right], \tag{F.106}$$

The border conditions for B_1 and B_2 are given by

$$B_{1,m}(\varphi)e^{\xi_1(\varphi)\underline{x}} + B_{2,m}(\varphi)e^{\xi_2(\varphi)\underline{x}} = -\sum_{i=0}^{m} b_{i,m}(\varphi)\underline{x}^i, \tag{F.107}$$

$$B_{1,m}(\varphi)e^{\xi_1(\varphi)\overline{x}} + B_{2,m}(\varphi)e^{\xi_2(\varphi)\overline{x}} = -\sum_{i=0}^{m} b_{i,m}(\varphi)\overline{x}^i. \tag{F.108}$$

Putting all the results togethers we have that

$$h_m(\varphi) = v_m(\hat{x}, \varphi) \tag{F.109}$$

$$= \frac{1}{\lambda - \varphi}\Big[- e^{\xi_1(\varphi)\hat{x}} \left[\overline{\alpha}_2(\varphi)\kappa_m^m(\underline{x}, \varphi) - \underline{\alpha}_2(\varphi)\kappa_m^m(\overline{x}, \varphi)\right] - e^{\xi_2\hat{x}} \left[\underline{\alpha}_1(\varphi)\kappa_m^m(\overline{x}, \varphi) - \overline{\alpha}_1(\varphi)\kappa_m^m(\underline{x}, \varphi)\right] + \overline{\kappa}_m^m(\hat{x})\Big] \tag{F.110}$$

□

Proposition F.7. *The following relations hold*

$$\mathbb{E}^{\hat{x}}\left[\int_0^\tau e^{\xi_1 x_t} dt\right] = -\frac{-e^{\xi_1\hat{x}}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\lambda_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_1\hat{x}}\hat{x}}{\sigma^2(\tilde{\nu} + \xi_1)} \tag{F.111}$$

$$\mathbb{E}^{\hat{x}}\left[\int_0^\tau e^{\xi_2 x_t} dt\right] = -\frac{-e^{\xi_1\hat{x}}(e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\lambda_2\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2\hat{x}}\left(e^{\xi_2\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_2\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_2\hat{x}}\hat{x}}{\sigma^2(\tilde{\nu} + \xi_2)} \tag{F.112}$$

Proof. The computation of the solution is equivalent for ξ_1 and ξ_2, so we only do it for ξ_1. Define $p(x) = \mathbb{E}^x\left[\int_0^\tau e^{\xi_1 x_t} dt\right]$. Then $p(x)$ satisfies the HJB

$$0 = e^{\xi_1 x} + \nu p'(x) + \frac{\sigma^2}{2} p''(x) + \lambda\left(-v_m(x)\right) \tag{F.113}$$

with the border conditions $p(\underline{x}) = 0$ and $p(\overline{x}) = 0$. The homogenous solution is given by

$$v^h(x) = A_1 e^{\xi_1 x} + A_2 e^{\xi_2 x}. \tag{F.114}$$

To compute the non-homogenous solution let us use a guess and verify $v^{nh}(x) = e^{\xi_1 x} C x$. Then

$$0 = e^{\xi_1 x} + \nu\left[e^{\xi_1 x} C + \xi_1 e^{\xi_1 x} C x\right] + \frac{\sigma^2}{2}\left[2\xi_1 e^{\xi_1 x} C + \xi_1^2 e^{\xi_1 x} C x\right] - \lambda e^{\xi_1 x} C x, \tag{F.115}$$

or equivalently

$$0 = \frac{1}{\sigma^2} + \tilde{\nu}\left[C + \xi_1 C x\right] + \frac{1}{2}\left[2\xi_1 C + \xi_1^2 C x\right] - \tilde{\lambda} C x, \tag{F.116}$$

$$= x(\tilde{\nu}\xi_1 C + \frac{\xi_1^2 C}{2} - \tilde{\lambda} C) + \frac{1}{\sigma^2} + \tilde{\nu} C + \xi_1 C \tag{F.117}$$

$$= x\underbrace{\left(\tilde{\nu}\lambda_1 + \frac{\lambda_1^2}{2} - \tilde{\lambda}\right)}_{=0} C + \frac{1}{\sigma^2} + \tilde{\nu} C + \xi_1 C \tag{F.118}$$

Since the previous equation must be satisfied for all x we have that

$$C = -\frac{1}{\sigma^2(\tilde{\nu} + \xi_1)}. \tag{F.119}$$

Therefore the solution is given by

$$v(x) = A_{1,1} e^{\xi_1 x} + A_{2,1} e^{\xi_2 x} + \frac{e^{\xi_1 x} x}{\sigma^2(\tilde{\nu} + \xi_1)}, \tag{F.120}$$

Using the boundaries conditions we have that

$$A_{1,1} = \left(\frac{e^{\xi_1\underline{x}}\underline{x}}{\sigma^2(\tilde{\nu} + \xi_1)}\right)\overline{\alpha}_2 - \left(\frac{e^{\lambda_1\overline{x}}\overline{x}}{\sigma^2(\tilde{\nu} + \xi_1)}\right)\underline{\alpha}_2, \tag{F.121}$$

$$A_{2,1} = \left(\frac{e^{\xi_1\overline{x}}\overline{x}}{\sigma^2(\tilde{\nu} - \xi_1)}\right)\underline{\alpha}_1 - \left(\frac{e^{\xi_1\underline{x}}\underline{x}}{\sigma^2(\tilde{\nu} + \xi_1)}\right)\overline{\alpha}_1. \tag{F.122}$$

Putting all the results together

$$v(x) = -\frac{-e^{\xi_1 x}(e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_2 - e^{\lambda_1\overline{x}}\overline{x}\underline{\alpha}_2) - e^{\xi_2 x}\left(e^{\xi_1\overline{x}}\overline{x}\underline{\alpha}_1 - e^{\xi_1\underline{x}}\underline{x}\overline{\alpha}_1\right) + e^{\xi_1 x} x}{\sigma^2(\tilde{\nu} + \xi_1)} \tag{F.123}$$

□

Verification of $\frac{\mathbb{E}[\Delta x]}{\mathbb{E}[\tau]} = -\nu$. From proposition F.2, we have that

$$\begin{aligned}\mathbb{E}^{\hat{x}}[\Delta x] &= -e^{\xi_1 \hat{x}}\left[\overline{\alpha}_2 \overline{\kappa}_0^1(\hat{x}-\underline{x}) - \underline{\alpha}_2 \overline{\kappa}_0^1(\hat{x}-\overline{x})\right] - e^{\xi_2 \hat{x}}\left[\underline{\alpha}_1 \overline{\kappa}_0^1(\hat{x}-\overline{x}) - \overline{\alpha}_1 \overline{\kappa}_0^1(\hat{x}-\underline{x})\right] + \overline{\kappa}_0^1(0) \\ &= -\frac{\tilde{\nu}}{\tilde{\lambda}}\left[-e^{\xi_1 \hat{x}}\left[\overline{\alpha}_2 - \underline{\alpha}_2\right] - e^{\xi_2 \hat{x}}\left[\underline{\alpha}_1 - \overline{\alpha}_1\right] + 1\right], \end{aligned} \tag{F.124}$$

where we have used that

$$\begin{aligned} &= \overline{\kappa}_0^1(\hat{x}-\underline{x}) = \overline{\kappa}_0^1(\hat{x}-\overline{x}) = \overline{\kappa}_0^1(0) \\ &= \frac{\xi_1 - \xi_1\xi_2 \frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1 - \xi_2}(-\xi_1)^{-1} + \frac{\xi_2 - \xi_1\xi_2 \frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2 - \xi_1}(-\xi_2)^{-1} \\ &= \frac{-1 + \xi_2 \frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1 - \xi_2} + \frac{1 - \xi_1 \frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1 - \xi_2} \\ &= -\frac{\tilde{\nu}}{\tilde{\lambda}} \end{aligned} \tag{F.125}$$

From proposition F.5, we have that

$$\mathbb{E}^{\hat{x}}[e^{\varphi\tau}\Delta x^0] = \frac{\lambda}{\lambda-\varphi}\overline{\kappa}_0^0(0) + \frac{\varphi}{\lambda-\varphi}\left[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi) - \underline{\alpha}_2(\varphi)\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi) - \overline{\alpha}_1(\varphi)\right]\right] \tag{F.126}$$

$$= \frac{\lambda}{\lambda-\varphi} + \frac{\varphi}{\lambda-\varphi}\underbrace{\left[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi) - \underline{\alpha}_2(\varphi)\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi) - \overline{\alpha}_1(\varphi)\right]\right]}_{=\mathcal{H}(\tilde{\nu},\tilde{\lambda}-\frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x})}. \tag{F.127}$$

Taking derivative with respect to φ and evaluating in zero

$$\begin{aligned} &= \left.\frac{d\mathbb{E}^{\hat{x}}[e^{\varphi\tau}\Delta x^0]}{d\varphi}\right|_{\varphi=0} \\ &= \left.\left[\frac{\lambda}{(\lambda-\varphi)^2} + \frac{\lambda}{(\lambda-\varphi)^2}\mathcal{H}(\tilde{\nu},\tilde{\lambda}-\frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x}) - \frac{\varphi}{\sigma^2(\lambda-\varphi)}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda}-\frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x})\right]\right|_{\varphi=0} \\ &= \frac{1}{\lambda}\left[-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2 - \underline{\alpha}_2\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1 - \overline{\alpha}_1\right] + 1\right], \end{aligned} \tag{F.128}$$

where we used $\mathcal{H}_2(\cdot)$ to denote the derivative with respect to the second argument.

From equations (F.124) and (F.128) we have

$$\frac{\mathbb{E}[\Delta x]}{\mathbb{E}[\tau]} = -\frac{\frac{\tilde{\nu}}{\tilde{\lambda}}\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + 1\right]}{\frac{1}{\tilde{\lambda}}\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + 1\right]} \tag{F.129}$$

$$= -\nu \tag{F.130}$$

Verification of $\sigma^2 = \frac{\mathbb{E}[\Delta x^2]}{\mathbb{E}[\tau]} + \frac{2\nu\mathbb{E}[\tau\Delta x]}{\mathbb{E}[\tau]} + \nu^2\frac{\mathbb{E}[\tau^2]}{\mathbb{E}[\tau]}$. From proposition F.2, we have that

$$\begin{aligned}\mathbb{E}^{\hat{x}}[\Delta x^2] &= -e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\overline{\kappa}_1^2(\hat{x}-\underline{x}) - \underline{\alpha}_2\overline{\kappa}_1^2(\hat{x}-\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\overline{\kappa}_1^2(\hat{x}-\overline{x}) - \overline{\alpha}_1\overline{\kappa}_1^2(\hat{x}-\underline{x})\right] + \overline{\kappa}_0^2(0) \\ &= 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + \frac{1}{2\tilde{\lambda}}\right]\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + 1\right] - 2\frac{\tilde{\nu}}{\tilde{\lambda}}\underbrace{\left[-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2(\hat{x}-\underline{x}) - \underline{\alpha}_2(\hat{x}-\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\hat{x}-\overline{x}) - \overline{\alpha}_1(\hat{x}-\underline{x})\right]\right]}_{=\mathcal{B}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})} \\ &= 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + \frac{1}{2\tilde{\lambda}}\right]\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + 1\right] - 2\frac{\tilde{\nu}}{\tilde{\lambda}}\mathcal{B}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}), \end{aligned} \tag{F.131}$$

where we have used that

$$\begin{aligned} &= \overline{\kappa}_1^2(x) \\ &= 2\left[\frac{\xi_1 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1-\xi_2}(-\xi_1)^{-2} + \frac{\xi_2 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2-\xi_1}(-\xi_2)^{-2}\right] + 2\left[\frac{\xi_1 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1-\xi_2}(-\xi_1)^{-1} + \frac{\xi_2 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2-\xi_1}(-\xi_2)^{-1}\right]x \\ &= 2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 + \frac{1}{2\tilde{\lambda}}\right] - 2\frac{\tilde{\nu}}{\tilde{\lambda}}x \end{aligned} \tag{F.132}$$

From proposition F.5 evaluating in $m = 0$

$$
\begin{aligned}
&= \frac{d^2\mathbb{E}^{\hat{x}}[e^{\varphi\tau}\Delta x^0]}{d\varphi^2}\bigg|_{\varphi=0} \\
&= \left[\frac{2\lambda}{(\lambda-\varphi)^3} + \frac{2\lambda\mathcal{H}(\tilde{\nu},\tilde{\lambda}-\frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x})}{(\lambda-\varphi)^3} - 2\frac{\lambda\mathcal{H}_2(\tilde{\nu},\tilde{\lambda}-\frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x})}{\sigma^2(\lambda-\varphi)^2} + \frac{\varphi}{\sigma^4(\lambda-\varphi)}\mathcal{H}_{2^2}(\tilde{\nu},\tilde{\lambda}-\frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x})\right]\bigg|_{\varphi=0} \\
&= \frac{2}{\lambda^2}\left[1+\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right] - \frac{2}{\lambda\sigma^2}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}). \qquad \text{(F.133)}
\end{aligned}
$$

From proposition F.5 evaluating in $m = 1$

$$
\begin{aligned}
\mathbb{E}^{\hat{x}}[e^{\varphi\tau}\Delta x] &= \frac{\lambda}{\lambda-\varphi}\Big[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\overline{\kappa}_0^1(\hat{x}-\underline{x},\varphi)-\underline{\alpha}_2(\varphi)\overline{\kappa}_0^1(\hat{x}-\overline{x},\varphi)\right]\dots \\
&\quad\dots - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\overline{\kappa}_0^1(\hat{x}-\overline{x},\varphi)-\overline{\alpha}_1(\varphi)\overline{\kappa}_0^1(\hat{x}-\underline{x},\varphi)\right]+\overline{\kappa}_0^1(0)\Big]\dots \\
&\quad\dots + \frac{\varphi}{\lambda-\varphi}\left[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)(\hat{x}-\underline{x})-\underline{\alpha}_2(\varphi)(\hat{x}-\overline{x})\right]-e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)(\hat{x}-\overline{x})-\overline{\alpha}_1(\varphi)(\hat{x}-\underline{x})\right]\right] \\
&= -\frac{\lambda}{\lambda-\varphi}\frac{\tilde{\nu}}{\tilde{\lambda}}\left[1+\mathcal{H}(\tilde{\nu},\tilde{\lambda}-\frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x})\right]+\frac{\varphi}{\lambda-\varphi}\mathcal{B}(\tilde{\nu},\tilde{\lambda}-\frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x}) \\
&= -\frac{\lambda\nu}{(\lambda-\varphi)^2}\left[1+\mathcal{H}(\tilde{\nu},\tilde{\lambda}-\frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x})\right]+\frac{\varphi}{\lambda-\varphi}\mathcal{B}(\tilde{\nu},\tilde{\lambda}-\frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x}) \qquad \text{(F.134)}
\end{aligned}
$$

and taking derivatives and evaluating at zero, we have

$$
\mathbb{E}^{\hat{x}}[\tau\Delta x] = -2\frac{\nu}{\lambda^2}\left[1+\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right]-\frac{\nu}{\lambda\sigma^2}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})+\frac{1}{\lambda}\mathcal{B}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}). \qquad \text{(F.135)}
$$

Using equations (F.124) to (F.135)

$$
\begin{aligned}
&= \frac{\mathbb{E}[\Delta x^2]}{\mathbb{E}[\tau]} + \frac{2\nu\mathbb{E}[\tau\Delta x]}{\mathbb{E}[\tau]} + \nu^2\frac{\mathbb{E}[\tau^2]}{\mathbb{E}[\tau]} \qquad \text{(F.136)} \\
&= \frac{2\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2+\frac{1}{2\tilde{\lambda}}\right]\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})+1\right]-2\frac{\tilde{\nu}}{\tilde{\lambda}}\mathcal{B}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})}{\frac{1}{\lambda}\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})+1\right]}\dots \\
&\quad\dots + 2\nu\frac{-2\frac{\nu}{\lambda^2}\left[1+\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right]-\frac{\nu}{\lambda\sigma^2}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})+\frac{1}{\lambda}\mathcal{B}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})}{\frac{1}{\lambda}\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})+1\right]}\dots \\
&\quad + \nu^2\frac{\frac{2}{\lambda^2}\left[1+\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right]-\frac{2}{\lambda\sigma^2}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})}{\frac{1}{\lambda}\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})+1\right]} \\
&= 2\lambda\left[\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2+\frac{1}{2\tilde{\lambda}}\right]-4\lambda\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2+2\lambda\left(\frac{\tilde{\nu}}{\tilde{\lambda}}\right)^2 \\
&= \sigma^2 \qquad \text{(F.137)}
\end{aligned}
$$

Verification that if $\hat{x} = \frac{\mathbb{E}[\tau\Delta x]}{\mathbb{E}[\tau]} + \nu\frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]}$, then $\mathcal{M}_1[x] = 0$. From equations (F.133) and (F.135)

$$
\begin{aligned}
\hat{x} &= \frac{\mathbb{E}[\tau\Delta x]}{\mathbb{E}[\tau]} + \nu\frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]} \\
&= \frac{-2\frac{\nu}{\lambda^2}\left[1+\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right]-\frac{\nu}{\lambda\sigma^2}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})+\frac{1}{\lambda}\mathcal{B}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})}{\frac{1}{\lambda}\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})+1\right]}\dots \\
&\quad + \nu\frac{\frac{2}{\lambda^2}\left[1+\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right]-\frac{2}{\lambda\sigma^2}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})}{\frac{2}{\lambda}\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})+1\right]} \\
&= -\frac{\tilde{\nu}}{\tilde{\lambda}} + \frac{\mathcal{B}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})}{\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})+1\right]} \qquad \text{(F.138)}
\end{aligned}
$$

From proposition F.3, we have that

$$\mathcal{M}_1[x] = \frac{-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(\underline{x})\right] + \kappa_1^1(\hat{x})}{\lambda\mathbb{E}[\tau]}. \tag{F.139}$$

Since the $\kappa_1^1(x)$ satisfies

$$\kappa_1^1(x) = \frac{1!}{0!}\left[\frac{\xi_1 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1 - \xi_2}(\xi_1)^{-1} + \frac{\xi_2 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2 - \xi_1}(\xi_2)^{-1}\right] + \frac{1!}{1!}\left[\frac{\xi_1 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_1 - \xi_2} + \frac{\xi_2 - \xi_1\xi_2\frac{\tilde{\nu}}{\tilde{\lambda}}}{\xi_2 - \xi_1}\right]x \tag{F.140}$$

$$= \frac{\tilde{\nu}}{\tilde{\lambda}} + x \tag{F.141}$$

Using this result

$$\begin{aligned}
\mathcal{M}_1[x] &= \frac{-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\kappa_1^1(\underline{x}) - \underline{\alpha}_2\kappa_1^1(\overline{x})\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\kappa_1^1(\overline{x}) - \overline{\alpha}_1\kappa_1^1(\underline{x})\right] + \kappa_1^1(\hat{x})}{\lambda\mathbb{E}[\tau]} \\
&= \frac{-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2 - \underline{\alpha}_2\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1 - \overline{\alpha}_1\right] + 1}{\lambda\mathbb{E}[\tau]} + \frac{-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x}\right)] + \hat{x}}{\lambda\mathbb{E}[\tau]} \\
&= \frac{\frac{\tilde{\nu}}{\tilde{\lambda}}\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + 1\right]}{\frac{\lambda}{\tilde{\lambda}}\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + 1\right]} + \frac{-\left[\mathcal{B}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right] + \hat{x}\left(-e^{\xi_1\hat{x}}\left[\overline{\alpha}_2\underline{x} - \underline{\alpha}_2\overline{x}\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1\overline{x} - \overline{\alpha}_1\underline{x}\right)]\right) + \hat{x}}{\frac{\lambda}{\tilde{\lambda}}\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + 1\right]} \\
&= \frac{\tilde{\nu}}{\tilde{\lambda}} - \frac{\mathcal{B}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})}{\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + 1\right]} + \hat{x} \\
&= 0
\end{aligned} \tag{F.142}$$

Observability with respect to $\mathcal{M}_1[a]$. We need to show that $\mathcal{M}_1[a] = \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]}$. Equations (F.128) and (F.133) shows the functional form for $\mathbb{E}[\tau]$ and $\mathbb{E}[\tau^2]$. Using these two equations we have that

$$\frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]} = \frac{\frac{2}{\lambda^2}\left[1 + \mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right] - \frac{2}{\lambda\sigma^2}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})}{2\frac{1}{\lambda}\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + 1\right]} = \frac{\frac{1}{\lambda}\left[1 + \mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right] - \frac{1}{\sigma^2}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})}{\left[\mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + 1\right]}. \tag{F.143}$$

From proposition F.6 we have that

$$\begin{aligned}
\mathcal{M}_{0,1}[x,a] &= \frac{h_0'(0)}{\mathbb{E}^{\hat{x}}[\tau]} \\
h_m(\varphi) &= \frac{1}{\lambda - \varphi}\Big[-e^{\xi_1(\varphi)\hat{x}}\left[\overline{\alpha}_2(\varphi)\kappa_0^0(\underline{x},\varphi) - \underline{\alpha}_2(\varphi)\kappa_0^0(\overline{x},\varphi)\right] - e^{\xi_2\hat{x}}\left[\underline{\alpha}_1(\varphi)\kappa_0^0(\overline{x},\varphi) - \overline{\alpha}_1(\varphi)\kappa_0^0(\underline{x},\varphi)\right] + \overline{\kappa}_0^0(\hat{x})\Big] \\
&= 1 + \mathcal{H}(\tilde{\nu},\tilde{\lambda} - \frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x}),
\end{aligned} \tag{F.144}$$

Where we have that

$$h_0'(0) = \frac{1}{\lambda - \varphi}\left(1 + \mathcal{H}(\tilde{\nu},\tilde{\lambda} - \frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x})\right)\Big|_{\varphi=0} \tag{F.145}$$

$$= \frac{1}{(\lambda - \varphi)^2}\left(1 + \mathcal{H}(\tilde{\nu},\tilde{\lambda} - \frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x})\right) + \frac{1}{(\lambda - \varphi)\sigma^2}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda} - \frac{\varphi}{\sigma^2},\hat{x},\underline{x},\overline{x})\Big|_{\varphi=0} \tag{F.146}$$

$$= \frac{1}{\lambda^2}\left(1 + \mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right) - \frac{1}{\lambda\sigma^2}\mathcal{H}_2(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}). \tag{F.147}$$

Therefore, using (F.143) we have that

$$\mathcal{M}_{0,1}[x,a] = \mathcal{M}_1[a] = \frac{\frac{1}{\lambda^2}\left(1 + \mathcal{H}(\tilde{\nu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})\right) - \frac{1}{\lambda\sigma^2}\mathcal{H}_2(\tilde{\mu},\tilde{\lambda},\hat{x},\underline{x},\overline{x})}{\frac{1}{\lambda}\left[\mathcal{H}(\tilde{\mu},\tilde{\lambda},\hat{x},\underline{x},\overline{x}) + 1\right]} = \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]} \tag{F.148}$$

F.2 Lumpy Investment in the Calvo-Taylor Model

This section describe briefly the environment with the change in the investment adjustment policy.

Environment. The main change in the environment is the decision problem of the firms, all other problems remain the same. Let $\{\tau_i\}$ be a sequence of stopping time describing the adjusment dates. We assume that $\tau_{i+1} = \inf_t\{t \geq t \geq \tau_i : \tilde{N}_{zs} - \tilde{N}_{zt} \geq 1 \; or \; t - \tau_i = T\}$, where $\{\tilde{N}\}$ is a Poisson counter with unit increase and arrival rate λ and T is fixed parameter. The firm's problem is to choose investment rates $\Delta K_{\tau_{zi}} = K_{\tau_{zi}} - K_{\tau_{zi}^-}$ that solve the following problem

$$\max_{\{\Delta K_{\tau_{\omega,i}}\}_{i=1}^{\infty}} \mathbb{E}\left[\int_0^{\infty} Q_s Y_{\omega,s}\, ds - \sum_{i=1}^{\infty} Q_{\tau_{\omega,i}}\left(\Delta K_{\tau_{\omega,i}}\right)\middle| \mathcal{F}_t\right], \tag{F.149}$$

where output is given by (1) and E_{zt} follows (2).

Controlled States. Let $S_t = (k_t, N_t, a_t)$ be the controlled state of the firm. k_t follows a Brownian motion with drift given by

$$dk_t = \nu dt + \sigma dW_t, \tag{F.150}$$

where W_t is a Brownian motion. N_t is a Poisson counter with arrival rate λ and a_t a satisfies the law of motion $da_t = dt$. The reset state is given by $\hat{S} = (\hat{k}, 0, 0)$ and the stopping time is given by

$$\tau = \inf\{t \geq 0 : N_t \geq 1 \; or \; a \geq T\} \tag{F.151}$$

Notice that this model has only 4 parameters: $(\nu, \sigma, \lambda, T)$ and the distribution of stopping times is independent of k state and it is given by

$$g(\tau) = \lambda e^{-\lambda\tau} I(\tau < T), \quad with \quad Pr(\tau = T) = e^{-\lambda T} \tag{F.152}$$

Policy. Using the assumption of constant real interest rate and writing (F.149) recursively with the capital-gaps as a state, we have that

$$\tilde{v}(k) = \mathbb{E}_0\left[\int_0^{\tau} e^{-\left(\rho-\mu-\sigma^2/2\right)} e^{\alpha k_s} ds + e^{-(\rho-\mu)\tau}\left[\max_{\hat{y}}\left\{-\left[e^{\hat{k}} - e^{k_\tau}\right] + \tilde{v}(\hat{k})\right]\right\}\right]. \tag{F.153}$$

Using the result above, the reset capital gap for teh adjusting firm $\hat{k}$, can be written as

$$\hat{k} = \arg\max_{\hat{k}}\left\{\mathbb{E}\left[\int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/2)t}\left(e^{\alpha\left(\hat{k}+\nu t+\sigma B_t\right)} + \lambda e^{\hat{k}+\nu t+\sigma_a B_t} + \left(v(\hat{k}) - \hat{k}\right)\right) dt\right] - e^{\hat{k}}\right\} \tag{F.154}$$

The next proposition characterizes the reset capital gap in the Calvo Taylor model.

Proposition F.8. *The reset state is given by*

$$\hat{k} = \log\left[\left(\frac{\alpha\frac{1-e^{-(\rho+\lambda-\mu-\sigma^2/2-\alpha\nu-\frac{\sigma^2\alpha^2}{2})T}}{\rho+\lambda-\mu-\sigma^2/2-\alpha\nu-\frac{\sigma^2\alpha^2}{2}}}{1-\lambda\frac{1-e^{-(\rho+\lambda-\mu-\sigma^2/2-\nu-\frac{\sigma^2}{2})T}}{\rho+\lambda-\mu-\sigma^2/2-\nu-\frac{\sigma^2}{2}}}\right)^{\frac{1}{1-\alpha}}\right] \tag{F.155}$$

Proof. Operating over the expectation

$$\begin{aligned}\mathbb{E}\left[\int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/2)t+\alpha\left(\hat{k}+\nu t+\sigma W_t\right)} dt\right] &= \int_0^T \mathbb{E}\left[e^{-(\rho+\lambda-\mu-\sigma^2/2)t+\alpha\left(\hat{k}+\nu t+\sigma W_t\right)}\right] dt \\ &= \int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/2-\alpha\nu-\frac{\alpha^2\sigma^2}{2})t+\alpha\hat{k}} dt \end{aligned} \tag{F.156}$$

$$\mathbb{E}\left[\int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/2)t+\left(\hat{k}+\nu t+\sigma B_t\right)} dt\right] = \int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/2-\nu-\frac{\sigma^2}{2})t+\hat{k}} dt \tag{F.157}$$

Therefore the optimality conditions for the reset state in the Calvo-Taylor model is given by

$$\hat{k} = \arg\max_{\hat{k}}\left[\int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/-\alpha\nu-\frac{\alpha^2\sigma^2}{2})t+\alpha\hat{k}} dt + \lambda\int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/2-\nu-\frac{\sigma^2}{2})t+\hat{k}} dt - e^{\hat{k}}\right]. \tag{F.158}$$

Taking FOC we have that

$$\alpha e^{\alpha\hat{k}}\int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/2-\alpha\nu-\frac{\alpha^2\sigma^2}{2})t} dt + \lambda e^{\hat{k}}\int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/2-\nu-\frac{\sigma^2}{2})t} dt - e^{\hat{k}} = 0, \tag{F.159}$$

and operating we have that

$$e^{(\alpha-1)\hat{k}} = \frac{1-\lambda\int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/2-\nu-\frac{\sigma_2^2}{2})t}dt}{\alpha\int_0^T e^{-(\rho+\lambda-\mu-\sigma^2/2-\alpha\nu-\frac{\alpha^2\sigma_a^2}{2})t}dt}, \tag{F.160}$$

or equivalently, (F.155). □

Re-normalization We normalize the policy function in the model to work with centralized moments. We redefine the reset state $\hat{x}$ in the following way

$$\hat{x} = \hat{k} - \mathcal{M}_1[k], \tag{F.161}$$

where $\mathcal{M}_1[k]$ is the mean capital-gap under the the firms' policies $\hat{k}$. It is easy to check that under the re-normalized policy $\mathcal{M}_1[x] = 0$.

Moments to verify observation property in the structural parameters and reset state The next proposition computes all the inputs to verify observability in the structural parameters: $\mathbb{E}[\tau]$, $\mathbb{E}[\tau^2]$,$\mathbb{E}[\tau\Delta x]$, $\mathbb{E}[\Delta x]$,$\mathbb{E}[\Delta x^2]$.

Proposition F.9. *In the CalvoTaylor model the following relationships hold*

$$\mathbb{E}\left[\tau^n\right] = \frac{n!\left[1-e^{-\lambda T}\sum_{i=0}^{n-1}\frac{(\lambda T)^i}{i!}\right]}{\lambda^n} \tag{F.162}$$

$$\mathbb{E}\left[\Delta x^n\right] = \sum_{k=0,k\ even}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)(-\nu)^{n-k}\sigma^k(k-1)!\mathbb{E}[\tau^{n-k/2}] \tag{F.163}$$

$$\mathbb{E}[\tau\Delta x] = -\nu\mathbb{E}[\tau^2] \tag{F.164}$$

Proof. In this proof, we will use the incomplete gamma function $\Gamma(\cdot)$, and recall that $\Gamma(n+1,x) = n!e^{-x}\sum_{k=0}^{n}\frac{x^k}{k!}$. By definition we have that

$$\mathbb{E}\left[\tau^n\right] = Pr(\tau=T)T^n + \lambda\int_0^T s^n e^{-\lambda s}ds = e^{-\lambda T}T^n + \left[-\lambda^{-n}\Gamma(n+1,\lambda x)\right]_0^T \tag{F.165}$$

$$= e^{-\lambda T}T^n + \lambda^{-n}\left[\Gamma(n+1,0) - \Gamma(n+1,\lambda T)\right] \tag{F.166}$$

$$= e^{-\lambda T}T^n + \lambda^{-n}\left[n! - n!e^{-\lambda T}\sum_{k=0}^{n}\frac{(\lambda T)^k}{k!}\right] \tag{F.167}$$

$$= e^{-\lambda T}T^n + \lambda^{-n}\left[n! - n!e^{-\lambda T}\sum_{k=0}^{n-1}\frac{(\lambda T)^k}{k!}\right] - n!e^{-\lambda T}\lambda^{-n}\frac{(\lambda T)^n}{n!} \tag{F.168}$$

$$= \frac{n!\left[1-e^{-\lambda T}\sum_{i=0}^{n-1}\frac{(\lambda T)^i}{i!}\right]}{\lambda^n} \tag{F.169}$$

To compute the moment generating function of the changes, note that by definition

$$\mathbb{E}\left[\Delta x^n\right] = \mathbb{E}\left[(-\nu\tau + \sigma B_\tau)^n\right] \tag{F.170}$$

$$= \mathbb{E}\left[\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)(-\nu\tau)^{n-k}(\sigma B_\tau)^k\right] \tag{F.171}$$

$$= \mathbb{E}\left[\sum_{k=0,k\ even}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)(-n\tau)^{n-k}(\sigma B_\tau)^k\right], \tag{F.172}$$

where in the last step we use that $\mathbb{E}\left[\tau^{n-k}B_\tau^k\right] = 0$ for k odd, since $t^{n-k}B_t^k$ is a martingale for k odd with zero initial condition and therefore $\tau^{n-k}B_\tau^k$ is a martingale. Thus

$$\mathbb{E}\left[\Delta x^n\right] = \mathbb{E}\left[\sum_{k=0,k\ even}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)(-\nu\tau)^{n-k}(\sigma B_\tau)^k\right] = \sum_{k=0,k\ even}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)(-\nu)^{n-k}\sigma^k\mathbb{E}\left[\tau^{n-k}B_\tau^k\right], \tag{F.173}$$

where B_τ is a normal random variable with mean 0 and variance τ. Using this property it is easy to check that $\mathbb{E}[\tau^{n-k}B_\tau^k] = \mathbb{E}[\tau^{n-k}\mathbb{E}[B_t^k|t=\tau]] = \mathbb{E}[\tau^{n-k}\tau^{k/2}(k-1)!]$, and therefore

$$\mathbb{E}\left[\Delta x^n\right] = \sum_{k=0,k\ even}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)(-\nu)^{n-k}\sigma^k(k-1)!\mathbb{E}[\tau^{n-k/2}]. \tag{F.174}$$

Finally with similar arguments as before

$$\mathbb{E}[\tau \Delta x] = (\psi + \mu)\mathbb{E}[\tau^2]. \tag{F.175}$$

□

Distribution of the State. To verify observability and representability, we first compute the ergodic distribution of the state. Let $h(x, a)$ denote the joint distribution of the capita gap and the time since the last adjustment and $f(x)$ the marginal distribution over capital gaps. The following proposition characterizes both objects.

Proposition F.10. *The distribution of the state and the marginal distribution of the capital-gap are given by*

$$h(x,a) = \frac{\lambda \exp\left(-\lambda a - \frac{(x-\nu a-\hat{x})^2}{2\sigma^2 a}\right)}{(1-e^{-\lambda T})\sqrt{2\pi\sigma^2 a}} \tag{F.176}$$

$$f(x) = \frac{\lambda}{(1-e^{-\lambda T})\sqrt{2\sigma^2}} \mathcal{H}_{\sqrt{\lambda+\frac{\nu^2}{2\sigma^2}}, \frac{-\nu}{\sqrt{2\sigma^2}}, T}\left(\frac{x-\hat{x}}{\sqrt{2\sigma^2}}\right) \tag{F.177}$$

$$H_{z,p,T}(x) = \frac{e^{-2(|x|z+px)}}{2|z|}\left[1 - erf\left(\frac{|x|-Tz}{\sqrt{T}}\right) + e^{4|x|z}\left(erf\left(\frac{|x|+Tz}{\sqrt{T}}\right) - 1\right)\right], \tag{F.178}$$

here $erf(x)$ is the error function.

Proof. To compute the distribution of relative price we need to extend the firm's state space. Let (a, x_t) be the state of the firm where a is the time since the last adjustment and follows the process $da = dt$ (notice that N has a probability atom in $N = 0$ of 1). Let $h(x, a)$ be the ergodic distribution. Then $h(x, a)$ satisfies

$$\lambda h(\hat{x}, a) = -\frac{\partial h(x,a)}{\partial a} - \nu \frac{\partial h(x,s)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 h(x,a)}{\partial x^2} \quad (a,x) \in (0,T) \times \mathbb{R} \tag{F.179}$$

with the border conditions $\lim_{t\to\infty} h(x,a) = \lim_{t\to-\infty} h(x,a) = 0$, $\int\int h(x,a)dxda = 1$ and the border condition $\lim_{a\to 0} u(x,a) = \delta_d(x-\hat{x})$ where $\delta_d(\hat{x})$ is the Direc delta function. Let $h(x,a) = \alpha e^{-\lambda a}\tilde{h}(x,a)$. It is easy to see that h is a solution for some α iff $\tilde{h}(x,a)$ satisfies the standard heat equation

$$\frac{\partial \tilde{h}(x,a)}{\partial a} = -\nu \frac{\partial \tilde{h}(x,a)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 \tilde{h}(x,a)}{\partial x^2} \quad (a,\hat{x}) \in (0,T) \times \mathbb{R}. \tag{F.180}$$

The solution is given by

$$\tilde{h}(x,a) = A\frac{\int_{-\infty}^{\infty} \delta_d(z-\hat{x}) \exp\left(-\frac{(x-\nu a-z)^2}{2\sigma^2 a}\right) dz}{\sqrt{a}} = A\frac{\exp\left(-\frac{(x-\nu a-\hat{x})^2}{2\sigma^2 a}\right)}{\sqrt{a}}. \tag{F.181}$$

Therefore

$$h(x,a) = A\frac{\exp\left(-\lambda a - \frac{(x-\nu a-\hat{x})^2}{2\sigma^2 a}\right)}{\sqrt{a}}, \tag{F.182}$$

and using the border condition

$$h(x,a) = \frac{\lambda \exp\left(-\lambda a - \frac{(x-\nu a-\hat{x})^2}{2\sigma^2 a}\right)}{(1-e^{-\lambda T})\sqrt{2\pi\sigma^2 a}}. \tag{F.183}$$

The marginal distribution of the estate is given by

$$\begin{aligned} f(x) &= \int_0^T h(x,a)da \\ &= \frac{\lambda}{(1-e^{-\lambda T})\sqrt{2\sigma^2}}\left[\int_0^T \frac{\exp\left(-\lambda a - \frac{(x-\nu a-\hat{x})^2}{2\sigma^2 a}\right)}{\sqrt{\pi a}} da\right] \\ &= \frac{\lambda}{(1-e^{-\lambda T})\sqrt{2\sigma^2}} \mathcal{H}_{\sqrt{\lambda+\frac{\nu^2}{2\sigma^2}}, \frac{-\nu}{\sqrt{2\sigma^2}}, T}\left(\frac{x-\hat{x}}{\sqrt{2\sigma^2}}\right) \end{aligned} \tag{F.184}$$

where $\mathcal{H}_{z,p,y}(x)$ satisfies (when $x \neq 0$)

$$
\begin{aligned}
&= H_{z,p,y}(x) && \text{(F.185)}\\
&= \int_0^y \frac{\exp\left(-(z^2-p^2)s - \frac{(x+ps)^2}{s}\right)}{\sqrt{\pi s}} ds \\
&= \frac{e^{-2(|x||z|+px)}}{2|z|}\left[1 - erf\left(\frac{|x|-sz}{\sqrt{s}}\right) + e^{4|x||z|}\left(erf\left(\frac{|x|+sz}{\sqrt{s}}\right)-1\right)\right]\Bigg|_{s=0}^{s=y} \\
&= \frac{e^{-2(|x||z|+px)}}{2|z|}\left[1 - erf\left(\frac{|x|-yz}{\sqrt{y}}\right) + e^{4|x|z}\left(erf\left(\frac{|x|+yz}{\sqrt{y}}\right)-1\right)\right] - \dots \\
&\cdots - \frac{e^{-2(|x||z|+px)}}{2|z|}\left[1 - \lim_{K\to\infty} erf(K) + e^{4|x|z}\left(\lim_{K\to\infty} erf(K) - 1\right)\right] \\
&= \frac{e^{-2(|x|z+px)}}{2|z|}\left[1 - erf\left(\frac{|x|-yz}{\sqrt{y}}\right) + e^{4|x|z}\left(erf\left(\frac{|x|+yz}{\sqrt{y}}\right)-1\right)\right], && \text{(F.186)}
\end{aligned}
$$

where in the last equality we used the result that $\lim_{K\to\infty} erf(K) = 1$. In the case $x = 0$, we have that

$$1 - erf(0) + e^0 (erf(0) - 1) = 0 \tag{F.187}$$

Thus, we have the result. □

Observability of structural parameters. With the results in Proposition F.9, we can check observability for the structural parameters.

- For the drift ν note that

$$\mathbb{E}[\tau] = \frac{1-e^{-\lambda T}}{\lambda} \tag{F.188}$$

$$\mathbb{E}[\Delta x] = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (-\nu)^1 (0-1)! \mathbb{E}[\tau^1] = -\nu \frac{1-e^{-\lambda T}}{\lambda} \tag{F.189}$$

and therefore

$$\frac{\mathbb{E}[\Delta x]}{\mathbb{E}[\tau]} = -\nu \frac{\frac{1-e^{-\lambda T}}{\lambda}}{\frac{1-e^{-\lambda T}}{\lambda}} = (\psi + \mu). \tag{F.190}$$

- For the volatility σ^2 note that

$$\mathbb{E}[\tau^2] = \frac{2\left(1 - e^{-\lambda T}(1+\lambda T)\right)}{\lambda^2} \tag{F.191}$$

$$\mathbb{E}[\Delta x \tau] = -\nu \mathbb{E}[\tau^2] = -\nu \frac{2\left(1 - e^{-\lambda T}(1+\lambda T)\right)}{\lambda^2} \tag{F.192}$$

$$\mathbb{E}[\Delta x^2] = \begin{pmatrix} 2 \\ 0 \end{pmatrix} (-\nu)^{2-0} \sigma^0 (0-1)! \mathbb{E}[\tau^2] + \begin{pmatrix} 2 \\ 2 \end{pmatrix} (-\nu)^{2-2} \sigma^2 (2-1)! \mathbb{E}[\tau^1] \tag{F.193}$$

$$= \nu^2 \mathbb{E}[\tau^2] + \sigma^2 \mathbb{E}[\tau] \tag{F.194}$$

$$= \nu^2 \frac{2\left(1 - e^{-\lambda T}(1+\lambda T)\right)}{\lambda^2} + \sigma^2 \frac{1-e^{-\lambda T}}{\lambda} \tag{F.195}$$

and therefore

$$
\begin{aligned}
\sigma^2 &= \frac{\mathbb{E}[\Delta x^2]}{\mathbb{E}[\tau]} - 2\frac{\mathbb{E}[\Delta x]\mathbb{E}[\Delta x \tau]}{\mathbb{E}[\tau]^2} + \frac{\mathbb{E}[\Delta x]^2 \mathbb{E}[\tau^2]}{\mathbb{E}[\tau]^3} \\
&= \nu^2 \frac{2\left(1 - e^{-\lambda T}(1+\lambda T)\right)}{\lambda\left(1 - e^{-\lambda T}\right)} + \sigma^2 - \nu^2 \frac{\frac{\left(1-e^{-\lambda T}\right)4\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda^3}}{\frac{\left(1-e^{-\lambda T}\right)^2}{\lambda^2}} + \nu^2 \frac{\frac{\left(1-e^{-\lambda T}\right)^2 2\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda^4}}{\frac{\left(1-e^{-\lambda T}\right)^3}{\lambda^3}} \\
&= \sigma^2 + 2\nu^2 \frac{2\left(1 - e^{-\lambda T}(1+\lambda T)\right)}{\lambda\left(1 - e^{-\lambda T}\right)} - 2\nu^2 \frac{2\left(1 - e^{-\lambda T}(1+\lambda T)\right)}{\lambda\left(1 - e^{-\lambda T}\right)} && \text{(F.196)}\\
&= \sigma^2 && \text{(F.197)}
\end{aligned}
$$

- For the reset state $\hat{x}$, the verification is a little more involved. Notice that if we apply the formula in the main text

we have that

$$\hat{x} = \frac{\mathbb{E}[\Delta x \tau]}{\mathbb{E}[\tau]} - \frac{\mathbb{E}[\Delta x]\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]^2} = \nu \frac{(1 - e^{-\lambda T}(1 + \lambda T))}{\lambda(1 - e^{-\lambda T})} \tag{F.198}$$

As we show $\hat{x}$ is given by the number s.t. $\int x h(x,a) dx da = 0$. As we show below $h(x,a) = \frac{\lambda \exp\left(-\lambda a - \frac{(x\nu a - \hat{x})^2}{2\sigma^2 a}\right)}{(1-e^{-\lambda T})\sqrt{2\pi\sigma^2 a}}$, therefore

$$\begin{aligned}
0 = \int x h(x,a) dx da &= \int_a \frac{\lambda \exp(-\lambda a)}{1 - e^{-\lambda T}} \int_x x \frac{\exp\left(-\frac{(x - \nu a - \hat{x})^2}{2\sigma^2 a}\right)}{\sqrt{2\pi\sigma^2 a}} dx da \\
&= -\int_0^T \frac{\lambda \exp(-\lambda a)}{1 - e^{-\lambda T}} (-\nu a - \hat{x})\, da \\
&= -\left[-\nu \int_0^T \frac{\lambda a \exp(-\lambda a)}{1 - e^{-\lambda T}} da - \hat{x} \int_0^T \frac{\lambda \exp(-\lambda a)}{1 - e^{-\lambda T}} da\right] \\
&= -\left[-\nu \int_0^T \frac{\lambda \left.\frac{(-e^{-\lambda a})(1+\lambda a)}{\lambda^2}\right|_0^T}{1 - e^{-\lambda T}} da - \hat{x} \left.\frac{(-e^{-\lambda a})}{1 - e^{-\lambda T}}\right|_0^T\right] \\
&= -\left[-\nu \frac{1 - e^{-\lambda T}(1 + \lambda T)}{\lambda(1 - e^{-\lambda T})} - \hat{x}\right]
\end{aligned} \tag{F.199}$$

From equations (F.198) and (F.199) we get the result between micro-price statistics and the zero ergodic mean of capital gaps.

Observability with respect to $\mathcal{M}_1[a]$ We need to show that $\mathcal{M}_1[a] = \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]}$. Using the joint distribution $h(x,a)$ from (F.176)

$$\int_z a_z dz = \int_0^T \int_{-\infty}^{\infty} a \frac{\lambda \exp\left(-\lambda a - \frac{(x - \nu a - \hat{x})^2}{2\sigma^2 a}\right)}{(1 - e^{-\lambda T})\sqrt{2\pi\sigma_a^2 a}} dx da = \int_0^T a \frac{\lambda \exp(-\lambda a)}{(1 - e^{-\lambda T})} da \tag{F.200}$$

$$= \frac{\int_0^T a \exp(-\lambda a)\, da}{\mathbb{E}[\tau]} = \frac{\left.-\frac{e^{-\lambda a}(\lambda x + 1)}{\lambda^2}\right|_{a=0}^{a=T}}{\mathbb{E}[\tau]} = \frac{1 - e^{-\lambda T}(1 + \lambda T)}{\lambda^2 \mathbb{E}[\tau]} = \frac{\mathbb{E}\left[\tau^2\right]}{2\mathbb{E}[\tau]} \tag{F.201}$$

Observability with respect to $\mathcal{M}_2[x]$ We need to verify that

$$\mathcal{M}_2[x] = \frac{\hat{x}^3 - \mathbb{E}[(\hat{x} - \Delta x)^3]}{\mathbb{E}[\Delta x]3} \tag{F.202}$$

Computing the left hand size

$$\begin{aligned}
\mathcal{M}_2[x] &= \int x^2 h(x,a) dx da \\
&= \int_a \frac{\lambda \exp(-\lambda a)}{1 - e^{-\lambda T}} \int_x x^2 \frac{\exp\left(-\frac{(x - \nu a - \hat{x})^2}{2\sigma^2 a}\right)}{\sqrt{2\pi\sigma^2 a}} dx da = \int_0^T \frac{\lambda \exp(-\lambda a)}{1 - e^{-\lambda T}} \left((\hat{x} + \nu a)^2 + \sigma^2 a\right) da \\
&= \hat{x}^2 + \frac{\lambda(\sigma^2 + 2\nu\hat{x})}{1 - e^{-\lambda T}} \int_0^T a \exp(-\lambda a)\, da - \frac{\lambda \nu^2}{1 - e^{-\lambda T}} \int_0^T a^2 \exp(-\lambda a)\, da \\
&= \hat{x}^2 + \frac{\lambda(\sigma^2 + 2\nu\hat{x})}{1 - e^{-\lambda T}} \frac{1 - e^{-\lambda T}(1 + \lambda T)}{\lambda^2} - 2\frac{\lambda \nu^2}{1 - e^{-\lambda T}} \left.\frac{e^{-\lambda a}(1 + \lambda a + \frac{(\lambda a)^2}{2})}{\lambda^3}\right|_0^T \\
&= \hat{x}^2 + \frac{(\sigma^2 + 2\nu\hat{x})\left(1 - e^{-\lambda T}(1 + \lambda T)\right)}{\lambda(1 - e^{-\lambda T})} + 2\frac{\nu^2}{1 - e^{-\lambda T}} \frac{1 - e^{-\lambda T}(1 + \lambda T + \frac{(\lambda T)^2}{2})}{\lambda^2} \\
&= \hat{x}^2 + \hat{x}\frac{2\nu\left(1 - e^{-\lambda T}(1 + \lambda T)\right)}{\lambda(1 - e^{-\lambda T})} + \frac{2\nu^2(1 - e^{-\lambda T}(1 + \lambda T + \frac{(\lambda T)^2}{2}))}{\lambda^2(1 - e^{-\lambda T})} + \frac{\sigma^2\left(1 - e^{-\lambda T}(1 + \lambda T)\right)}{\lambda(1 - e^{-\lambda T})}
\end{aligned} \tag{F.203}$$

Computing the right hand size

$$
\begin{aligned}
&= \frac{\hat{x}^3 - \mathbb{E}[(\hat{x}-\Delta x)^3]}{\mathbb{E}[\Delta x]3} \qquad \text{(F.204)}\\
&= \frac{\hat{x}^3 - \left(\hat{x}^3 - 3\hat{x}^2\mathbb{E}[\Delta x] + 3\hat{x}\mathbb{E}[\Delta x^2] - \mathbb{E}[\Delta x^3]\right)}{\mathbb{E}[\Delta x]3}\\
&= \hat{x}^2 - \hat{x}\frac{\mathbb{E}[\Delta x^2]}{\mathbb{E}[\Delta x]} + \frac{\mathbb{E}[\Delta x^3]}{3\mathbb{E}[\Delta x]}\\
&= \hat{x}^2 - \hat{x}\left[\frac{\nu^2\frac{2\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda^2} + \sigma^2\frac{1-e^{-\lambda T}}{\lambda}}{-\nu\frac{1-e^{-\lambda T}}{\lambda}}\right] + \frac{-\nu^3\mathbb{E}[\tau^3] - 3\nu\sigma^2\mathbb{E}[\tau^2]}{-3\nu\frac{1-e^{-\lambda T}}{\lambda}}\\
&= \hat{x}^2 + \hat{x}\frac{2\nu\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda\left(1-e^{-\lambda T}\right)} + \hat{x}\frac{\sigma^2}{\nu} + \frac{2\sigma^2(1-e^{-\lambda T}(1+\lambda T))}{\lambda(1-e^{-\lambda T})} + \frac{32\nu^2(1-e^{-\lambda T}(1+\lambda T+\frac{(\lambda T)^2}{2}))}{3\lambda^2(1-e^{-\lambda T})}\\
&= \hat{x}^2 + \hat{x}\frac{2\nu\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda\left(1-e^{-\lambda T}\right)} + \frac{2\nu^2(1-e^{-\lambda T}(1+\lambda T+\frac{(\lambda T)^2}{2}))}{\lambda^2(1-e^{-\lambda T})}\cdots\\
&\cdots - \nu\frac{(1-e^{-\lambda T}(1+\lambda T))}{\lambda(1-e^{-\lambda T})}\frac{\sigma^2}{\nu} + \frac{2\sigma^2(1-e^{-\lambda T}(1+\lambda T))}{\lambda(1-e^{-\lambda T})}\\
&= \hat{x}^2 + \hat{x}\frac{2\nu\left(1-e^{-\lambda T}(1+\lambda T)\right)}{\lambda\left(1-e^{-\lambda T}\right)} + \frac{2\nu^2(1-e^{-\lambda T}(1+\lambda T+\frac{(\lambda T)^2}{2}))}{\lambda^2(1-e^{-\lambda T})} + \frac{\sigma^2(1-e^{-\lambda T}(1+\lambda T))}{\lambda(1-e^{-\lambda T})} \qquad \text{(F.205)}
\end{aligned}
$$

Representation with respect to Γ_1 By definition

$$\Gamma_1 = \frac{\mathbb{E}\left[\int_0^\tau \left(\mathbb{E}^{S_t}[x_\tau] - x_t\right)\right]}{\nu\mathbb{E}[\tau]}. \qquad \text{(F.206)}$$

Operating, we have that

$$\Gamma_1 = \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau (\mathbb{E}^{S_t}[x_\tau] - x_t)dt\right]}{\nu\mathbb{E}^{\hat{S}}[\tau]} = \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \mathbb{E}[\tau|a_t]dt\right]}{\mathbb{E}^{\hat{S}}[\tau]} = \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \frac{1-e^{-\lambda(T-a_t)}dt}{\lambda}dt\right]}{\mathbb{E}^{\hat{S}}[\tau]} \qquad \text{(F.207)}$$

Let us define $V(a) = \mathbb{E}^a\left[\int_0^\tau \frac{1-e^{-\lambda(T-a_t)}dt}{\lambda}dt\right]$. Then $v(a)$ satsfies the HJB equation

$$\lambda v(a) = \frac{1-e^{-\lambda(T-a)}}{\lambda} + v'(a) \qquad \text{(F.208)}$$

with $v(T) = 0$. It is easy to check that the solution is given by $v(a) = \frac{1}{\lambda^2} + Ae^{\lambda a} + \frac{ae^{-\lambda(T-a)}}{\lambda}$ and with the border condition $v(a) = \frac{1-e^{-\lambda(T-a)}(1+\lambda T)}{\lambda^2} + \frac{ae^{-\lambda(T-a)}}{\lambda}$ and we have that $v(0) = \frac{\mathbb{E}[\tau^2]}{2}$. Therefore

$$\Gamma_1 = \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]} = \mathcal{M}_1[a], \qquad \text{(F.209)}$$

where we used observability of $\mathcal{M}_1[a]$ in the last step.

Aggregation Using the direct approach we have that $\mathcal{A}_1(\delta) = \delta \times \int v'(x,a)h(x,a)dx + o(\delta)$ where $v_1(x,a)$ satisfies

$$v_1(S) = \mathbb{E}^S\left[\int_0^\tau x_t dt\right] = \mathbb{E}^S\left[\int_0^\tau (x + \nu t + \sigma_a B_t)\,dt\right] = x\mathbb{E}^S[\tau|S] + \nu\frac{\mathbb{E}[\tau^2|S]}{2} + \sigma\mathbb{E}[\tau B_\tau]. \qquad \text{(F.210)}$$

Taking the derivative with respect to x we have that $\frac{\partial v_1(x,a)}{\partial x} = \mathbb{E}^S[\tau|S] = \frac{1-e^{-\lambda(T-a)}}{\lambda}$. To compute the ergodic distribution of a note that

$$f(a) = \int_{-\infty}^{\infty} \frac{\lambda\exp\left(-\lambda a - \frac{(x-\nu a-\hat{x})^2}{2\sigma^2 a}\right)}{(1-e^{-\lambda T})\sqrt{2\pi\sigma_a^2 a}}dx = \frac{\lambda e^{-\lambda a}}{1-e^{-\lambda T}}\int_{-\infty}^{\infty}\frac{\exp\left(-\frac{(x-\nu a-\hat{x})^2}{2\sigma^2 a}\right)}{\sqrt{2\pi\sigma_a^2 a}}dx = \frac{\lambda e^{-\lambda a}}{1-e^{-\lambda T}}$$

Taking the integral

$$\begin{aligned}
\frac{\mathcal{A}_1(\delta)}{\delta} &= \int_0^T \frac{1-e^{-\lambda(T-a)}}{\lambda}\frac{\lambda e^{-\lambda a}}{1-e^{-\lambda T}}da + o(\delta) = \frac{1}{1-e^{-\lambda T}}\int_0^T \left(e^{-\lambda a} - e^{-\lambda T}\right)da + o(\delta) \\
&= \frac{\lambda^{-1}}{1-e^{-\lambda T}}\left[1-e^{-\lambda T}(1+\lambda T)\right] + o(\delta) = \frac{1}{2}\frac{\lambda}{1-e^{-\lambda T}}2\frac{1-e^{-\lambda T}(1+\lambda T)}{\lambda^2} + o(\delta) \\
&= \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]} + o(\delta) = \Gamma_1 + o(\delta)
\end{aligned} \tag{F.211}$$

F.3 Constructing covariance and variance of ergodic moments.

For a given $(\underline{x}, \hat{x}, \overline{x})$, we can define the expected time of investment as

$$\tilde{\nu} = -\frac{\psi + \mu}{\sigma^2} \ ; \ \nu = -\psi + \mu \tag{F.212}$$

$$\tilde{\lambda} = \frac{\lambda}{\sigma^2} \tag{F.213}$$

$$\xi_1 = -\tilde{\nu} - \sqrt{\tilde{\nu}^2 + 2\tilde{\lambda}} \ ; \ \xi_2 = -\tilde{\nu} + \sqrt{\tilde{\nu}^2 + 2\tilde{\lambda}}, \tag{F.214}$$

$$\overline{\alpha}_1 = \frac{e^{\xi_1 \overline{x}}}{e^{\xi_1 \underline{x} + \xi_2 \overline{x}} - e^{\xi_2 \underline{x} + \xi_1 \overline{x}}} \ ; \ \overline{\alpha}_2 = \frac{e^{\xi_2 \overline{x}}}{e^{\xi_1 \underline{x} + \xi_2 \overline{x}} - e^{\xi_2 \underline{x} + \xi_1 \overline{x}}}, \tag{F.215}$$

$$\underline{\alpha}_1 = \frac{e^{\xi_1 \underline{x}}}{e^{\xi_1 \underline{x} + \xi_2 \overline{x}} - e^{\xi_2 \underline{x} + \xi_1 \overline{x}}} \ ; \ \underline{\alpha}_2 = \frac{e^{\xi_2 \underline{x}}}{e^{\xi_1 \underline{x} + \xi_2 \overline{x}} - e^{\xi_2 \underline{x} + \xi_1 \overline{x}}}, \tag{F.216}$$

$$\mathbb{E}[\tau] = \frac{1}{\lambda} \left[-e^{\xi_1 \hat{x}} \left[\overline{\alpha}_2 - \underline{\alpha}_2 \right] - e^{\xi_2 \hat{x}} \left[\underline{\alpha}_1 - \overline{\alpha}_1 \right] + 1 \right], \tag{F.217}$$

Now if we operate, we have that

$$\mathbb{E}[\tau] = \frac{1}{\lambda} \left[-e^{\xi_1 \hat{x}} \left[\overline{\alpha}_2 - \underline{\alpha}_2 \right] - e^{\xi_2 \hat{x}} \left[\underline{\alpha}_1 - \overline{\alpha}_1 \right] + 1 \right], \tag{F.218}$$

$$= \frac{1}{\lambda} \left(\frac{-e^{\xi_1 \hat{x}} \left(e^{\xi_2 \overline{x}} - e^{\xi_2 \underline{x}} \right) - e^{\xi_2 \hat{x}} \left(e^{\xi_1 \underline{x}} - e^{\xi_1 \overline{x}} \right)}{e^{\xi_1 \underline{x} + \xi_2 \overline{x}} - e^{\xi_2 \underline{x} + \xi_1 \overline{x}}} + 1 \right] \tag{F.219}$$

$$= \frac{1}{\lambda} \left(\frac{-e^{\xi_1 \hat{x}} \left(e^{\xi_2 \overline{x}} - e^{\xi_2 \underline{x}} \right) - e^{\xi_2 \hat{x}} \left(e^{\xi_1 \underline{x}} - e^{\xi_1 \overline{x}} \right)}{e^{(\xi_1 + \xi_2) \hat{x}} \left[e^{\xi_1 (\underline{x} - \hat{x}) + \xi_2 (\overline{x} - \hat{x})} - e^{\xi_2 (\underline{x} - \hat{x}) + \xi_1 ((\overline{x} - \hat{x})} \right]} + 1 \right] \tag{F.220}$$

$$= \frac{1}{\lambda} \left(\frac{-e^{-\xi_2 \hat{x}} \left(e^{\xi_2 \overline{x}} - e^{\xi_2 \underline{x}} \right) - e^{-\xi_1 \hat{x}} \left(e^{\xi_1 \underline{x}} - e^{\xi_1 \overline{x}} \right)}{\left[e^{\xi_1 (\underline{x} - \hat{x}) + \xi_2 (\overline{x} - \hat{x})} - e^{\xi_2 (\underline{x} - \hat{x}) + \xi_1 ((\overline{x} - \hat{x})} \right]} + 1 \right] \tag{F.221}$$

$$= \frac{1}{\lambda} \left(\frac{-\left(e^{\xi_2 (\overline{x} - \hat{x})} - e^{\xi_2 (\underline{x} - \hat{x})} \right) - \left(e^{\xi_1 (\underline{x} - \hat{x})} - e^{\xi_1 (\overline{x} - \hat{x})} \right)}{e^{\xi_1 (\underline{x} - \hat{x}) + \xi_2 (\overline{x} - \hat{x})} - e^{\xi_2 (\underline{x} - \hat{x}) + \xi_1 ((\overline{x} - \hat{x})}} + 1 \right] \tag{F.222}$$

G Quantitative Models of Investment

In this section, we explain the role of each of the three simplifications to the benchmark model in Khan and Thomas (2008), Bachmann, Caballero and Engel (2013), and Winberry (2016)—eliminating free constrained investment, mean reversion and convex adjustment costs—in terms of the cross-sectional implications and transitional dynamics. To facilitate the comparison with the literature, we conduct the analysis in discrete time. As in the model presented in Section 2, we focus on a partial equilibrium environment.

G.1 Quantitative Model

Environment. Time is discrete and infinite. Consider a steady state economy with a unit mass of production units. Each unit or establishment produces output y using its predetermined capital stock k and labor n via an increasing and concave production function F, such that $y = \overline{z}eF(k,n)$, where $\overline{z}$ denotes aggregate productivity and e denotes idiosyncratic productivity. We assume that F is a decreasing returns Cobb-Douglas production function with capital share θ and labor share ν, such that $\theta + \nu < 1$; e follows an AR(1) process in logs $\log e_t = \rho_e \log e_{t-1} + \sigma_e \eta_t$, where $\eta_t \sim_{iid} \mathcal{N}(0,1)$, and $\bar{z}$ grows at a deterministic rate γ. All variables are normalized by the aggregate growth rate γ. Plants discount the future at a rate β.

At the beginning of each period, a plant hires labor at a wage w; then it chooses how much capital to purchase, subject to different types of adjustment costs. A plant can undertake an unconstraint investment $i \in \mathbb{R}$ by paying a fixed adjustment cost $\xi_t \in [0, \bar{\xi}]$ measured in labor units. The fixed cost ξ is uniformly distributed and *iid* across firms and time; its realization is unknown at the moment of making employment decisions. Alternatively, a plant may undertake free but constrained investments, that require that the investment rate is sufficiently small, i.e. $i \in [-ak, ak]$. Finally, plants face quadratic adjustment costs in the form of $\frac{\phi}{2}\left(\frac{i}{k}\right)^2 k$, measured in units of output. Given the investment policy, the plant's capital stock evolves according $(1+\gamma)k' = (1-\delta)k + i$, where i is its current investment, δ is the depreciation rate, and γ denotes aggregate productivity growth.

In each period, a plant is defined by its predetermined stock of capital $k \geq 0$ and its idiosyncratic productivity level e. Since the fixed cost realization ξ is *iid*, it is not included in the plant's state. Let $v(k,e)$ be the present discounted value of the firms optimal plan, $v^u(k,e)$ the value when undertaking an unconstrained investment (excluding the adjustment cost) and $v^c(k,e)$ the value of undertaking a constrained investment. Then $v(k,e)$ satisfies the following Bellman equations:

$$v(e,k) = \max_n \left\{ \bar{z}e^{1-\theta-\nu}k^\theta n^\nu - wn \right\} + \mathbb{E}\left[\max\left\{v^u(e,k) - \xi, v^c(e,k)\right\}\right], \tag{G.223}$$

$$v^u(e,k) = \max_i \left\{ -i - \frac{\phi}{2}\left(\frac{i}{k}\right)^2 k + \beta\mathbb{E}\left[v(e',k')\right] \right\}, \tag{G.224}$$

$$v^c(e,k) = \max_{i\in[-ak,ak]} \left\{ -i - \frac{\phi}{2}\left(\frac{i}{k}\right)^2 k + \beta\mathbb{E}\left[v(e',k')\right] \right\}, \tag{G.225}$$

where capital's law of motion is $k' = \frac{(1-\delta)k+i}{1+\gamma}$. The static labor choice yields a labor demand $n(k,e) = \left(e^{1-\theta-\nu}k^\theta \nu/w\right)^{\frac{1}{1-\nu}}$. Substituting it back, and normalizing the wage by a factor $\left[\nu^{\frac{\nu}{1-\nu}} - \nu^{\frac{1}{1-\nu}}\right]^{\frac{1-\nu}{\nu}}$, we can rewrite the plant's problem exclusively in terms of capital as follows:

$$v(e,k) = \bar{z}e^{1-\frac{\theta}{1-\nu}}k^{\frac{\theta}{1-\nu}} + \mathbb{E}\left[\max\left\{v^u(e,k) - \xi, v^c(e,k)\right\}\right] \tag{G.226}$$

where $v^u(e,k)$ and $v^c(e,k)$ remain as before.

Calibration. We follow the calibration in Winberry (2016), that targets the average investment rate, the frequency of positive investments, the inaction rate, and positive spikes in the US tax record data as reported by Zwick and Mahon (2017). Column (1) Benchmark in Table I and Table II summarize, respectively, the parametrization and the moments generated by this calibration of the model; and Figure I plots the cross-sectional distribution of capital gaps $\Delta x = 100\log(1 + i/100)$ in the model and in the data. We observe that while the calibration matches the moments mentioned before, it has trouble in generating very small and very large investment rates and other higher moments of the empirical distribution.

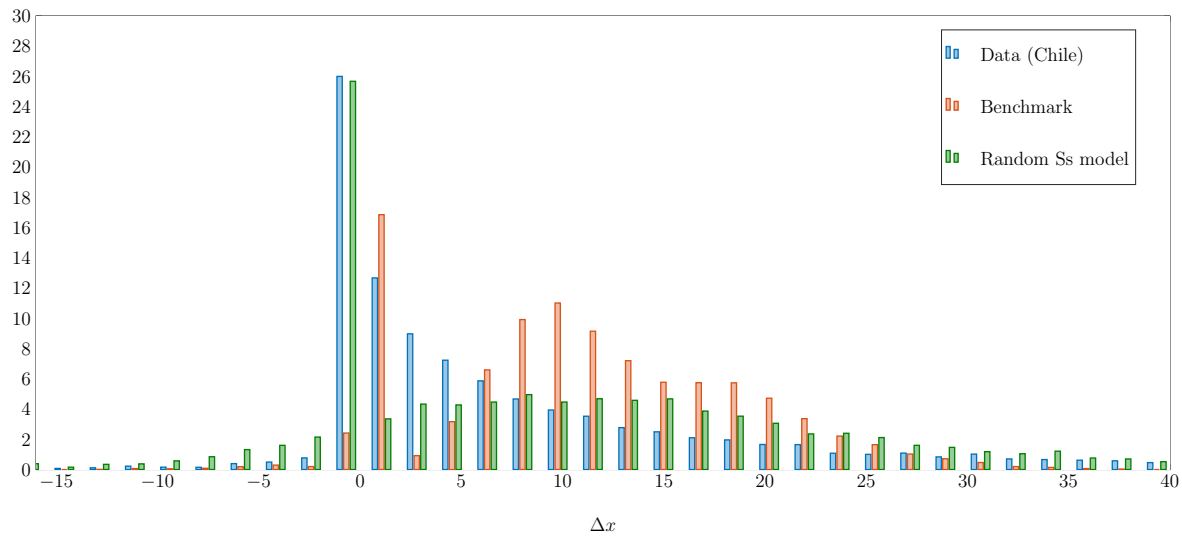
Regarding macro dynamics, Figure III plots the impulse-response of aggregate investment to an unanticipated permanent 1% increase in productivity $\bar{z}$. Capital takes around 10 quarters to converge to its new steady state.

Table I – Calibration (Benchmark and simplifications)

		(1) Benchmark	(2) No Free Investment ($a = 0$)	(3) High Persistence ($\rho = 0.99$)	(4) No Quadratic Adjustment Cost ($\phi = 0$)	(5) Random Ss Model ($\lambda > 0$)
Common Parameters						
Discount rate	β	0.99				
Capital share	θ	0.21				
Labor share	ν	0.64				
Aggregate growth	γ	0.004				
Depreciation rate	δ	0.025				
Highest fixed adj. cost	$\overline{\xi}$	0.0004				**0.002**
Volatility	σ_e	0.053				
Simplifications						
Size of constrained inv.	a	0.002	**0**	0.002	0.002	**0**
Convex adj. costs	ϕ	0.025	0.025	0.025	**0**	**0**
Persistence	ρ_e	0.94	0.94	**0.99**	0.94	**1**
Prob. zero adj. cost	λ					**0.3**

Note: Benchmark calibration from Winberry (2016).

Figure I – Investment Rate Distributions



G.2 Role of each simplification.

Next, we shut down each friction at a time to gauge its effect on the steady state micro-level distribution and on the macro dynamics after an aggregate shock (convergence towards the steady state). First, we shut down the possibility of free constrained investments by setting $a = 0$; we label this exercise "No Free Investment". Second, we increase the persistence of idiosyncratic shocks to $\rho = 0.99$; we label this exercise "High Persistence". Third and last, we eliminate the quadratic adjustment costs $\phi = 0$; we label this exercise "No Quadratic Adjustment Cost". Columns (2)-(4) in Tables Table I and II describe the parametrization and implied moments for each of the experiments, Figure II shows the distributions and Figure III the impulse-responses.

Simplification I: No free investment. Motivated by the presence of small investments, Khan and Thomas (2008) introduce the possibility that plants undertake small investments without incurring on the adjustment costs. Two consequences of eliminating free constrained investments are a decrease in the number of investments below than 1% and a shift towards investment rates near the average; both changes push the model further away from the data (and highlight why this feature was introduced). An additional consequence is an increase in the number of plants with exactly zero investment. Interestingly, while the fraction of plants with zero investment increases (there is more inaction), the quantitative response

to an aggregate shock is almost identical to the benchmark. This exercise shows that constrained investment reacts mainly to idiosyncratic shocks, and therefore it is harmless to eliminate this assumption for aggregate dynamics.

Simplification II: Highly persistent productivity. While there is little agreement on the persistence of the idiosyncratic productivity process, the autocorrelation of investment rates can provide some guidance. In the US data, the serial correlation of investment rates ranges between 0.06 in Cooper and Haltiwanger (2006)'s and 0.4 in Zwick and Mahon (2017). In contrast, the benchmark calibration with $\rho_e = 0.94$ (as well as the experiments with no free investment and no quadratic adjustment costs) generates a small but negative autocorrelation for investment rates. When we increase the persistence of idiosyncratic shocks to $\rho_e = 0.99$, the serial correlation of investment rates remains small but now positive as in the data. To understand this, consider the limit with productivity shocks that follow a random-walk. Then in the absence of adjustment costs, the investment rate becomes *iid*. With some adjustment costs (particularly of the convex type), the investment rate becomes positively serially correlated as investments are spread across several periods. With respect to the response to aggregate shocks, we see that there are no quantitative differences in the IRF with respect to the baseline case. As with the previous simplification, eliminating mean-reversion and opting for a highly persistent productivity process brings the model closer to the data and does not change quantitatively the aggregate response of the economy.

Simplification III: No quadratic adjustment costs. The early introduction of quadratic adjustment costs into representative agent investment models was aimed at eliminating the excess volatility of the aggregate investment rate generated by a frictionless model. Cooper and Haltiwanger (2006) show that a model with both convex and non-convex adjustments costs provides a good fit of the micro-data as well. What are the consequences of eliminating this assumption? At the micro level, eliminating convex costs significantly shifts the distribution to the right and increases the fraction of investment rates near zero. Overall, average moments are not affected except for fraction of positive investments (that falls) and positive spikes (that increases). In the aggregate, we see that in the absence of adjustment costs the convergence speed toward the new steady state increases dramatically.

Table II – Investment Moments (Data and Models)

	Data	(1) Benchmark	(2) No Free Investment ($a = 0$)	(3) High Persistence ($\rho = 0.99$)	(4) No Quadratic Adjustment Cost ($\phi = 0$)	(5) Random Ss model
Investment Moments	US data					
Average investment	$(0.122; 0.106)$	0.114	0.114	0.115	0.113	0.101
Inaction rate	$(0.081; 0.237)$	0.188	0.166	0.196	0.281	0.259
Positive Spike rate	$(0.186; 0.144)$	0.161	0.161	0.179	0.191	0.214
Negative Spike rate	$(0.018; -)$	0.000	0.000	0.000	0.000	0.002
Positive rate	$(0.815; 0.602)$	0.642	0.664	0.606	0.494	0.431
Negative rate	$(0.104; -)$	0.009	0.010	0.019	0.033	0.095
Serial correlation	$(0.058; 0.400)$	-0.058	-0.083	0.027	-0.192	-0.171
Std investment rates	$(0.000; 0.160)$	0.081	0.082	0.088	0.102	0.128
Zero investment	–	0.000	0.166	0.000	0.000	0.221
Higher moments	Chilean data					
$\mathbb{E}[\Delta x]$	0.136	0.131	0.129	0.134	0.146	0.128
$\mathbb{E}\left[\Delta x^2\right]$	0.051	0.021	0.021	0.023	0.028	0.031
$\mathbb{E}\left[\Delta x^3\right]$	0.029	0.004	0.004	0.004	0.006	0.008
$\mathbb{E}\left[\Delta x^4\right]$	0.021	0.001	0.001	0.001	0.001	0.002
$\mathbb{P}erc^5[\Delta x]$	-0.012	0.049	0.045	0.043	0.021	-0.064
$\mathbb{P}erc^{10}[\Delta x]$	0.014	0.062	0.057	0.059	0.064	-0.027
$\mathbb{P}erc^{25}[\Delta x]$	0.028	0.086	0.081	0.085	0.103	0.050
$\mathbb{P}erc^{50}[\Delta x]$	0.070	0.120	0.117	0.123	0.145	0.119
$\mathbb{P}erc^{75}[\Delta x]$	0.171	0.175	0.174	0.181	0.192	0.206
$\mathbb{P}erc^{90}[\Delta x]$	0.344	0.217	0.218	0.229	0.244	0.290
$\mathbb{P}erc^{95}[\Delta x]$	0.486	0.242	0.245	0.258	0.280	0.341

Notes: Micro investment moments in US reflects the computed by computed moments by Cooper and Haltiwanger (2006) (first column, first entry in parentheses) and Zwick and Mahon (2017) (first column, second entry in parentheses). The moments in US computed in the model are calculated as in their empirical target, $i_t = I_t/K_t$; as the moments in Chile $\Delta x_t = \log(1 + I/0.5 * (K_{t-1} + K_t))$.

Figure II – Investment Rate Distributions

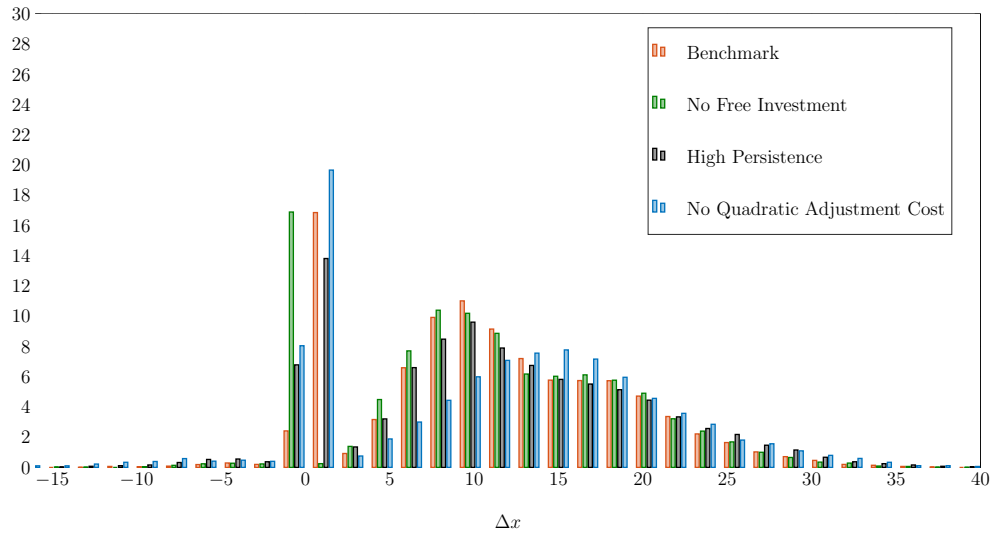
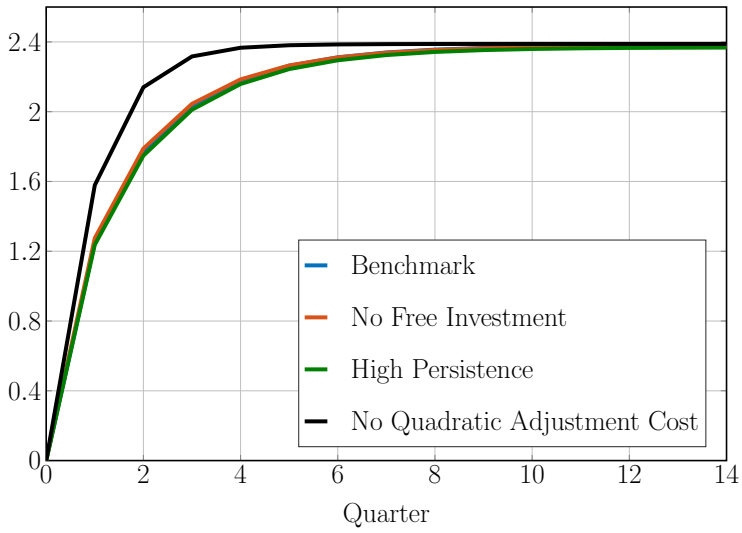


Figure III – IRF for Different Calibrations



G.3 Fit of the random Ss model

Lastly, we present the fit of our random Ss model. The model is the same as in the previous section but with $\rho_e = 1$ and a random fixed cost re-scale by e. Additional the distribution of ξ is given by a binomial random variable s.t. with probability λ, $\xi = 0$; and with $1-\lambda$, $\xi = \overline{\xi}$. In the case it is easy to show that the value function satisfies $v(e,k) = e\tilde{v}\left(\frac{k}{e}\right)$ is given by

$$\tilde{v}(\tilde{k}) = \bar{z}\tilde{k}^{\alpha} + \mathbb{E}\left[\max\left\{\tilde{v}^{u}(\tilde{k}) - \xi, \tilde{v}^{c}(\tilde{k})\right\}\right], \tag{G.227}$$

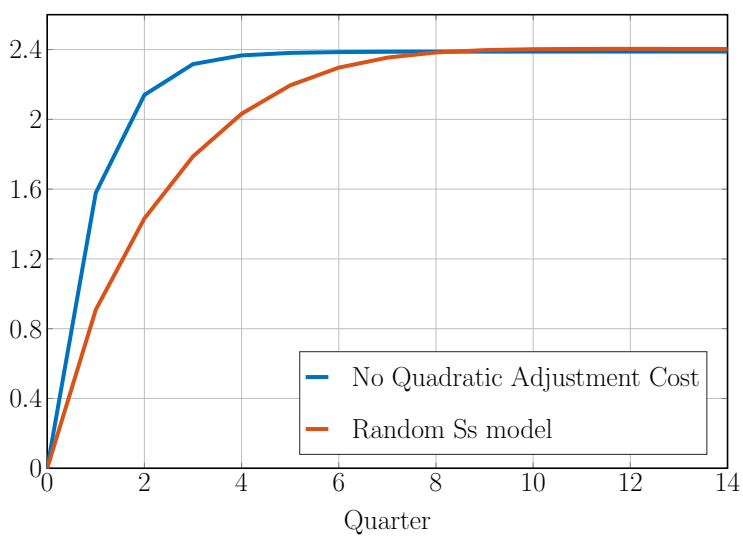
$$\tilde{v}^{u}(\tilde{k}) = \max_{\tilde{i}}\left\{-\tilde{i} - \frac{\phi}{2}\left(\frac{\tilde{i}}{\tilde{k}}\right)^{2}\tilde{k} + \beta\mathbb{E}\left[e^{\sigma_\eta \eta}\tilde{v}(\tilde{k}')\right]\right\}, \tag{G.228}$$

$$\tilde{v}^{c}(\tilde{k}) = \max_{\tilde{i}\in\{0\}}\left\{-\tilde{i} - \frac{\phi}{2}\left(\frac{\tilde{i}}{\tilde{k}}\right)^{2}\tilde{k} + \beta\mathbb{E}\left[e^{\sigma_\eta \eta}\tilde{v}(\tilde{k}')\right]\right\}, \tag{G.229}$$

where we define $\alpha = \frac{\theta}{1-\nu}$ and $k' = e^{-\sigma_\eta \eta}\frac{(1-\delta)k+i}{1+\gamma}$. For simplicity we shut down free investment and the quadratic adjustment cost, thus we have two parameters to calibrate, $\overline{\xi}$ and λ, that we choose to match the histogram of investment rates—see table I.

As we can see, this calibration of the model does a better match in generating the distribution of investment rates—the target in the calibration. We compare the transition dynamics with the transition dynation generates by the model without adjustment cost.

Figure IV – IRF for Random Ss and No Adjustment Cost models



Aggregate Dynamics in Lumpy Economies

Isaac Baley and Andrés Blanco

Data Web Appendix: Not for Publication

A Data description

The objective of the empirical work is to construct the capital stock series at the firm level in order to identify the lumpiness associated to its dynamics. This section describe the sources, the construction of variables, and the filters we apply to clean the data.

Chile. The data comes from the *Encuesta Nacional Industrial Anual* (ENIA). The sample period covers 17 years, from 1995 to 2011, with an average of 744 manufacturing plants per year. From an overall number of plant-year observations of 87,191, we first drop the 3,373 permanently small firms (i.e. with less than 10 workers throughout the sample period, about 4% of sample). This filter is motivated by the lack of good quality data with respect to these firms since ENIA is directed to plant with more than 10 workers. Second, we drop 3,101 observations with non-positive total value of book capital, wage bill or sales (about 3.5% of the sample). Third, we drop 7,070 observations that had a frequency of non-zero investment lower than 10% of the sample period (about 8% of the sample). Finally, we drop plants with less than 3 years of coverage (about 6% of the sample). Note that we consider as new plants (and give a new ID) those that disappear from the sample more than three years and reappear in the sample after that. In total, we drop about 22% of the year-plant observations and keep 68,441 observations. Within this remaining sample, a balanced panel would maintain 23% of observations or 15,691 plant-years.

The following table summarizes the data cleaning process, providing the number of plant-year observations dropped in each step.

Table I – Data cleaning

Description	**Chile**	**Colombia**
Start year	1979	—
End year	2011	—
Avg. number of plants per year	543	—
Plant-year observations	154591	—
Cleaning	Number of removed obs.	
Less than 10 employees	3984	—
Non-positive wage bill, capital, or sales	5659	—
Frequency of non-zero investment less than 10	12781	—
Less than 3 years of coverage	6222	—
Remaining observations	125945	—
% of total	81	—
With more than 10 years of data	98327	—
% of remaining observations	78	—

Sources: Authors calculations using establishment-level survey data from Chile and Colombia. Less than 10 employees refers to plants with less than 10 employees for all the years in the sample.

A.1 Perpetual Inventory Method

In order to deal with reporting and measurement errors in the surveys, we construct capital series using the standard perpetual inventory method (PIM).

Capital stocks Let firm's i stock of capital of type j on year t be given by:

$$K_{i,j,t} = (1-\delta_j)K_{i,j,t-1} + \frac{I_{i,j,t}}{D_{j,t}} \quad \text{for } K_{i,j,t_0} \text{ given.} \tag{A.1}$$

We consider the following elements to construct the capital series:

- Capital types considered are $j \in \{\text{structures}, \text{machinery and equipment}, \text{vehicles}\}$.
- Gross investment: $I_{i,j,t}$ is gross nominal investment into capital of type j at time t, and it is based on the information on purchases, reforms and improvements, and sales of fixed assets reported by each plant in the ENIA and EAM data sets.

  $$I_{i,j,t} = puchases_{i,j,t} + reforms_{i,j,t} + improvements_{i,j,t} - sales_{i,j,t} \tag{A.2}$$

  See Section A.2 for details and one alternative way to construct investment using reported data.
- Depreciation rate: $\delta_j \in (0,1)$ is a type-specific time-invariant depreciation rate described in Table II. See Section A.3 for details.

Table II – Depreciation rates

	Structures	Machinery & Equipment	Vehicles
Chile	3 %	11 %	15 %
Colombia	5 %	10 %	20 %

- Price deflators: $D_{j,t}$ are gross fixed capital formation deflators by capital type from Penn World Tables (PWT). See Section A.4 for details.
- Initial capital: K_{i,j,t_0} is given by:

  $$K_{i,j,t_0} = \frac{\tilde{K}_{i,j,t_0}}{D_{t_0}} \quad \text{if } \tilde{K}_{i,j,t_0} \geq 0, \tag{A.3}$$

  where $\tilde{K}_{i,j,t_0}$ is firm i's self-reported nominal stock of capital of type j at current prices on the starting year $t_0 = t_{0,i,j}$, which is the first year in which firm i reports a non-negative capital stock of type j.

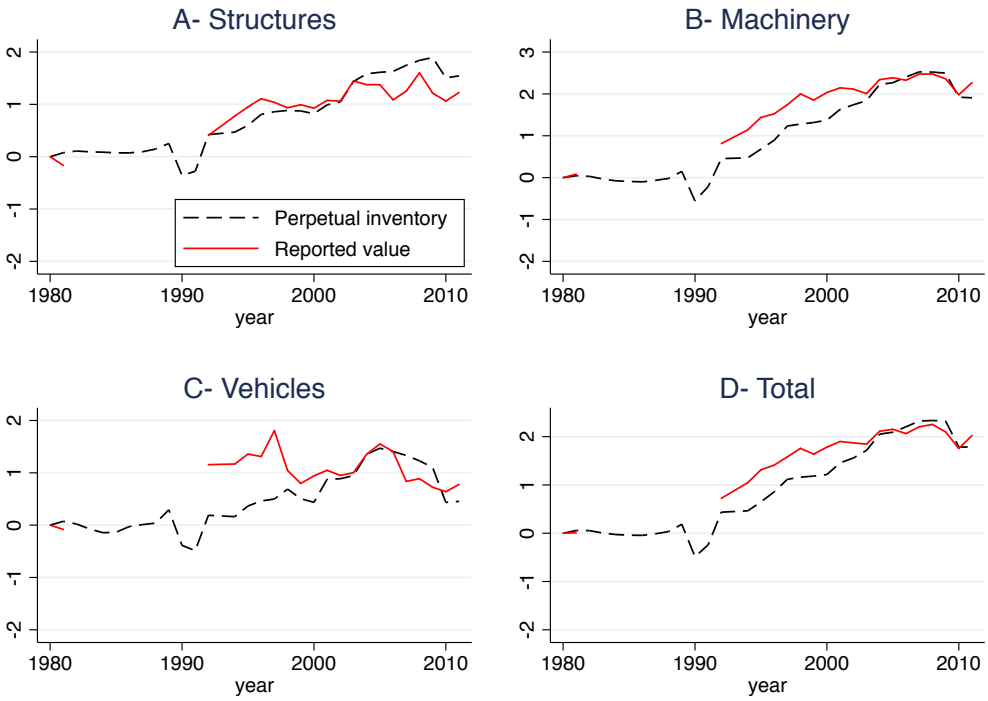
Investment rates Once we construct the investment and capital stock series, we generate the investment rate $i_{i,j,t}$ by dividing investment by initial capital:

$$i_{i,j,t} = \frac{I_{i,j,t}}{K_{i,j,t}}, \tag{A.4}$$

Outliers. In both countries, once we generate the series of investment rates, we eliminate 2% of observations considered outliers, with investment rates below the 1st percentile and above the 99th percentile of the investment rate distribution.

Figure I plots the aggregate capital stock computed with the perpetual inventory method and compares it to the reported book value. We observe that, in the aggregate, the reported book value is consistent with the PIM series, for each capital-type and for the total stock. This shows the good quality of the micro-data. Moreover, given the similarity in the series, we validate our choice of using the initial book value reported by the a plant as the initial condition for the PIM construction.

Figure I – Reported Book Value vs Perpetual Inventory in Chile



Notes: Aggregate capital stock in manufacturing sector reported by plants and computed through the PIM. All the variables are in logs and real terms, normalized to zero in 1980.

A.2 Investment and Capital Stocks

Chile. Table A.I describes the variables from the *Encuesta Nacional Industrial Anual* used to construct the firms' gross investments on structures, vehicles and machinery. In the case of structures and vehicles (purchases of new and old buildings, sales of used buildings, and the value of works in progress) definitions and availability of variables are similar across years without substantial changes. The only exception is in 2008 and 2009, in which reforms and improvements are not disaggregated, but since we focus on totals, there is no substantial change in the definition. The value of works in progress is included for the available years—from 1996 to 2011.

When constructing machinery and equipment, we take into account a substantial change in the way the data is disaggregated. The period 1995-2007 disaggregates the data into 4 categories: i) land, ii) buildings, iii) vehicles, and iv) machinery, equipment and others; in contrast, for the period 2008-2011, there are two additional categories considered, namely (v) furniture and supplies and vi) software. We include furniture and supplies in machinery, and exclude software all together from the analysis. Figure II explains our choice. It plots investment and capital stock of machinery under three alternative ways to construct machinery. Construction 1 (C1) only considers machinery and equipment, Construction 2 (C2) adds furniture and supplies, and Construction 3 (C3) adds software. After 2008, we observe a permanent drop in C1 (this drop happens before the 2008-2009 Chilean recession, so cannot be fully attributed to it). The drop is mitigated under C2, which is almost identical to C3. Since we do not see a substantial change in the dynamics of machinery when adding software, we exclude this item for constructing the capital stock for machinery.

There is a new category of investment for different type of capitals after 2008: Retirement due to capital obsolescence, dismantling, abandonment or conversion to scrap. We do not use this variable for constructing investment for two reasons. First, this variables is only available after 2008, an small part of our sample. Second, this variable is included in the perpetual inventory method whenever we consider depreciation.

Colombia. Table A.II describes the variables from the *Encuesta Anual Manufacturera* used to construct the firms' capital stock. Again, we focus on three types of capital: structures, vehicles and machinery. For all types of capital, we add up the purchase of new and old assets, costs and expenses of assets produced or built for own use, the value of upgrades and improvements of assets, and the acquisitions, transfers received and goods produced for own use; then we subtract the book value of sold assets and sales, withdrawals and adjusted transfers. The acquisitions, transfers received and goods produced for own use as well as withdrawals and adjusted transfers are only available for the earlier period of 1996-2003.

Table A.I – Construction of Capital Stock: Chile

Label	Short description	Years in data
Construction of Investment in Structures		
+ cbnedi	Purchase of new goods: buildings	1979-2011
+ cbuedi	Purchase of used goods: buildings	1979-2011
− vbuedi	Sale of used goods: buildings	1979-2011
+ refedi	Reforms and improvements made by third parties: buildings	1979-2007/2010-2011
+ vbnedi	Reforms and improvements made by the establishment: buildings	1979-2007/2010-2011
+ rmedi	Reforms and improvements: buildings	2008-2009
+ vocedi	Value of works in progress: buildings	1996-2011
Construction of Investment in Vehicles		
+ cbnveh	Purchase of new goods: vehicles	1979-2011
+ cbuveh	Purchase of used goods: vehicles	1979-2011
− vbuveh	Sale of used goods: vehicles	1979-2011
+refveh	Reforms and improvements made by third parties: vehicles	1979-2007/2010-2011
+ vbnveh	Reforms and improvements made by the establishment: vehicles	1979-2007/2010-2011
+ rmveh	Reforms and improvements: vehicles	2008-2009
+ vocveh	Value of works in progress: vehicles	1996-2011
Construction of Investment in Machinery		
+ cbnmaq	Purchase of new goods: machines, equipment etc.	1979-2011
+ cbumaq	Purchase of used goods: machines, equipment etc.	1979-2011
− vbumaq	Sale of used goods: machines, equipment etc.	1979-2011
+ refmaq	Reforms and improvements made by third parties: machines, equipment etc.	1979-2007/2010-2011
+ vbnmaq	Reforms and improvements made by the establishment: machines, equipment etc.	1979-2007/2010-2011
+ rmmaq	Reforms and improvements: machines, equipment etc.	2008-2009
+ vocmaq	Value of works in progress: machines, equipment etc.	1996-2011
+ cbnmue (+cbnotr)	Purchase of new goods: furniture and supplies (etc.)	2008-2011
+ cbumue (+cbuotr)	Purchase of used goods: furniture and supplies (etc.)	2008-2011
− vbumue (+cbuotr)	Sale of used goods: furniture and supplies (etc.)	2008-2011
+ refmue	Reforms and improvements made by third parties: furniture and supplies	2010-2011
+ rmmue (+rmotr)	Reforms and improvements: furniture and supplies (etc.)	2008-2009
+ vocmue (+vocotr)	Value of works in progress: furniture and supplies (etc.)	2008-2011
+ vbnmue	Reforms and improvements made by the establishment: furniture and supplies	2010-2011

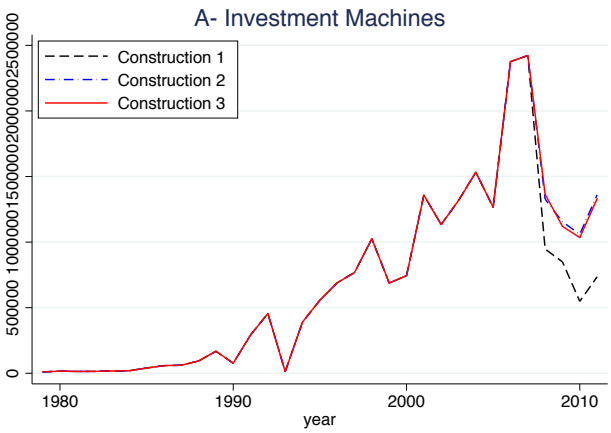
Notes: The table describes the variables used in the construction of the different types of capital. Machinery includes machinery, equipment and others, and for 2008-2011, it also includes furniture and supplies; we do not include software and intangibles.

Table A.II – Construction of Capital Stock: Colombia

Label	Short description	Years in data
Construction of Investment in Structures		
+ C7C2R2	Purchase of assets – new – buildings and structures	2004-2016
+ C7C2R3	Purchase of assets – used – buildings and structures	2004-2016
+ C7C2R4	Costs and expenses of assets produced or built for own use – buildings and structures	2004-2016
+ C7C2R7	Value of upgrades and improvements of assets – buildings and structures	2004-2016
− C7C2R12	Book value of sold assets – buildings and structures	2004-2016
+ C7R1C2	Acquisitions, transfers received and goods produced for own use – buildings and structures	1996-2003
− C7R1C5	Sales, withdrawals and adjusted transfers – buildings and structures	1996-2003
Construction of Investment in Vehicles		
+ C7C6R2	Purchase of assets – new – transportation equipment	2004-2016
+ C7C6R3	Purchase of assets – used – transportation equipment	2004-2016
+ C7C6R4	Costs and expenses of assets produced or built for own use – transportation equipment	2004-2016
+ C7C6R7	Value of upgrades and improvements of assets – transportation equipment	2004-2016
− C7C6R12	Book value of sold assets – transportation equipment	2004-2016
+ C7R8C2	Acquisitions, transfers received and goods produced for own use – transportation equipment	1996-2003
− C7R8C5	Sales, withdrawals and adjusted transfers – transportation equipment	1996-2003
Construction of Investment in Machines		
+ C7C6R2	Purchase of assets – new – machinery and industrial equipment	2004-2016
+ C7C6R3	Purchase of assets – used – machinery and industrial equipment	2004-2016
+ C7C6R4	Costs and expenses of assets produced or built for own use – machinery and industrial equipment	2004-2016
+ C7C6R5	Value of machinery under assembly during the year – used – machinery and industrial equipment	2004-2016
+ C7C6R6	Value of machinery under assembly during the year – used – machinery and industrial equipment	2004-2016
+ C7R6C2	Acquisitions, transfers received and goods produced for own use – machinery and industrial equipment	1996-2003
− C7R6C5	Sales, withdrawals and adjusted transfers – machinery and industrial equipment	1996-2003

Notes: The table describes the variables used in the construction of the different types of capital. Machines incorporate machines, equipment and others, and takes into account the change in the survey in the time period 2008-2011. In these periods, we include other type of asset and we do not include software and intangible .

Figure II – Machinery: Investment, Alternative Specifications



Notes: The figure describes aggregate investment in millions of pesos under different specifications for *machinery*. Black dashed line (construction 1) describes the construction of investment and capital machines using only variables associated with machines and equipment. Blue dashed-dotted line (construction 2) describes the construction of investment and capital machines using variables associated with machines and equipment, furniture and supplies and others. Red solid line (construction 3) adds variables associated with software.

Correcting observations with payment dues. We identify one challenge within the Chilean dataset. For some plants, there is a sequence of almost identical positive investments that share similar growth rates across firms. These payments are not associated with the rental of capital services, since there is a specific variable that captures rents and it is not correlated with these types of investments. We understand this information as payment dues: an increase in the capital stock that is registered in fractions spread through several years for book-keeping purposes (the small differences in value are part of monetary adjustments). For these reason, we substitute this investments by summing them up and assigning them to the first date in which we observe a substantial increase in the book value of this asset. The key assumption that capital is integrated into the firm's production at that first investment date. Table V describes two examples for different plants of investments in structures and machines. Table VI describes the percentage of observations with this problem, which is small.

Table V – Example of correction of payment dues: Chile

Year	Inv. structures	Book value struc.	Redefined struc.	Inv. machines	Book value mach.	Redefined mach.
1995	2274	3911	16077	393	30536	2381
1996	2279	3931	0	394	30689	0
1997	2290	3951	0	396	30842	0
1998	2299	3967	0	399	30965	0
1999	2304	3975	0	401	31027	0
2000	2313	3991	0	-377	31151	-377
2001	2318	3991	0	624	29035	624
2002	2378	6878	2378	0	21722	0
2003	2445	10068	2445	–	–	–
2004	2472	11356	2472	–	–	–

Table VI – Number of Observation with Payment Dues

	Structures	Machinery	Vehicle
Percentage of obs. with constant inv.	0.27%	1.05%	0.36%

A.3 Depreciation rates

In this section we explain the choices for type-specific depreciation rates used for each country. While it is standard in the literature to use 5% for structures, 10% for machinery and 20% for vehicles (see Oberfield (2013)), we decide to use country-specific information and different sources to set their values.

Chile. For the case of Chile, Henríquez (2008) uses tax records and other studies to obtain the average duration of different types of capital goods, as replicated here in Table VII, and derive implied geometric depreciation rates of 1.8%, 7.2% and 6%, respectively. These numbers are much smaller than the standard ones referred above. We also construct depreciation rates by capital type using information on capital stocks from the Penn World Tables (PWT) and from National Accounts, which yield higher rates than in Henríquez (2008). With this information, we decide to take middle values across sources, yielding 3% for structures, 11% for machinery and 15% for vehicles, respectively.

Table VII – Capital Depreciation in Chile

Sources	Structures	Machinery	Vehicles
Standard values			
Geometric Depreciation	0.05	0.1	0.2
Henríquez (2008)			
Lifetime 1974-1984	60	16	18
Lifetime 1985-1995	56	14	17
Lifetime 1996-2002	52	12	15
Lifetime 2003-2005	50	11	15
Average Lifetime	55.87	13.93	16.67
Geometric Depreciation (1974-2005)	0.018	0.072	0.06
PWT			
Geometric Depreciation (1979-2014)	0.025	0.155	0.224
National Accounts			
Geometric Depreciation (1985-2011)	0.029	0.12	-

Notes: Depreciation rates by capital type in Chile across different sources. Average life of machinery means average life of electric machinery.

Colombia. We obtain average depreciation rates in Colombia by type of capital from Pombo (1999). The table also shows the imputed depreciation rates in the PWT and National Accounts.

Table VIII – Capital Depreciation in Colombia

	Buildings	Machinery	Vehicles
Pombo (1999)			
Cross-sectoral mean, observed (1974-1994)	0.045	0.112	0.186
PWT			
Geometric Depreciation (1980-2014)	0.025	0.139	0.189
National Accounts			
Geometric Depreciation (1985-2011)			

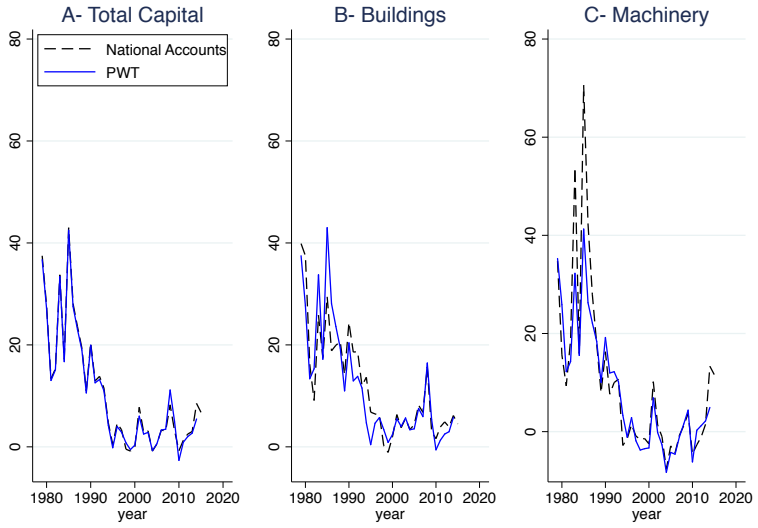
Notes: The table describes depreciation rates across different sources for Colombia.

A.4 Deflators by capital types

We use the Penn World Table (PWT) investment deflators by capital type, see Feenstra, Inklaar and Timmer (2015).[27] There are two advantages of using PWT prices instead of National Accounts. First, they are consistent with the investment types in our micro-data, and second, their methodological construction is consistent across countries and time. We verify that the data is consistent between National Accounts and PWT for the period of analysis for total capital stock and for each capital type.

Figures III and IV describe investment inflation for different capital categories.[28] We compute total investment inflation from PWT using the prices indexes for structures, machinery, vehicles and other, together with their total nominal expenditure to construct their relative weights. In National Accounts, we use "formación bruta de capital". In the case of Chile, National Accounts does not provide separately the vehicles category.

Figure III – Capital Inflation in Chile



Notes: Panels A, B and C describe inflation of total capital stock, structures and machinery. Black dashed line describes inflations computed from National Accounts and blue solid line describes inflations computed from PWT.

A.5 Comparison with National Accounts

This section verifies that the information contained in the survey data is consistent with aggregated information from National Accounts.

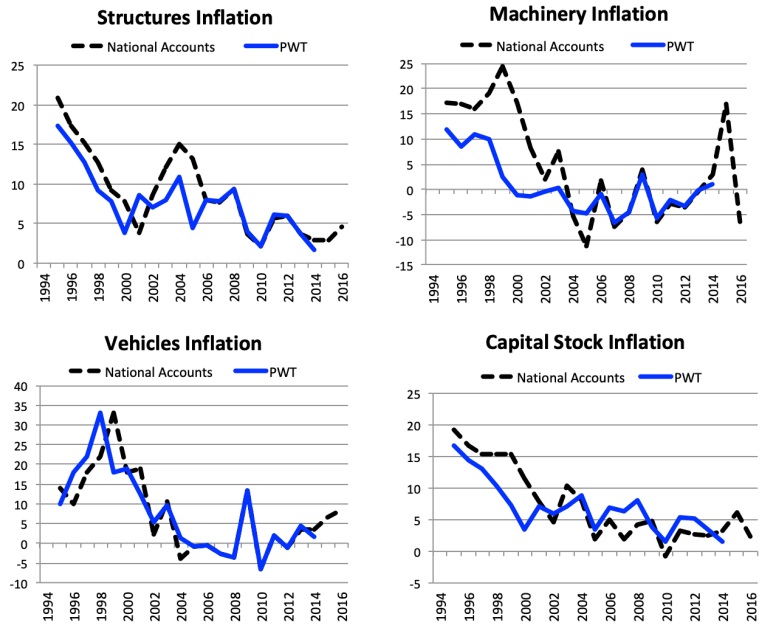
Chile. The national account office in Chile uses the ENIA survey as a source to compute several indices, such as variations in inventories or value added by type of industry. Nevertheless, National Accounts does not use ENIA to compute total investment or investment in the manufacturing sector; for that purpose, it uses sources related to the supply of capital goods (i.e., balance of payments, national statistical institute, Corporacion de Desarrollo Tecnologico de Bienes de Capital). Therefore, National Accounts serve as an orthogonal source to verify that the micro-data from the survey is consistent with the total investment in the manufacturing sector.

Panel A of Figure V describes total nominal investment constructed from the ENIA (dashed black line) and the total nominal investment in the manufacturing sector constructed using National Accounts (solid blue line), in current millions of pesos. As we can see, the two series are very close to each other with a correlation of 0.62. Total investment from the micro-data for the period 2005-2009 seems to growth at a much faster rate than National Accounts, but we found that this is mainly explained by a few outliers. For example, if we drop observations with investment rates larger than 5% of aggregate investment (dashed-dotted red line), then the fit between the aggregate investment from the micro-data and the national account is much better, both in levels and cyclicality. Finally, Panel B of V describes the proportion of total investment that is done in the manufacturing sector, which represents on average 7% in the sample period, but has been declining.

[27] PWT data is downloaded from here.

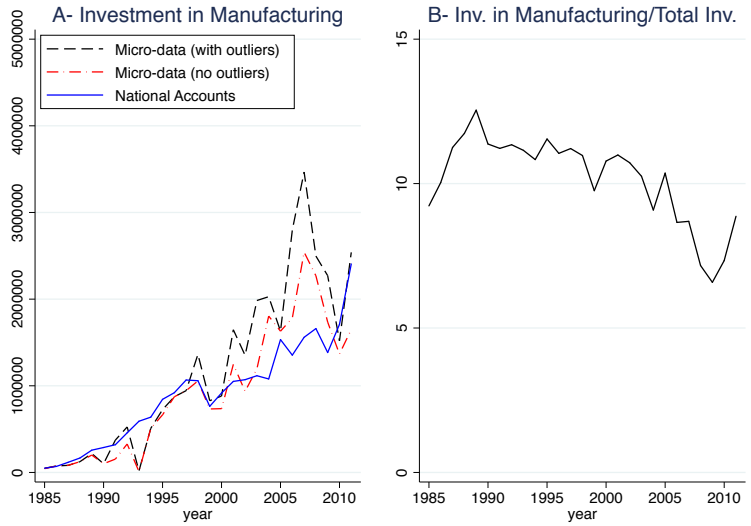
[28] The Chilean National Account data can be downloaded from here and for some years, the data is found in the "Anuario de Cuentas Nacionales" here.

Figure IV – Capital Inflation in Colombia



Notes: Inflation of total capital stock, structures, machinery and vehicles. Black dashed lines describe inflation computed from National Accounts and blue solid lines describe inflation computed from PWT.

Figure V – Investment in Chilean Manufacturing Sector: Micro-data vs. National Accounts



Notes: Panels A describes investments in the manufacturing sector. Blacked dashed line plots aggregate nominal investment constructed from ENIA, red dashed-dotted line plots the same variable but dropping outliers (i.e., investment larger than 5% of aggregate investment), and the blue solid line plots the total investment from National Accounts. Panels B describes investment in the manufacturing sector over total investment. Nominal investment from national account uses the concatenated investments from the base year 2015.

For the year 2003-2009, the National Accounts calculates the distribution of investment by capital types at the sector level. Table IX describes the composition of capital across different types from the ENIA and from National Accounts. We find that the proportions invested in structures are similar between National Accounts and in ENIA, but the decomposition between machinery and vehicles is different across datasets.

Table IX – Distribution of Investment Across types of Capital

	Structures	Machinery	Vehicle
National Accounts	35.4	51.4	13.1
ENIA	29.1	68.6	2.1

Notes: The table describes average percentage of investments across different types of capital in the ENIA and national accounts.

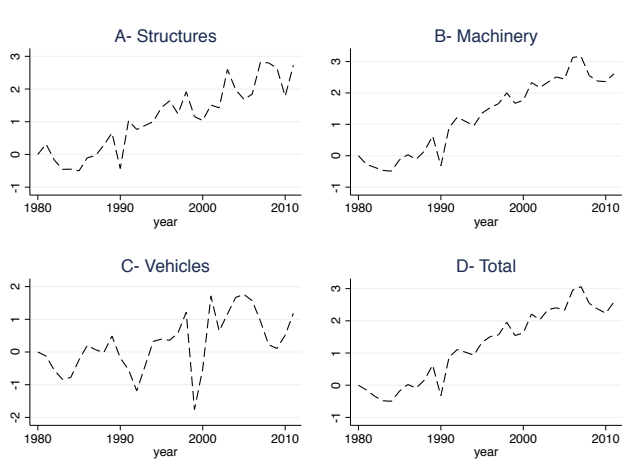
B Investment Rates Statistics

B.1 Aggregate Time Series

This section plots aggregate time series, constructed as the sum of the plant level variable $X_{j,t} = \sum_{i=1}^{N_t} x_{i,j,t}$.

Investment. Figures VI shows aggregate investment series by type of capital. All the variables are expressed in logs and in real terms, normalized in 1995.

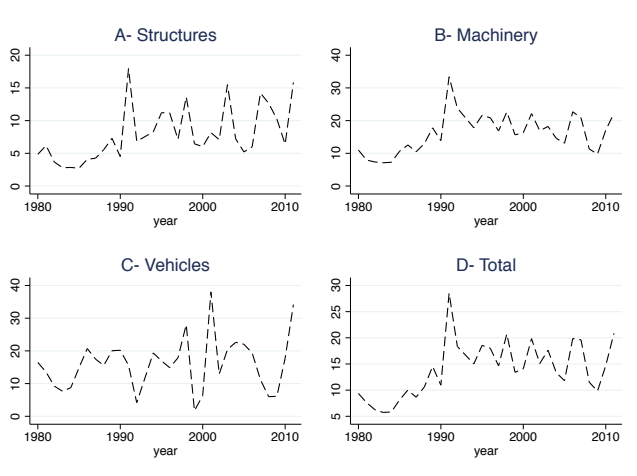
Figure VI – Investment in Chile



Notes: Panels A, B, C, and D describe aggregate manufacturing investments in buildings, machines, vehicles and total. All the variables are in log and in real values.

Investment/Capital Ratios. Figure VII shows the aggregate investment/capital ratio by type of capital. Capital stocks in both series are constructed using the PIM.

Figure VII – Investment to Capital Ratio in Chile



Notes: Panels A, B, C, and D investments over capital in buildings, machines, vehicles and total. All the variables are in log and in real values.

B.2 Cross-sectional statistics

Now we focus on the distributions of investment rates across plants for all time periods. We present yearly average of cross-sectional statistics for a variety of samples:

1. By capital type: structures, machinery and equipment, and vehicles.
2. By plant size, measured as number of workers.
3. By sector: 2-digit manufacturing sectors from CIIU classification.

Statistics by capital-type Table X presents the yearly average of cross-sectional statistics by capital type for a balanced panel within the ENIA establishment-level survey data for Chile. Table XI repeats the information for an unbalanced panel. Inaction frequency is defined as the fraction of observations with investment below 1% in absolute value; positive spikes are investments above 20% and negative spikes below −20%. Note that the column total considers the statistics for the total capital stock, and it is not the average of the statistics by capital type. For comparison, we include information for the US in Cooper and Haltiwanger (2006) and Zwick and Mahon (2017).

We find that vehicles has the largest investment rate of almost 20%. In terms of frequency of investment, the fraction of positive investments is largest in machinery, while the fraction of negative investments is largest in vehicles. The inaction rate (investment rates lower than 1% in absolute value) is largest for structures and vehicles, above 60%. Vehicles is the category with the largest spike rates (investment rates larger than 20%), both for positive and negative spikes. Across all categories, investment rates appear to be very asymmetric (the frequency of positive investments is larger than the frequency of negative investments) and serially uncorrelated.

Table X – Investment Rates Statistics: **Capital Type** (Balanced Panel)

			Chile			
	Structures	**Machinery**	**Vehicles**	**Total**	**US I**	**US II**
Average Investment	10.3	19.9	21.3	18.3	12.2	10.4
Positive Fraction ($i > 1\%$)	22.8	54.8	26.0	56.7	81.5	—
Negative Fraction ($i < -1\%$)	1.4	2.4	6.1	4.0	10.4	—
Inaction rate ($\|i\| <= 1\%$)	75.8	42.9	67.9	39.3	8.1	23.7
Spike rate ($\|i\| > 20\%$)	10.2	24.2	22.7	24.3	20.4	14.4
Positive spikes ($i > 20\%$)	9.7	23.7	19.3	23.2	18.6	—
Negative spikes ($i < -20\%$)	0.5	0.6	3.4	1.0	1.8	—
Serial correlation	0.0	0.0	0.0	0.0	0.1	0.4

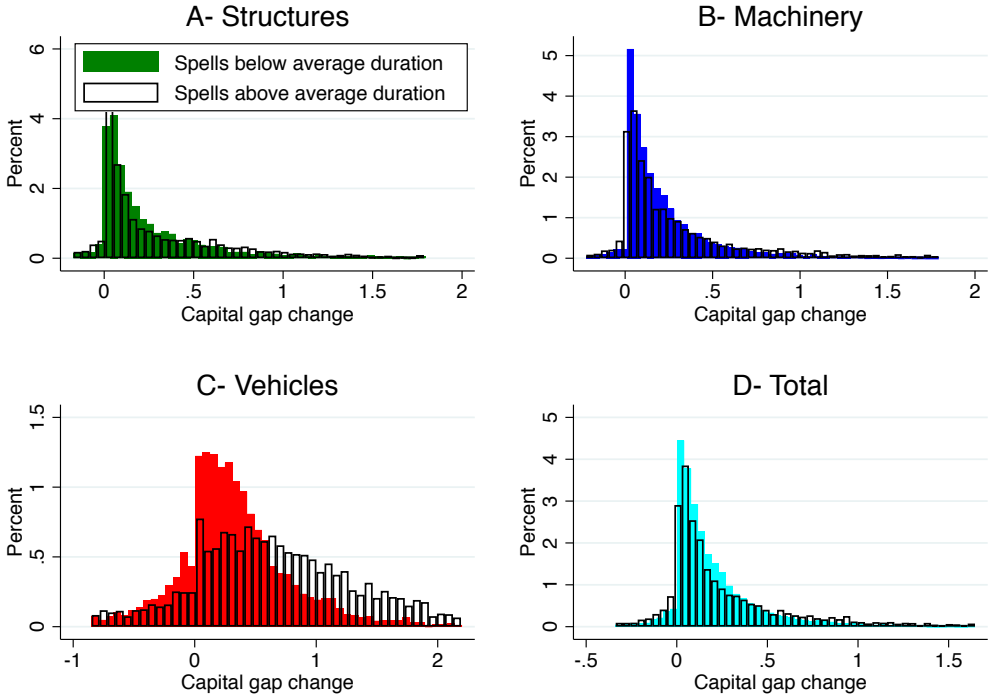
Sources: Authors calculations using establishment-level survey data for Chile. US I shows data from Cooper and Haltiwanger (2006) and US II shows data reported in Zwick and Mahon (2017) for the balanced panel. Following these papers, investment rates reported in this table are computed as Investment divided by Initial Capital. We use perpetual inventories to compute capital stock.

Table XI – Investment Rates Statistics: **Capital Type** (Unbalanced Panel)

	Chile			
	Structures	**Machinery**	**Vehicles**	**Total**
Average Investment	10.6	20.3	21.3	18.7
Positive Freq ($i > 1\%$)	22.9	53.6	25.7	55.6
Negative Freq ($i < -1\%$)	1.5	2.5	6.2	4.2
Inaction Freq ($\|i\| <= 1\%$)	75.6	43.9	68.0	40.2
Spike rate ($\|i\| > 20\%$)	10.5	24.3	22.8	24.6
Positive spikes ($i > 20\%$)	10.0	23.7	19.3	23.5
Negative spikes ($i < -20\%$)	0.5	0.6	3.6	1.1
Serial correlation	0.0	0.0	0.0	0.0

Sources: Own calculations using establishment-level survey data for Chile. Capital stocks computed with PIM.

Figure VIII – Distribution of capital gap changes by average duration



Notes: Panels A, B, C, and D describe aggregate manufacturing investments in buildings, machines, vehicles and total. All the variables are in log and in real values.

Statistics by plant size We define plant size in terms of the average number of workers during the sample period and then consider four quartiles: small plants (0-25%, S), medium plants (25-50%, M), large plants (50-75%, L), and very large plants (75-100%, XL). Table XII shows statistics by plant size for the total capital stock and Table XIII presents more detail by capital-type and plant size. In all capital categories, average investment, the frequency of non-zero investments, and the fraction of spikes increase with plant size. In contrast, the inaction rate decreases with size.

Table XII – Investment Rate Statistics: **Plant Size** (Unbalanced Panel)

	Chile				
	Small	**Medium**	**Large**	**XLarge**	**All**
Average Investment	14.8	17.4	20.2	22.3	18.7
Positive Freq ($i > 1\%$)	39.1	48.4	59.5	75.0	55.6
Negative Freq ($i < -1\%$)	3.8	4.2	4.7	4.1	4.2
Inaction Freq ($\|i\| <= 1\%$)	57.1	47.4	35.8	21.0	40.2
Spike rate ($\|i\| > 20\%$)	18.7	22.2	26.3	31.0	24.6
Positive spikes ($i > 20\%$)	17.6	21.2	25.1	29.9	23.5
Negative spikes ($i < -20\%$)	1.1	1.0	1.1	1.1	1.1
Serial correlation	-0.0	0.0	0.0	0.0	0.0

Sources: Own calculations using establishment-level survey data for Chile. Capital stocks computed with PIM.

Table XIII – Investment Rate Statistics: **Plant Size and Capital Type** (Unbalanced Panel)

	Structures				
	S	**M**	**L**	**XL**	**All**
Average Investment	5.8	7.6	11.0	15.0	18.7
Positive Freq ($i > 1\%$)	9.0	12.4	21.1	39.1	55.6
Negative Freq ($i < -1\%$)	0.6	1.0	1.3	2.5	4.2
Inaction Freq ($\|i\| <= 1\%$)	90.5	86.6	77.5	58.4	40.2
Spike rate ($\|i\| > 20\%$)	5.2	6.9	10.5	15.8	24.6
Positive spikes ($i > 20\%$)	4.9	6.4	10.0	15.1	23.5
Negative spikes ($i < -20\%$)	0.3	0.5	0.5	0.7	1.1
Serial correlation	0.0	0.0	0.0	0.0	0.0

	Machinery				
	S	**M**	**L**	**XL**	**All**
Average Investment	14.6	18.6	22.2	25.4	18.7
Positive Freq ($i > 1\%$)	35.4	45.7	57.9	74.5	55.6
Negative Freq ($i < -1\%$)	1.8	2.0	3.0	3.1	4.2
Inaction Freq ($\|i\| <= 1\%$)	62.7	52.3	39.1	22.4	40.2
Spike rate ($\|i\| > 20\%$)	16.3	20.7	26.4	33.5	24.6
Positive spikes ($i > 20\%$)	15.8	20.1	25.6	32.8	23.5
Negative spikes ($i < -20\%$)	0.6	0.5	0.8	0.7	1.1
Serial correlation	0.0	0.0	0.0	0.0	0.0

	Vehicles				
	S	**M**	**L**	**XL**	**All**
Average Investment	16.4	19.9	23.1	24.8	18.7
Positive Freq ($i > 1\%$)	15.1	20.3	26.4	38.6	55.6
Negative Freq ($i < -1\%$)	3.6	4.4	6.3	9.9	4.2
Inaction Freq ($\|i\| <= 1\%$)	81.3	75.3	67.2	51.5	40.2
Spike rate ($\|i\| > 20\%$)	15.3	19.6	24.5	30.2	24.6
Positive spikes ($i > 20\%$)	12.8	16.7	20.7	25.5	23.5
Negative spikes ($i < -20\%$)	2.6	2.9	3.8	4.7	1.1
Serial correlation	0.0	-0.0	-0.0	0.0	0.0

Sources: Authors' calculations using establishment-level survey data for Chile. Capital stocks computed with PIM.

Statistics by subsector We consider 8 major subsectors within the manufacturing sector: (1) Food and beverages; (2) Textile, clothing and leather; (3) Wood and furniture; (4) Paper and printing; (5) Chemistry, petroleum, rubber and plastic; (6) Manufacture of non-metallic mineral products; (7) Basic metal; (8) Metal products, machinery and equipment. Each sector contains around X firms or X% of the sample. Table XIV summarizes the investment rate statistics by subsector for the total capital stock, while Table XV provides more details by capital type. We observe that, besides the textile sector, there are no significant differences in the size of investments, the relative frequencies of positive, negative and zero investments, or in the spike rates.

Table XIV – Investment Rate Statistics: **Major Sectors** (Unbalanced Panel)

					Chile				
	Food	**Textile**	**Wood**	**Paper**	**Chem**	**Mineral**	**Metal**	**Machine**	**All**
Average Investment	18.1	15.6	20.1	21.1	21.4	20.9	19.1	18.7	18.7
Positive Freq ($i > 1\%$)	54.4	48.4	53.1	56.3	65.9	56.6	62.0	56.6	55.6
Negative Freq ($i < -1\%$)	3.3	4.9	5.8	4.7	4.1	3.8	4.4	4.2	4.2
Inaction Freq ($\|i\| <= 1\%$)	42.2	46.7	41.1	39.0	30.0	39.7	33.6	39.2	40.2
Spike rate ($\|i\| > 20\%$)	23.9	20.4	25.5	26.0	28.9	25.2	27.0	24.9	24.6
Positive spikes ($i > 20\%$)	23.0	19.2	24.1	25.1	27.7	24.2	25.6	24.0	23.5
Negative spikes ($i < -20\%$)	0.9	1.2	1.4	0.9	1.2	1.0	1.4	1.0	1.1
Serial correlation	0.0	0.0	0.1	0.0	-0.0	0.0	0.0	0.0	0.0

Sources: Own calculations using establishment-level survey data for Chile. Capital stocks computed with PIM.

Table XV – Investment Rate Statistics: **Major Sectors** (Unbalanced Panel)

	Food	Textile	Wood	Paper	Chem	Mineral	Metal	Machine	All
Structures									
Average Investment	10.6	7.7	12.0	10.7	13.1	11.6	11.6	10.0	18.7
Positive Freq ($i > 1\%$)	25.2	13.7	22.8	21.7	30.0	24.3	25.8	20.0	55.6
Negative Freq ($i < -1\%$)	1.4	1.4	1.6	1.6	1.9	1.5	1.8	1.2	4.2
Inaction Freq ($\|i\| <= 1\%$)	73.4	84.9	75.6	76.7	68.1	74.2	72.4	78.8	40.2
Spike rate ($\|i\| > 20\%$)	10.9	7.5	11.3	9.6	13.1	10.3	12.0	9.6	24.6
Positive spikes ($i > 20\%$)	10.5	6.8	10.6	9.2	12.4	9.9	11.3	9.1	23.5
Negative spikes ($i < -20\%$)	0.4	0.7	0.7	0.4	0.7	0.4	0.6	0.4	1.1
Serial correlation	0.0	0.1	-0.0	-0.0	0.0	0.0	0.0	0.0	0.0
Machinery									
	Food	Textile	Wood	Paper	Chem	Mineral	Metal	Machine	All
Average Investment	19.2	17.2	22.0	22.2	23.5	21.6	21.2	21.0	18.7
Positive Freq ($i > 1\%$)	51.3	47.4	50.7	54.6	64.5	53.5	59.9	55.2	55.6
Negative Freq ($i < -1\%$)	1.9	3.1	3.5	3.6	2.6	2.3	2.3	2.2	4.2
Inaction Freq ($\|i\| <= 1\%$)	46.7	49.4	45.8	41.8	32.9	44.2	37.7	42.6	40.2
Spike rate ($\|i\| > 20\%$)	23.0	20.7	24.2	25.4	30.4	24.6	27.0	24.5	24.6
Positive spikes ($i > 20\%$)	22.5	20.0	23.2	24.7	29.6	23.9	26.3	24.0	23.5
Negative spikes ($i < -20\%$)	0.5	0.7	1.0	0.7	0.8	0.7	0.7	0.5	1.1
Serial correlation	0.0	0.0	0.0	-0.0	0.0	0.0	0.0	0.0	0.0
Vehicles									
	Food	Textile	Wood	Paper	Chem	Mineral	Metal	Machine	All
Average Investment	21.2	16.9	22.8	21.0	24.5	19.0	22.8	23.3	18.7
Positive Freq ($i > 1\%$)	27.3	17.4	27.1	21.8	30.6	25.6	27.6	27.8	55.6
Negative Freq ($i < -1\%$)	5.8	4.8	7.9	5.3	7.9	6.7	6.5	6.3	4.2
Inaction Freq ($\|i\| <= 1\%$)	66.8	77.7	65.0	72.9	61.4	67.7	65.9	65.8	40.2
Spike rate ($\|i\| > 20\%$)	23.3	17.6	23.7	19.8	27.2	21.1	24.2	24.8	24.6
Positive spikes ($i > 20\%$)	20.2	14.3	19.2	16.6	22.6	17.3	20.6	21.1	23.5
Negative spikes ($i < -20\%$)	3.1	3.3	4.5	3.1	4.6	3.8	3.6	3.6	1.1
Serial correlation	0.0	-0.0	-0.0	0.0	0.0	0.0	-0.0	-0.0	0.0

Sources: Own calculations using establishment-level survey data for Chile. Capital stocks computed with PIM.

C Inputs from data

In this section, we construct cross-sectional statistics using the panel data on $(\Delta x, \tau)$ and then use them as inputs into our observation formulas to recover parameters, steady state moments and the CIR. First, we explain how to use the data on investment rates to construct changes in capital gaps, and second, we explain how to construct the stopping time distribution and some challenges.

Capital gaps. Recall that the change in capital gaps is given by the log difference in the capital stock between an adjustment date $\tau_{i,j,t}$ and immediately before adjustment $\tau^-_{i,j,t}$:

$$\Delta x_{i,j,t} = \hat{x} - x_{\tau^-_{i,j,t}} = \log\left(K_{\tau_{i,j,t}}/K_{\tau^-_{i,j}}\right) = \log\left(1 + i_{i,j,t}\right) \tag{C.5}$$

Using the information on investment rates, we construct capital gaps as:

$$\Delta x_{i,j,t} = \begin{cases} \log\left(1 + i_{i,j,t}\right) & \text{if} \quad |i_{i,j,t}| > \underline{i} \\ 0 & \text{if} \quad |i_{i,j,t}| < \underline{i}, \end{cases} \tag{C.6}$$

where $\underline{i} > 0$ is a parameter that captures the idea that small maintenance investments do not incur the fixed cost of investment. Following Cooper and Haltiwanger (2006), we set $\underline{i} = 0.01$, such that all investments smaller than 1% in absolute value are excluded and considered as part of the inaction frequency.

Stopping Times Distribution. There are two challenges to estimate moments in the stopping time distribution: i) Right-censoring, and ii) Heterogeneity.

(1) Right censoring Right censoring comes for the fact that the panel is finite in the time dimension. Thus, instead of estimating unconditional moments, we estimate moments conditional of being in the sample. For example, if we work in a panel data with T years, then using the simple mean across stopping times implies that we estimate $\mathbb{E}[\tau_{ij} | \sum_j \tau_{ij} < T]$, where i stands for the plant and j for the j-th adjustment. To deal with this challenge, we implement a series of random samplings of stopping times, one by firm, and then taking the average stopping time of each firm. As a way to corroborate the validity of this strategy, we compute average capital age directly from the data and then compare it to the value obtained by using the relationship between age and stopping times $\mathbb{E}[a] = \frac{1}{2}\mathbb{E}^{\hat{x}}[\tau]\left(1 + \mathbb{CV}^2[\tau]\right)$. With the random sampling, both numbers are very close to each other; we take this as a sign that the concerns regarding right-censoring are alleviated this way.

(2) Heterogeneity Statistics related to duration, in particular stopping times, are very sensitive to the degree of heterogeneity in the sample. To alleviate this concern, one ideally report statistics for very disaggregated data. The challenge here lies in that, in order to compute each stopping time, we require at least two investment observations to determine the beginning and the end of the inaction period. While this is not a concern when considering the dataset as a whole, it becomes a problem when computing statistics for smaller subsamples. Considering these issues, we deal with heterogeneity in the following way. When computing statistics by plant size, we pool small and medium plants together and large and extra large plants together. Regarding subsectors within manufacturing, our analysis in the previous section shows that investment behavior at the plant level across sectors is very homogenous, and for that reason, heterogeneity does not appear as a concern in that case.

C.1 Results by capital type

Table XIX shows results by capital type and is divided in two parts. In the upper part, we show the cross-sectional statistics for frequency, capital gaps, and covariances between them, which serve as inputs into our observation formulas. The lower part of the table shows the estimated parameters $\nu, \sigma, \hat{x}$ and ergodic moments $\mathcal{M}_2[x], \mathcal{M}_{1,1}[x, a]$, and $\text{CIR}_1(\delta)$ implied by our formulas.

Table XVI – Inputs from Micro Data and Outputs from the Theory (Balanced Panel)

		Chile		
Inputs from Micro Data	**Structures**	**Machinery**	**Vehicles**	**Total**
Frequency				
$\mathbb{E}^{\hat{x}}[\tau]$	2.44	1.75	2.44	1.71
$\mathbb{CV}^2[\tau]$	1.09	0.93	0.90	0.85
Capital Gaps				
$\mathbb{E}^{\hat{x}}[\Delta x]$	0.27	0.25	0.42	0.22
$\mathbb{E}^{\hat{x}}[\Delta x^2]$	0.19	0.16	0.46	0.13
$\mathbb{E}^{\hat{x}}[(\hat{x} - \Delta x)^3]$	-0.19	-0.13	-0.33	-0.10
Covariances				
$\mathbb{C}ov^{\hat{x}}[\tau, \Delta x]$	0.06	0.08	0.36	0.04
$\mathbb{E}^{\hat{x}}[\tau(\hat{x} - \Delta x)^2]$	0.53	0.34	1.32	0.25
Outputs from Theory	**Structures**	**Machinery**	**Vehicles**	**Total**
Parameters				
ν	-0.11	-0.14	-0.17	-0.13
σ^2	0.08	0.08	0.13	0.07
$\hat{x}$	0.01	0.05	0.17	0.04
Steady State Moments				
$\mathbb{V}ar[x]$	0.23	0.17	0.26	0.15
$\mathbb{C}ov[x, a]$	0.91	0.38	0.08	0.42
Transitional Dynamics				
$\mathbb{V}[x]/\sigma^2$	3.01	2.25	2.03	2.24
$-\nu\mathbb{C}ov[x, a]/\sigma^2$	1.34	0.71	0.10	0.80
$\text{CIR}_1(\delta)$ in random fixed cost model	4.35	2.96	2.13	3.04

Sources: Authors calculations using establishment-level survey data for Chile.

C.2 Results by firm size

Table XVII – Inputs from Micro Data and Outputs from the Theory (Balanced Panel, Small Firms)

		Chile		
Inputs from Micro Data	**Structures**	**Machinery**	**Vehicles**	**Total**
Frequency				
$\mathbb{E}^{\hat{x}}[\tau]$	2.87	2.06	2.83	1.97
$\mathbb{CV}^2[\tau]$	1.17	0.99	0.89	0.94
Capital Gaps				
$\mathbb{E}^{\hat{x}}[\Delta x]$	0.32	0.27	0.51	0.23
$\mathbb{E}^{\hat{x}}[\Delta x^2]$	0.22	0.18	0.56	0.15
$\mathbb{E}^{\hat{x}}[(\hat{x}-\Delta x)^3]$	-0.21	-0.15	-0.41	-0.12
Covariances				
$\mathbb{C}ov^{\hat{x}}[\tau, \Delta x]$	0.09	0.11	0.48	0.07
$\mathbb{E}^{\hat{x}}[\tau(\hat{x}-\Delta x)^2]$	0.78	0.46	1.77	0.34
Outputs from Theory	**Structures**	**Machinery**	**Vehicles**	**Total**
Parameters				
ν	-0.11	-0.13	-0.18	-0.12
σ	0.08	0.07	0.13	0.07
$\hat{x}$	0.01	0.05	0.20	0.04
Steady State Moments				
$\mathcal{M}_2[x]$	0.22	0.19	0.27	0.16
$\mathcal{M}_{1,1}[x,a]$	0.83	0.47	-0.03	0.48
Transitional Dynamics				
$\mathcal{M}_2[x]/\sigma^2$	2.87	2.61	2.14	2.53
$-\nu\mathcal{M}_{1,1}[x,a]/\sigma^2$	1.20	0.82	-0.05	0.88
$\mathrm{CIR}_1(\delta)$	4.07	3.43	2.09	3.40

Sources: Authors calculations using establishment-level survey data for Chile.

Table XVIII – Inputs from Micro Data and Outputs from the Theory (Balanced Panel, Large Firms)

		Chile		
Inputs from Micro Data	**Structures**	**Machinery**	**Vehicles**	**Total**
Frequency				
$\mathbb{E}^{\hat{x}}[\tau]$	2.29	1.47	2.15	1.45
$\mathbb{CV}^2[\tau]$	1.02	0.69	0.81	0.58
Capital Gaps				
$\mathbb{E}^{\hat{x}}[\Delta x]$	0.26	0.24	0.36	0.21
$\mathbb{E}^{\hat{x}}[\Delta x^2]$	0.19	0.14	0.40	0.11
$\mathbb{E}^{\hat{x}}[(\hat{x} - \Delta x)^3]$	-0.19	-0.10	-0.29	-0.08
Covariances				
$\mathbb{C}ov^{\hat{x}}[\tau, \Delta x]$	0.03	0.04	0.23	0.01
$\mathbb{E}^{\hat{x}}[\tau(\hat{x} - \Delta x)^2]$	0.46	0.22	0.97	0.16
Outputs from Theory	**Structures**	**Machinery**	**Vehicles**	**Total**
Parameters				
ν	-0.11	-0.16	-0.17	-0.14
σ	0.08	0.08	0.14	0.06
$\hat{x}$	0.01	0.06	0.14	0.05
Steady State Moments				
$\mathcal{M}_2[x]$	0.24	0.15	0.26	0.12
$\mathcal{M}_{1,1}[x, a]$	1.01	0.28	0.23	0.30
Transitional Dynamics				
$\mathcal{M}_2[x]/\sigma^2$	3.13	1.89	1.93	1.90
$-\nu\mathcal{M}_{1,1}[x, a]/\sigma^2$	1.45	0.59	0.29	0.66
$\text{CIR}_1(\delta)$	4.58	2.48	2.22	2.56

Sources: Authors calculations using establishment-level survey data for Chile.

C.3 Results with Weights

Table XIX – Inputs from Micro Data and Outputs from the Theory (Balanced Panel, Weighted by Capital)

		Chile		
Inputs from Micro Data	**Structures**	**Machinery**	**Vehicles**	**Total**
Frequency				
$\mathbb{E}^{\hat{x}}[\tau]$	2.21	1.29	1.60	1.27
$\mathbb{CV}^2[\tau]$	0.87	0.37	0.74	0.34
Capital Gaps				
$\mathbb{E}^{\hat{x}}[\Delta x]$	0.17	0.21	0.33	0.18
$\mathbb{E}^{\hat{x}}[\Delta x^2]$	0.10	0.10	0.34	0.07
$\mathbb{E}^{\hat{x}}[(\hat{x}-\Delta x)^3]$	-0.09	-0.06	-0.41	-0.04
Covariances				
$\mathbb{C}ov^{\hat{x}}[\tau, \Delta x]$	-0.07	-0.00	0.11	0.00
$\mathbb{E}^{\hat{x}}[\tau(\hat{x}-\Delta x)^2]$	0.18	0.11	0.66	0.08
Outputs from Theory	**Structures**	**Machinery**	**Vehicles**	**Total**
Parameters				
ν	-0.08	-0.16	-0.21	-0.14
σ	0.05	0.06	0.16	0.04
$\hat{x}$	-0.02	0.06	0.11	0.06
Steady State Moments				
$\mathcal{M}_2[x]$	0.18	0.09	0.42	0.07
$\mathcal{M}_{1,1}[x,a]$	1.24	0.20	0.56	0.15
Transitional Dynamics				
$\mathcal{M}_2[x]/\sigma^2$	3.87	1.64	2.51	1.65
$-\nu\mathcal{M}_{1,1}[x,a]/\sigma^2$	2.06	0.53	0.67	0.50
$\mathrm{CIR}_1(\delta)$	5.93	2.17	3.18	2.15

Sources: Authors calculations using establishment-level survey data for Chile.